

## Certain conditions for a Riemannian manifold to be isometric with a sphere

Dedicated to Professor Kentaro Yano on his fiftieth birthday

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### Introduction.

In this paper, by a Riemannian manifold we always mean a connected  $C^\infty$ -manifold of dimension  $n$  ( $\geq 2$ ) with a positive definite  $C^\infty$ -Riemannian metric. A transformation of a Riemannian manifold is said to be *conformal* if it preserves the angle defined by the Riemannian metric. Evidently a conformal transformation with respect to one Riemannian metric is also conformal with respect to one conformally related to the original one. It is known [5] that if a manifold is compact and of dimension greater than two, every Riemannian metric on it can be conformally deformed into one with constant scalar curvature. Thus as far as a conformal transformation is concerned, we may assume the constancy of the scalar curvature in the above case. Then the scalar curvature, which is constant, is preserved by the transformation. This property seems to help the study of conformal transformations. Indeed, a conformal transformation between two compact Riemannian manifolds of non-positive scalar curvature, not necessarily constant, is isometric if and only if it carries one scalar curvature into another [4]. Thus a compact Riemannian manifold of constant scalar curvature admits a conformal transformation, which is not isometric, only if the scalar curvature is positive. Therefore we may restrict our consideration to Riemannian manifolds of positive constant scalar curvature as far as a conformal transformation is concerned, provided that the manifold is compact and of dimension  $> 2$ .

Furthermore, if a Riemannian manifold of constant scalar curvature  $k$  admits an infinitesimal conformal transformation  $u$  with  $\mathcal{L}_u g = 2\phi g$ , where  $g$  is the Riemannian metric,  $\mathcal{L}_u$  the operation of Lie derivatives corresponding to  $u$  and  $\phi$  a function, then  $\phi$  satisfies the equation  $\Delta\phi = nk\phi$ , where  $k = K/n(n-1)$ ,  $K$  being the contracted curvature scalar (See for example [2, 6]). The existence of such a function might give some informations about the topological structure of the Riemannian manifold. Indeed we are going to show that *a compact Einstein manifold of constant scalar curvature  $k$  admits a non-constant*

function  $\phi$  such that  $\Delta\phi = nk\phi$ , if and only if the manifold is isometric with a sphere  $S^n(\sqrt{k})$  with radius  $1/\sqrt{k}$  in the  $(n+1)$ -dimensional Euclidean space (the main theorem).

In the course of the proof it will be shown that such a function satisfies the condition  $\nabla_X d\phi = -k\phi X$  for any tangent vector  $X$  of the manifold,  $\nabla_X$  being the operator of covariant differentiation in the direction of  $X$ . Then we use

**THEOREM A.** *In order for a complete Riemannian manifold of dimension  $n \geq 2$  to admit a non-constant function  $\phi$  with  $\nabla_X d\phi = -c^2\phi X$  for any vector  $X$ , it is necessary and sufficient that the manifold be isometric with a sphere  $S^n(c)$  of radius  $1/c$  in the  $(n+1)$ -Euclidean space.*

This theorem has been proved and used in the Einstein case [7] and also in the case of constant scalar curvature [1]. Even though the existence of such a function is implied by the existence of an infinitesimal conformal transformation in the Einstein case, essential is the former (with neither assumption of Einstein manifold nor of constant scalar curvature). We are going to give an elementary proof of the theorem in the coordinate free method in §2.

§1 is devoted to the proof of the above main theorem. In the course of it, it will also be shown that the minimum eigenvalue of the Laplacian restricted to the functions on a compact Einstein space is just equal to  $nk$ . This is a generalization of a result of Nagano [3]. The main theorem is just the case where the Laplacian admits such value as an eigenvalue.

1. Let  $M$  be a compact orientable Riemannian manifold of dimension  $n \geq 2$  with metric  $g: \langle, \rangle$  or  $ds^2 = g_{ji} dx^j dx^i$ . In the following, covariant and contravariant tensors are identified in the canonical manner.

The global inner product of two tensor fields  $u: u_{i_p \dots i_1}$  and  $v: v_{i_p \dots i_1}$  of degree  $p$  is defined by

$$(u, v) = (v, u) = \int_M u_{i_p \dots i_1} v^{i_p \dots i_1} dM,$$

$dM$  being the volume element of  $M$ . We denote by  $\nabla$  the operator of covariant differentiation and by  $\nabla_X$  that of covariant differentiation in the direction  $X$ ,  $X$  being a vector on  $M$ . We denote by  $\delta$  the operator acting on a tensor field  $u$  of degree  $p$  to give a tensor field of degree  $p-1$ ,  $\delta u: -\nabla_i u^{i i_{p-1} \dots i_1}$ . The Laplace operator for a scalar field is written as  $\Delta = \delta \nabla$ . It is well-known that  $\nabla$  and  $\delta$  are dual to each other:

$$(\nabla u, v) = (u, \delta v).$$

The Ricci curvature tensor  $K_{ji}$  defines a transformation  $R$  of a vector  $R: v \rightarrow R \cdot v$ , i. e.  $v^i \rightarrow K^i_j v^j$ . If the manifold is Einsteinian,  $R$  is the  $(n-1)k$ -times the identity.

Assuming that a non-constant function  $\phi$  satisfies the equation  $\Delta\phi = n\lambda\phi$  for some real  $\lambda \neq 0$ , we prepare some formulas. First of all it is known that  $\lambda$  is positive.

$$(1.1) \quad (\nabla\phi, \nabla\phi) = (\Delta\phi, \phi) = n\lambda(\phi, \phi)$$

$$(1.2) \quad (\nabla^2\phi, \phi g) = -(\Delta\phi, \phi) = -(\nabla\phi, \nabla\phi)$$

$$(1.3) \quad \delta\nabla^2\phi = n\lambda\nabla\phi - R \cdot \nabla\phi$$

In fact, it has components :

$$\begin{aligned} -\nabla_k\nabla_j\nabla_i\phi g^{kj} &= -\nabla_k\nabla_i\nabla_j\phi g^{kj} \\ &= -\nabla_i\nabla_k\nabla_j\phi g^{kj} - K_i^j\nabla_j\phi \\ &= n\lambda\nabla_i\phi - (R \cdot \nabla\phi)_i \end{aligned}$$

$$(1.4) \quad (\nabla^2\phi, \nabla^2\phi) = (\nabla\phi, \delta\nabla^2\phi) = n\lambda(\nabla\phi, \nabla\phi) - (R \cdot \nabla\phi, \nabla\phi)$$

On putting  $v = \nabla^2\phi + \lambda\phi g$ , we have, from the above formulas,

$$(1.5) \quad \begin{aligned} (v, v) &= (\nabla^2\phi, \nabla^2\phi) + 2\lambda(\nabla^2\phi, \phi g) + \lambda^2(\phi g, \phi g) \\ &= (n-1)\lambda(\nabla\phi, \nabla\phi) - (R \cdot \nabla\phi, \nabla\phi). \end{aligned}$$

Since  $(v, v) \geq 0$  and  $(\nabla\phi, \nabla\phi) > 0$ , we must have

$$(1.6) \quad \lambda \geq \frac{(R \cdot \nabla\phi, \nabla\phi)}{(n-1)(\nabla\phi, \nabla\phi)}.$$

The equality holds only when

$$\nabla^2\phi + \lambda\phi g = 0,$$

which implies that  $M$  is isometric with  $S^n(\sqrt{\lambda})$  by Theorem A.

If  $M$  is non-orientable, we take the orientable double covering  $\bar{M}$  of  $M$  and induce, in the natural manner, the Riemannian metric  $\bar{g}$  and the function  $\bar{\phi}$  from  $g$  and  $\phi$  respectively. Then  $M$  and  $\bar{M}$  have the same local geometry. Denoting by  $\bar{\nabla}$  and  $\bar{\Delta}$  the operator of covariant differentiation and the Laplace operator respectively corresponding to  $\bar{g}$ , we have

$$\begin{aligned} \bar{\Delta}\bar{\phi} = n\lambda\bar{\phi} &\quad \text{if and only if} \quad \Delta\phi = n\lambda\phi \\ \bar{\nabla}^2\bar{\phi} + \lambda\bar{\phi}\bar{g} = 0 &\quad \text{if and only if} \quad \nabla^2\phi + \lambda\phi g = 0. \end{aligned}$$

Thus if  $\nabla^2\phi + \lambda\phi g = 0$ , we know that  $M$  and  $\bar{M}$  are both isometric with  $S^n(\sqrt{\lambda})$ .

Therefore the assumption of orientability is removed and we have

**THEOREM 1.** *Let  $M$  be a connected compact Riemannian manifold of dimension  $n \geq 2$ . If  $M$  admits a non-constant function  $\phi$  such that  $\Delta\phi = n\lambda\phi$ ,  $\lambda \neq 0$ , then  $\lambda$  is positive and*

$$(1.6) \quad \lambda \geq \frac{(R \cdot \nabla\phi, \nabla\phi)}{(n-1)(\nabla\phi, \nabla\phi)}.$$

THEOREM 2. *In order for a compact Riemannian manifold  $M$  of dimension  $n \geq 2$  to be isometric with  $S^n(c)$  it is necessary and sufficient that  $M$  admit a non-constant function  $\phi$  with*

$$\Delta\phi = n c^2\phi$$

and

$$(n-1)c^2(\nabla\phi, \nabla\phi) = (R \cdot \nabla\phi, \nabla\phi).$$

If  $M$  is Einsteinian, (1.6) turns out to be  $\lambda \geq k$  where  $k$  is the constant scalar curvature (in case  $n = 2$ ,  $k$  is assumed to be constant). This means that the smallest possible value of  $\lambda$  in the Einstein case is just equal to the scalar curvature.

THEOREM 3. *Let  $M$  be a compact Einstein manifold of dimension  $n \geq 2$  with positive constant scalar curvature  $k$ . If  $M$  admits a non-constant function  $\phi$  such that  $\Delta\phi = n\lambda\phi$ ,  $\lambda \neq 0$ , then*

$$\lambda \geq k.$$

Now let  $P_i$  ( $i = 0, 1, 2, 3$ ) be the following properties of an Einstein manifold  $M$  with constant scalar curvature  $k > 0$ .

$P_0$ :  $M$  is isometric with a sphere  $S^n(\sqrt{k})$ .

$P_1$ :  $M$  admits a non-homothetic infinitesimal conformal transformation.

$P_2$ :  $M$  admits a non-constant function  $\phi$  satisfying

$$\nabla^2\phi + k\phi g = 0.$$

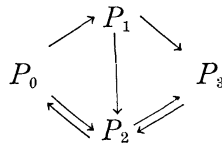
$P_3$ :  $M$  admits a non-constant function  $\phi$  satisfying

$$\Delta\phi = nk\phi.$$

Then Theorem 2 can be stated in the following form in the Einstein case:

THEOREM 4. *Let  $M$  be a compact Einstein manifold with positive constant scalar curvature  $k$ . Then the conditions  $P_0, P_1, P_2$  and  $P_3$  are equivalent with each other.*

In case  $\dim M = 2$ ,  $M$  is of constant curvature and the theorem is obvious, even though the implication  $P_1 \rightarrow P_2$  is not evident. Namely we have the diagram



It is also stated in the following form:

THEOREM 5. *In order for a compact Einstein manifold of dimension  $n$  with constant scalar curvature  $c^2$  to admit a non-constant function  $\phi$  with  $\Delta\phi = n c^2\phi$ , it is necessary and sufficient that it be isometric with a sphere  $S^n(c)$ .*

The above gives essentially the equivalence of  $P_2$  and  $P_3$  in the Einstein

case, whereas Theorem A shows the equivalence of  $P_0$  and  $P_2$  in the most general case.

**2. Proof of Theorem A.** The sufficiency is obvious. In fact,  $S^n(c)$  in the  $(n+1)$ -Euclidean space is defined by the equation  $\sum_{i=0}^n x_i^2 = 1/c^2$ . Then it is known that the function  $\phi = x_0$ , considered as a function on  $S^n(c)$ , satisfies the required condition.

Conversely, assume that a complete Riemannian manifold  $M$  admits a non-constant function  $\phi$  satisfying

$$(2.1) \quad \nabla^2 \phi + c^2 \phi g = 0.$$

It turns out  $\frac{d^2 \phi}{ds^2} + c^2 \phi = 0$  on each geodesic  $l(s)$ ,  $s$  being the arc length from a certain point  $Q_0$  on  $l(s)$ . Thus  $\phi$  is given by

$$\phi = A \cos cs + B \sin cs$$

on  $l(s)$ , where  $A = \phi(Q_0)$  and  $B = (1/c) \nabla_{X_0} \phi$ ,  $X_0$  being the unit vector at  $Q_0$  tangent to  $l(s)$ . Since  $M$  is connected and complete, every geodesic can be extended for any value of  $s$ . Therefore on each geodesic  $\phi$  takes the maximum value at some point. Furthermore on the geodesic through  $Q_0$  in the direction of the vector  $(d\phi)_{Q_0}$  there exists a point  $P_+$  at which  $\phi$  takes the maximum on  $M$ . The minimum is also taken at a point  $P_-$  on  $M$ . Without loss of generality we may assume that the maximum is 1 and then evidently the minimum is  $-1$ . From the equation (2.1),  $P_+$  and  $P_-$  are both non-singular critical points for  $\phi$  and hence they are isolated.

Now let us consider any geodesic through  $P_+$ . Then  $\phi$  is written as  $\phi = \cos cs$  on it,  $s$  being measured from  $P_+$  along it. Let  $M_s$  be the set of all points  $P$  being at distance  $s$  from  $P_+$  on geodesics through  $P_+$ .

LEMMA 1.  $M_{\pi/c}$  consists of a single point  $P_-$  and it is the only point such that  $\phi(P_-) = -1$ .

PROOF. Since  $\phi$  takes the value  $-1$  on  $M_{\pi/c}$ , the points of it are non-singular critical points for  $\phi$ . It follows that  $M_{\pi/c}$  is discrete. On the other hand, it is a continuous image of the tangent unit sphere at  $P_+$  and hence is connected. It follows then that  $M_{\pi/c}$  consists of a single point  $Q$ , which means that all the geodesics issuing from  $P_+$  meet at  $Q$ . Since  $M$  is a manifold, there exists no point whose distance from  $P_+$  exceeds  $\pi/c$ . Thus  $Q = P_-$ .

In like manner we have

LEMMA 2.  $P_+$  is the only point such that  $\phi(P_+) = 1$ .

Since the shortest geodesic joining any point  $P (\neq P_+, P_-)$  of  $M$  with  $P_+$ , which exists because of completeness, is uniquely determined in the direction  $(d\phi)_P$ , no two distinct such geodesics through  $P_+$  meet except at  $P_+$  and  $P_-$ . This implies

LEMMA 3.  $M_s$ ,  $0 < s < \pi/c$ , is homeomorphic with an  $(n-1)$ -dimensional sphere.

Now every point  $P$  on  $M_s$  ( $0 < s < \pi/c$ ) is determined uniquely by the pair  $(s, v)$ ,  $v$  being a unit vector at  $P_+$  tangent to the geodesic  $l_v$  joining  $P_+$  and  $P$  and  $s$  the arc length between  $P_+$  and  $P$ . In the sequel, we use the notation  $(s, v)$  to represent the point  $P$ .

Next consider a sphere  $S^n(c)$  in the  $(n+1)$ -Euclidean space and take an arbitrarily fixed point  $\bar{P}_+$  and its antipode  $\bar{P}_-$ . Since there is an isometry between the tangent unit sphere at  $P_+$  on  $M$  and that at  $\bar{P}_+$  on  $S^n(c)$ , we fix one and denote by  $\bar{v}$  the unit vector at  $\bar{P}_+$  corresponding to  $v$  at  $P_+$ . In the same manner as above every point  $\bar{P}$  on  $S^n(c)$  is represented by  $(\bar{s}, \bar{v})$  by using the great circles through  $P_+$  and  $P_-$ . (Of course  $\bar{M}_{\pi/c}$  consists of the point  $P_-$  alone).

Now define the mapping  $h: M \rightarrow S^n(c)$  by  $h(P_+) = \bar{P}_+$ ,  $h(s, v) = (\bar{s}, \bar{v})$  and  $h(P_-) = \bar{P}_-$ . Then evidently we have

LEMMA 4.  $h$  is a diffeomorphism of  $M$  onto  $S^n(c)$

Now if we define the function  $\bar{\phi}$  by  $\bar{\phi}(\bar{P}_+) = 1$ ,  $\bar{\phi}(\bar{P}_-) = -1$  and  $\bar{\phi}(\bar{s}, \bar{v}) = \cos cs$ , then  $\bar{\phi}$  satisfies (2.1) on  $S^n(c)$  and  $\phi = \bar{\phi} \circ h$ , which implies  $d\phi = h^{-1}(d\bar{\phi})$ . Since  $d\phi = (-c \sin cs) ds$  and  $d\bar{\phi} = (-c \sin cs) ds$ , we have  $|d\bar{\phi}| = |d\phi|$ ,  $| \cdot |$  denoting the length of vectors.

Next we are going to show that  $h$  is an isometry. To do this let us consider unit vectors  $X_0$  and  $v$  at  $P_+$  which are perpendicular to each other and  $X_s$  the vector at  $P = (s, v)$  obtained by parallel displacement along the geodesic  $l_v$ . Then  $X_s$  is tangent to  $M_s$ . Next consider any differentiable function  $f$  on  $M_{\pi/2c}$  and then define the function  $F$  on  $M - \{P_+, P_-\}$  by  $F(s, v) = f(\pi/2c, v)$ . The function  $sF$  is differentiable on  $M - \{P_-\}$  and every differentiable function on  $M - \{P_-\}$  is obtained as a function of functions of this type. (This is the idea of the normal coordinates!)

LEMMA 5.  $cX_0(sF) = \sin cs X_s(F)$ ,  $0 < s < \pi/c$ .

PROOF. From the definition we have  $\nabla_{ds} X_s = 0$ , which implies

$$[ds, X_s] = \nabla_{ds} X_s - \nabla_{X_s} ds = \nabla_{X_s} \left( \frac{1}{c \sin cs} d\phi \right).$$

Since  $\langle X_s, ds \rangle = 0$ , we have  $\nabla_{X_s} \left( \frac{1}{c \sin cs} \right) = 0$  and  $[ds, X_s] = -\frac{c \cos cs}{\sin cs} X_s$ .

On the other hand,

$$\begin{aligned} [ds, X_s](F) &= (ds)(X_s(F)) - X_s(ds(F)) \\ &= (ds)(X_s(F)) = \nabla_{ds} X_s(F). \end{aligned}$$

Therefore along  $l_v$  we have

$$\frac{d}{ds} X_s(F) = -\frac{c \cos cs}{\sin cs} X_s(F).$$

It follows then that

$$(2.2) \quad X_s(F) = \frac{1}{\sin cs} X_{\pi/2c}(F).$$

Thus we have  $X_s(sF) = sX_s(F) = \frac{s}{\sin cs} X_{\pi/2c}(F)$  and we obtain

$$(2.3) \quad X_0(sF) = \lim_{s \rightarrow 0} X_s(sF) = \frac{1}{c} X_{\pi/2c}(F).$$

From (2.2) and (2.3) the lemma is proved.

On  $S^n(c)$  we have the lemma corresponding to Lemma 5, which has just the same form, since  $\bar{\phi}$  satisfies (2.1).

We are now in a position to prove that  $h$  is isometric.

Let us take unit vectors  $X_0$  and  $v$  at  $P_+$  as above and consider the corresponding ones  $\bar{X}_0$  and  $\bar{v}$  at  $P_+$  on  $S^n(c)$ . Of course we have  $|X_0| = |\bar{X}_0| = 1$ . Then  $X_s$ , the vector at  $P = (s, v)$  obtained by parallel displacement along  $l_v$  on  $M$ . Then  $|X_0| = |X_s|$ . Now consider any function  $\bar{f}$  on  $\bar{M}_{\pi/2c}$  in  $S^n(c)$  and construct the functions  $\bar{F}$  and  $s\bar{F}$  as in  $M$ . It follows then from Lemma 5 and the corresponding lemma in  $S^n(c)$  that

$$\begin{aligned} dh(X_s)(\bar{F}) &= X_s(\bar{F} \circ h) = \frac{c}{\sin cs} X_0(s\bar{F} \circ h) \\ &= \frac{c}{\sin cs} \bar{X}_0(s\bar{F}) = \bar{X}_s(\bar{F}). \end{aligned}$$

Namely we have  $dh(X_s) = \bar{X}_s$ . Thus we have

$$|dh(X_s)| = |\bar{X}_s| = |\bar{X}_0| = |X_0| = |X_s|.$$

By the linearity of  $dh$  at any point  $P (\neq P_-)$ , we have  $|dh(X)| = |X|$  for any vector  $X$ , not necessarily tangent to  $M_s$ , at  $P$  on  $M$ . Changing the roles of  $P_+$  and  $P_-$ , we know that  $dh$  keeps the lengths of the vectors at every point, that is  $h$  is an isometry between  $M$  and  $S^n(c)$ , completing the proof of Theorem A.

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