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Results on the order of holomorphic functions defined in the unit disk[1]

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§ 1. Introduction

1. Let D denote the open unit disk and C the unit circle in the complex plane. Further, let $S(e^{i\theta}, \alpha)$ denote the symmetric Stolz domain at $e^{i\theta}$ of opening 2α lying in D. For certain classes of functions, holomorphic in D, results are known concerning the order, or growth, of functions belonging to the class. These results usually take two forms. One group of theorems gives a type of global order. For example, if $f(z)$ is univalent and holomorphic in D then Koebe's distortion theorem gives that $|f'(z)| \leqq \frac{(1+|z|)}{(1-|z|)^3}$. However if we restrict the choice of z somewhat a better estimate on the order can be given. Seidel and Walsh ([**15**], p. 338) showed that $|f'(z)|(1-|z|)^{\frac{1}{2}} \to 0$ as z tends to $e^{i\theta}$, $z \in S(e^{i\theta}, \alpha)$, for any $\alpha > 0$ and almost all $\theta \in [0, 2\pi)$. This type of result has been called a "statistical" result on order by J. Lelong-Ferrand.

If $P(z)$ is any function, holomorphic in D, which omits in D the values 0 and 1, then, as is well known, Schottky's theorem gives a global order for $P(z)$ to the effect that $|P(z)| \leqq e^{\frac{A}{(1-|z|)}}$ where A is a positive constant depending on $P(0)$. The main result in this paper will be to give a statistical theorem concerning the order of $P(z)$. In its simplest form the theorem states that for almost all $\theta \in [0, 2\pi)$, any fixed $\mu > 0$ and $\varepsilon > 0$, $|P(z)|\, e^{\frac{-\mu}{(1-|z|)^{1/2+\varepsilon}}}$ tends to 0 as z tends to $e^{i\theta}$ in any Stolz domain at $e^{i\theta}$. Thus, as in the case of univalent functions, a smaller estimate can be given for almost all $\theta \in [0, 2\pi)$, on sequences approaching $e^{i\theta}$ within any Stolz domain at $e^{i\theta}$.

In § 2 we deduce the fundamental theorem used to prove the main theorem. This fundamental theorem is similar in content to a result of Lelong-Ferrand ([**10**], p. 23).

Some results are given in § 3 on the order of functions holomorphic in D for which information is known concerning the order of their Taylor coeffi-

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cients. From these theorems we derive in § 4 estimates on the order of holomorphic functions which omit the values $\pm 2\pi in$ $(n=0,1,\cdots)$, which lead to the result on functions omitting 0 and 1 stated above.

§ 2. Fundamental Theorem

2. The method we use is essentially due to Lelong-Ferrand ([**10**], 20-23). Let $\bar{D}=D\cup C$; and set, for $\rho>0$, $z_0\in\bar{D}$, $D(z_0,\rho)=\{z\in D\,;\,|z-z_0|<\rho\}$. In the sequel set $z=x+iy=re^{i\theta}$. Now define the following outer measure for sets in the plane:

DEFINITION 1. Let $h(r)$ be a real-valued, non-decreasing, continuous function defined for $r\geqq 0$, satisfying the conditions $h(0)=0$, $h(r)>0$, for $r>0$, $h(\infty)>1$. For any set E in the plane and any $0<\rho<\infty$ let $h_\rho{}^*(E)$ denote the greatest lower bound of the quantities $\sum_{\nu=1}^{\infty} h(r_\nu)$ for all countable systems of open circular disks D_ν, with radius $0<r_\nu<\rho$, which cover E. Now define the h-measure of the set E to be the $\lim_{\rho\to 0} h_\rho{}^*(E)$. We denote this value by $h^*(E)$. It is easy to show that this defines on the plane an outer measure in the sense of Carathéodory[2]. In the case $h(r)=r^k$, $0<k<2$, this defines the k-dimensional outer measure.

We state now a lemma of Ahlfors, as formulated by Lelong-Ferrand ([**10**], p. 20)[3]. This lemma is based on ideas of Sire, Boutroux, H. Cartan, and Bloch. For references see ([**13**], p. 143).

LEMMA 1. *Let $g(e)$ be a non-negative, countably additive set function (mass distribution) defined for all measurable[4] subsets of a measurable set E in the plane, such that $g(E)=K<\infty$. Let $\mu(z,r)$ denote the quantity of mass contained in the intersection of the set E and the open disk with center z and radius r. (Of course we take the mass of the empty set to be zero.) Let A be a positive number satisfying $AK<1$. We then have, for all points of the plane,*

$$\mu(z,r)<\frac{h(r)}{A},$$

except possibly for a set of points S which can be covered by a sequence of

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2) μ is called an outer measure in the sense of Carathéodory if i) for every set A, $0\leqq\mu(A)\leqq\infty$, ii) if $B\subset A$, then $\mu(B)\leqq\mu(A)$, iii) if $A=\bigcup_{n=1}^{\infty}A_n$ then $\mu(A)\leqq\sum_{n=1}^{\infty}\mu(A_n)$, iv) if the distance between A and B is positive then $\mu(A\cup B)=\mu(A)+\mu(B)$. Note that the outer measure h_ρ^* satisfies properties (i)-(iii), but not (iv). For a discussion of these outer measures see, for example, ([**12**]).

3) The continuity of $h(r)$ is inadvertently omitted in the hypothesis.

4) By "measurable" we will mean Lebesgue-measurable unless otherwise stated. All integrals used are to be considered as Lebesgue integrals also.

circles C_ν, with radius r_ν, such that $\sum_{\nu=1}^{\infty} h(r_\nu) < 6KA$.

The lemma, as stated, gives no information concerning the size of the radius r_ν. However, given a fixed $\rho_0 > 0$, it is possible to choose the constant A so that the exceptional set S is such that $h^*_{\rho_0}(S) < 6KA$. We indicate the necessary modification to the proof as given by Lelong-Ferrand ([**10**], p. 20-23).

As usual, for each $r > 0$, set $\lambda_1(r) = l.u.b.\ (\mu(a, r))$. Given $\rho_0 > 0$, choose A so that

$$\text{(2.0)} \qquad \begin{aligned} &\text{i)} \quad AK < h(\rho_0); \\ &\text{ii)} \quad AK < 1. \end{aligned}$$

For all sufficiently large r

$$\text{(2.1)} \qquad h(r) > AK \geqq \lambda_1(r)A.$$

If (2.1) holds for all r the lemma is proved and the exceptional set is vacuous. Otherwise let r_1 be the least upper bound of all values r for which $\lambda_1(r) \geqq \frac{h(r)}{A}$. Now $r_1 < \rho_0$ since for all $r \geqq \rho_0$, $h(r) \geqq h(\rho_0)$. The proof now follows exactly as given by Lelong-Ferrand. Notice that the exceptional set S is now covered by a sequence of circles C_ν, with radius r_ν, $r_\nu \leqq r_1 < \rho_0$, such that $\sum_{\nu=1}^{\infty} h(r_\nu) < 6KA$; hence $h^*_{\rho_0}(s) \leqq 6KA$. The lemma now reads:

Let $g(e)$ be a non-negative, countably additive set function defined for all measurable subsets of a measurable set E in the plane such that $g(E) = K < \infty$. Let $\mu(z, r)$ denote the quantity of mass contained in the intersection of the set E and the open disk with center z and radius r. Let $h(r)$ be any function satisfying Definition 1, and let $\rho_0 > 0$ be given. If we choose a positive constant A so that $AK < 1$ and $AK < h(\rho_0)$ then we have, for all points of the plane,

$$\mu(z, r) < \frac{h(r)}{A},$$

except possibly for a set of points S such that $h^*_{\rho_0}(S) < 6KA$.

3. Applying the lemma in a manner analogous to Lelong-Ferrand although in a slightly more general form, we have:

THEOREM 1. *Let $h(r)$ be a function satisfying the conditions of Definition 1, and let $U(z)$ be a real-valued, non-negative, measurable function defined in D such that*

$$\text{(3.0)} \qquad \iint_D U(z)dxdy < \infty.$$

We then have

$$\lim_{r \to 0} \frac{1}{h(r)} \iint_{D(e^{i\theta}, r)} U(z)dxdy = 0,$$

except for at most a set of $e^{i\theta}$ *of h-measure* 0.

PROOF. This proof is patterned after the one given by Lelong-Ferrand ([**10**], p. 23), who did not use quite so general a function as $U(z)$. We give the proof since our basic lemma, Lemma 1, is slightly revised from the one used by Lelong-Ferrand; and also since the proof that the exceptional set has h-measure 0 is not clear to the writer.

Define for $n=1, 2, \cdots$,

$$\Omega_n=\left\{z\in D; 1-\frac{1}{n}<|z|<1\right\},$$

and set

$$\varepsilon_n=\iint_{\Omega_n} U(z)dxdy.$$

We note that condition (3.0) implies that

$$\lim_{n\to\infty}\varepsilon_n=0, \tag{3.1}$$

and since $U(z)$ is non-negative we also have that $\varepsilon_n\geqq\varepsilon_{n+1}$ for $n=1, 2, \cdots$. We consider two cases:

Case i) $\varepsilon_n=0$ for some $n=N_0$. Then, for all $n\geqq N_0$, $\varepsilon_n=0$ and the theorem is trivially true.

Case ii) $\varepsilon_n>0$ for all n. Consider an arbitrary but fixed function $h(r)$ and let $\rho_0>0$ be given. Choose N_0 so large that $\varepsilon_n<1$ and $\sqrt{\varepsilon_n}<h(\rho_0)$, for $n>N_0$. Restrict n to be always greater than N_0. We now apply Lemma 1, in its second form, choosing $E=\Omega_n$ for a fixed value of $n>N_0$. Define the mass function in Ω_n as follows: for any measurable subset e of Ω_n let

$$g(e)=\iint_e U(z)dxdy.$$

We then have $g(\Omega_n)=K=\varepsilon_n$ and take $A=\dfrac{1}{\sqrt{\varepsilon_n}}$. For this choice of A, AK $=\sqrt{\varepsilon_n}<h(\rho_0)$ for any $n>N_0$. Applying the result in Lemma 1, we can write, for all $r>0$,

$$\frac{\mu(z, r)}{h(r)}<\sqrt{\varepsilon_n}, \tag{3.2}$$

except for a set S_n of points z such that $h^*_{\rho_0}(S_n)<6\sqrt{\varepsilon_n}$. We shall show that, except for at most a set of $e^{i\theta}$ of h-measure zero,

$$\lim_{r\to 0}\frac{1}{h(r)}\iint_{D(e^{i\theta}, r)} U(z)dxdy=0.$$

Indeed, let a subsequence of natural numbers $\{n_k\}$ be chosen so that $n_1>N_0$, $n_k\geqq n_{k+1}$, for all k and

$$\iint_{\Omega_{n_k}} U(z)dxdy = \varepsilon_{n_k} < \frac{1}{k^4},$$

which is possible because of (3.1). From the above and (3.2) it follows that, for arbitrary Ω_{n_k} and all $r>0$,

$$\frac{\mu(z,r)}{h(r)} < \sqrt{\varepsilon_{n_k}} < \frac{1}{k^2},$$

except for a set S_{n_k} such that

$$h^*_{\rho_0}(S_{n_k}) < 6\sqrt{\varepsilon_{n_k}} < \frac{6}{k^2}. \tag{3.3}$$

Let S be the set of $e^{i\theta}$ for which

$$\lim_{r\to 0}\sup \frac{1}{h(r)}\iint_{D'e^{i\theta},r)} U(z)dxdy \equiv C_\theta > 0.$$

This implies, for every fixed $e^{i\theta}\in S$, the existence of a sequence $\{r_j\}$, with $\lim_{j\to\infty} r_j = 0$, such that

$$\frac{1}{h(r_i)}\iint_{D(e^{i\theta},r_j)} U(z)dxdy > \frac{C_\theta}{2}.$$

Let the natural number p be chosen so that $\sqrt{\varepsilon_{n_p}} < \frac{C_\theta}{2}$. We then have the inequality

$$\frac{1}{h(r_j)}\iint_{D(e^{i\theta},r_j)} U(z)dxdy > \sqrt{\varepsilon_{n_p}} \geqq \sqrt{\varepsilon_{n_k}}, \tag{3.4}$$

for $k\geqq p$ and all j since $\{\varepsilon_{n_k}\}$ is a non-increasing sequence of non-negative numbers.

Recall that we applied lemma 1 by choosing $E=\Omega_n$ for any $n>N_0$. If we restrict r so that $0\leqq r\leqq \frac{1}{n}$, then, by our definition of the mass distribution in Ω_n, the quantity of mass, $\mu(e^{i\theta},r)$, contained in $D(e^{i\theta},r)$ is given by

$$\iint_{D(e^{i\theta},r)} U(z)dxdy.$$

Formulate (3.4) as follows: given any fixed n_k, $k>p$,

$$\frac{\mu(e^{i\theta},r_j)}{h(r_j)} > \sqrt{\varepsilon_{n_k}}$$

for all r_j, such that $0<r_j\leqq \frac{1}{n_k}$. This implies $e^{i\theta}\in S_{n_k}$, $k\geqq p$, hence $S\subset\bigcup_{k=m}^{\infty} S_{n_k}$ for every $m=1,2,\cdots$. By the subadditivity of the outer measure $h^*_{\rho_0}$, and by (3.3),

$$h^*_{\rho_0}(S) \leqq \sum_{k=m}^{\infty} h^*_{\rho_0}(S_{n_k}) < 6\sum_{k=m}^{\infty}\frac{1}{k^2}.$$

This holds for every m, hence $h^*_{\rho_0}(S)=0$. Since ρ_0 was arbitrary $h^*(S) = \lim_{\rho\to 0} h^*_\rho(S) = 0$. This completes the proof of Theorem 1.

§ 3. Applications to functions belonging to the class A^s

4. We apply Theorem 1 to obtain information on the growth of functions holomorphic in D and which satisfy the following condition:

DEFINITION 2. Let $f(z)$ be holomorphic in D. If $s>-1$, we say that $f(z)\in A^s$ if

$$\iint_D |f'(z)|^2(1-|z|)^s dxdy<\infty\,.$$

Similar conditions on $f(z)$ we studied by Paley and Littlewood [see Zygmund ([**17**], p. 210)], Lelong-Ferrand ([**9**], p. 49), Dufresnoy ([**2**], p. 395), and Flett ([**3**], p. 3).

In Theorem 2 we consider a slightly more general class of functions than A^s, and this theorem generalizes a result of Flett's ([**3**], p. 3).

THEOREM 2. *Let $f(z)$ be holomorphic in D. If, for $s>-1$ and $k>0$,*

$$\iint_D |f'(z)|^k(1-|z|)^s dxdy<\infty\,,$$

then

$$\lim_{z\to e^{i\theta}}\frac{|f'(z)|^k(1-|z|)^{s+2}}{h(|z-e^{i\theta}|)}=0\,,$$

except for at most a set of $e^{i\theta}$ of h-measure zero.

REMARK 1. $h(r)$, of course, satisfies Definition 1, but in addition we require that

$$h(\alpha r)\leqq C_\alpha h(r)\,, \tag{4.0}$$

for all $\alpha>0$ where C_α is a positive constant depending only on α. This property is not essential in the following theorems. However the statement of the theorems would become somewhat more complicated if this property were not assumed. This property is satisfied, for example, in the case $h(r)=r^k$.

REMARK 2. In the case $k=2$, the theorem was proved in a similar manner by Lelong-Ferrand ([**9**], p. 49). The applications of the theorem differ.

PROOF. Let ζ be an arbitrary point of D and let $D(\zeta,r)\subset D$. Recalling that $|f'(z)|^k$, for $k>0$, is subharmonic in D, then

$$\pi r^2|f'(\zeta)|^k\leqq\iint_{D(\zeta,r)}|f'(z)|^k dxdy\,. \tag{4.1}$$

If $z\in D(\zeta,(1-|\zeta|)t)$, $0<t<1$, it is easy to show that

$$(1-|z|)^s\geqq C_{t,s}(1-|\zeta|)^s\,, \tag{4.2}$$

where $C_{t,s}$ is a positive constant depending only on t and s.

Utilizing (4.1) and (4.2) gives

$$\pi|f'(\zeta)|^k(1-|\zeta|)^{s+2}t^2C_{t,s}\leqq\iint_{D(\zeta,(1-|\zeta|)t)}|f'(z)|^k(1-|z|)^k dxdy\,. \tag{4.3}$$

Applying Theorem 1, with $U(z)=|f'(z)|^k(1-|z|)^s$, and $r=2|\zeta-e^{i\theta}|$, yields

$$\lim_{\zeta\to e^{i\theta}}\left(\frac{\iint_{D(e^{i\theta},2|\zeta-e^{i\theta}|)}|f'(z)|^k(1-|z|)^s dxdy}{h(2|\zeta-e^{i\theta}|)}\right)=0, \tag{4.4}$$

except for at most a set of $e^{i\theta}$ of h-measure 0. Since $D(\zeta,(1-|\zeta|)t)\subset D(e^{i\theta}, 2|\zeta-e^{i\theta}|)$, $0<t<1$, (4.3) and (4.4) imply, for any fixed $s>-1$, and $k>0$,

$$\lim_{\zeta\to e^{i\theta}}\frac{|f'(\zeta)|^k(1-|\zeta|)^{s+2}}{h(|\zeta-e^{i\theta}|)}=0,$$

except for at most a set of $e^{i\theta}$ of h-measure 0.

5. In the subsequent investigations we restrict ourselves to the class A^s. A result of Heywood ([8], p. 303) can be used to give a necessary and sufficient condition on $f(z)$ that it belongs to the class A^s. Thus we state:

THEOREM 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be holomorphic in D. Then $f(z)\in A^s$ if and only if $\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{s-1}}<\infty$.*

Theorem 2, which gives information on the order of the first derivative of a function belonging to the class A^s, can be generalized to derivatives of higher order. In customary fashion we let $\Gamma(x)$ and $B(x,y)$ denote the Gamma and Beta function respectively. If $f(x)$ and $g(x)$ are two real-valued functions defined for all $x\geqq A$, where A is some finite number, and $f(x)/g(x)\to 1$ as $x\to\infty$ we write $f(x)\cong g(x)$.

As before, let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be holomorphic in D. For any real number β let

$$f^{(\beta)}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}a_nz^{n-\beta},$$

where $z^{-\beta}$ has its principal value. This definition originated with Riemann ([**14**], 353-356). Hadamard ([**5**], p. 154 ff.), and Hardy and Littlewood ([**6**], p. 409 ff.), among others, discussed properties of $f^{(\beta)}(z)$. If β is an integer then $f^{(\beta)}(z)$ becomes the ordinary derivative or integral of $f(z)$ according as $\beta>0$ or $\beta<0$. If $\Gamma(n+1-\beta)$ becomes infinite for any value of n we let the corresponding term of the series be zero. We further note that if β is not an integer then, of course, $f^{(\beta)}(z)$ is not single-valued, but $|f^{(\beta)}(z)|$ is.

For convenience let $a_n^{(\beta)}=\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}a_n$; then, by use of Stirling's formula

$$a_n^{(\beta)}\cong n^\beta a_n. \tag{5.0}$$

With these preliminaries finished we are now ready to give a theorem estimating the order for the generalized derivative of a function belonging to the class A^s.

THEOREM 4. *If* $s+2\beta>1$, $s>-1$, *and* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *belongs to* A^s *then*

$$\lim_{z\to e^{i\theta}} \frac{|f^{(\beta)}(z)|(1-|z|)^{\frac{s+2\beta}{2}}}{\sqrt{h(|z-e^{i\theta}|)}}=0,$$

except for at most a set of $e^{i\theta}$ *of* h*-measure* 0.[5]

PROOF. Let

$$g(z)=\sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{n\Gamma(n+1-\beta)} a_n z^n .$$

That $g(z)\in A^{s+2\beta-2}$ for $s+2\beta>1$, can easily be deduced from (5.0) and Theorem 3. Hence, by Theorem 2,

$$\lim_{z\to e^{i\theta}} \frac{|g'(z)|(1-|z|)^{\frac{s+2\beta}{2}}}{\sqrt{h(|z-e^{i\theta}|)}}=0, \tag{5.1}$$

except for at most a set of $e^{i\theta}$ of h-measure 0. This establishes the theorem when we recall that $|g'(z)|=|z^{\beta-1}f^{(\beta)}(z)|$.

§ 4. Functions holomorphic in D omitting 0 and 1

6. Theorems 3 and 4 can be used to give estimates of a statistical nature on the order of certain other classes of functions holomorphic in D for which a global-type order is known.

Let

$$\begin{aligned} Q(z) &= \sum_{n=0}^{\infty} q_n z^n ; \\ R(z) &= \sum_{n=0}^{\infty} r_n z^n ; \\ P(z) &= \sum_{n=0}^{\infty} p_n z^n ; \end{aligned} \tag{6.0}$$

denote *any* function, holomorphic in D, which omits there the values $\pm 2\pi ni$ $(n=0,1,2,\cdots)$; $-4\pi^2 n^2$ $(n=0,1,\cdots)$; and $0,1$; respectively. It is evident that a necessary and sufficient condition for a function to be of the type $P(z)$ is that it be of the form $e^{Q(z)}$ for some function $Q(z)$; and for a function to be of the type $R(z)$ that if be of the form $(Q(z))^2$. (For a discussion of the properties of these functions, which we shall assume known, see Littlewood ([**11**], p. 185 ff)).

Since $|q_n|\leqq A_{q_0}\log n$, $n>1$, where A_{q_0} is a positive constant depending on $Q(0)$, then $\sum_{n=1}^{\infty}\left(\frac{|q_n|^2}{n^{1+\varepsilon}}\right)<\infty$. Theorems 3 and 4 imply

5) For $\beta=1,2,\cdots$ this was proved by Lelong-Ferrand ([**9**], p. 52) using somewhat different methods.

THEOREM 5. *If* $Q(z)$ *is defined as in* (6.1) *then for any* $\beta \geqq -1/2$, *and any* $\varepsilon > 0$,

$$\lim_{\substack{z \to e^{i\theta} \\ z \in D}} \frac{|Q^{(\beta)}(z)|(1-|z|)^{1+\beta+\varepsilon}}{\sqrt{h(|z-e^{i\theta}|)}} = 0,$$

except for at most a set of $e^{i\theta}$ *of h-measure* 0.

COROLLARY 1. *If* $R(z)$ *is defined as in* (6.0), *then for any* $\varepsilon > 0$,

$$\lim_{\substack{z \to e^{i\theta} \\ z \in D}} \frac{|R(z)|(1-|z|)^{2+\varepsilon}}{h(|z-e^{i\theta}|)} = 0,$$

except for at most a set of $e^{i\theta}$ *of h-measure* 0.

PROOF. Since $|R(z)| = |Q(z)|^2$, for some function $Q(z)$, we can apply Theorem 5 with $\beta = 0$.

REMARK. It is known that

$$|Q(z)| \leqq \frac{A}{(1-|z|)};$$

$$|R(z)| \leqq \frac{B}{(1-|z|)^2};$$

where A and B are positive constants depending only on $Q(0)$ and $R(0)$ respectively. If, in the above results, we take $h(r) = r$, $\beta = 0$, and $z \in S(e^{i\theta}, \alpha)$, we then have, for any $\varepsilon > 0$,

$$\lim_{z \to e^{i\theta}} |Q(z)|(1-|z|)^{1/2+\varepsilon} = 0,$$

and

$$\lim_{z \to e^{i\theta}} |R(z)|(1-|z|)^{1+\varepsilon} = 0,$$

for almost all $\theta \in [0, 2\pi)$. Thus the radial order of $Q(z)$ and $R(z)$, for almost all radii, is smaller than the global order.

We now formulate our main result which gives a statistical type order for functions $P(z)$.

THEOREM 6. *If* $P(z)$ *is holomorphic in* D *and omits there the values* 0 *and* 1, *then for any* $\varepsilon > 0$ *and* $\mu > 0$

$$\lim_{z \to e^{i\theta}} |P(z)| \exp\left(\frac{-\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon}}\right) = 0,$$

except for at most a set of $e^{i\theta}$ *of h-measure* 0.

PROOF. If the h-measure of the circumference C is zero then the statement of the theorem is vacuous; hence we assume that $h^*(C) > 0$. We know that $P(z) = e^{Q(z)}$ for some function $Q(z)$. In Theorem 5 let $e^{i\theta}$ be a point such that

$$\lim_{z \to e^{i\theta}} \frac{Q(z)(1-|z|)^{1+\varepsilon}}{\sqrt{h(|z-e^{i\theta}|)}} = 0.$$

Given $\mu > 0$, there is a $\delta > 0$, such that, if $|z-e^{i\theta}| < \delta$, then

$$0 < \frac{Q(z)(1-|z|)^{1+\varepsilon}}{\sqrt{h(|z-e^{i\theta}|)}} < \mu. \tag{6.1}$$

Since $0 < |P(z)| \leqq \exp |Q(z)|$, (6.1) gives

$$\varlimsup_{z\to e^{i\theta}} |P(z)| \exp\left[\frac{-\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon}}\right] \leqq 1, \tag{6.2}$$

except for at most a set of $e^{i\theta}$ of h-measure 0. Finally to prove the theorem[6] apply logarithms to both sides of (6.2) and suppose that for some $\varepsilon_0 > 0$,

$$-\infty < L_\theta \leqq \varliminf_{z\to e^{i\theta}} \left(\log |P(z)| - \frac{\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon_0}}\right) \leqq 0 \tag{6.3}$$

holds on a set E_0 of $e^{i\theta}$ of positive h-measure. Fix $\varepsilon, 0 < \varepsilon < \varepsilon_0$. Let E_1 be the set of all $e^{i\theta}$ for which

$$\varlimsup_{z\to e^{i\theta}} \left(\log |P(z)| - \frac{\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon}}\right) \leqq 0. \tag{6.4}$$

On account of (6.2), $h^*(E_1) = h^*(C)$, and setting $E_1' = C - E_1$, $h^*(E_1') = 0$. In addition on the circumference C, if $h^*(C) > 0$, we have

$$\lim_{\substack{z\to e^{i\theta} \\ z\in D}} \frac{1-|z|}{\sqrt{h(|z-e^{i\theta}|)}} = 0. \tag{6.5}$$

This follows immediately from Theorem 2 with $f(z) = z$, $s = 0$, and the remark that (6.5) is precisely the same for all points of C, hence if it holds at one point of C it holds at every point on C.

If we let $B = E_0 \cap E_1$, utilizing the subadditivity of the h-measure and the fact that $h^*(E_1') = 0$, gives $h^*(B) > 0$. Hence B is a non-empty subset of both E_0 and E_1. If we notice that

$$\begin{aligned} &\log |P(z)| - \frac{\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon}} \\ &\quad = \log |P(z)| - \frac{\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon_0}} + \frac{\mu\sqrt{h(|z-e^{i\theta}|)}}{(1-|z|)^{1+\varepsilon_0}}\left[1 - \frac{1}{(1-|z|)^{\varepsilon-\varepsilon_0}}\right], \end{aligned}$$

then (6.3) shows that, as $z \to e^{i\theta}$, $e^{i\theta} \in B$, the sum of the first two terms stays away from $-\infty$, while by (6.5), and the observation that $\varepsilon - \varepsilon_0 < 0$, the third term tends to $+\infty$. Hence the whole expression tends to $+\infty$. But this contradicts (6.4) and the theorem is proved.

If we set $h(r) = r$, and let $z \in S(e^{i\theta}, \alpha)$, then, for any $\mu > 0$ and $\varepsilon > 0$,

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6) I am indebted to Prof. W. Seidel for showing that (6.2) can be improved to give the final statement of the theorem.

$$\lim_{z \to e^{i\theta}} | P(z) | \exp\left(\frac{-\mu}{(1-|z|)^{1/2+\varepsilon}} \right) = 0 \tag{6.6}$$

for almost all $\theta \in [0, 2\pi)$. Given a function of the type $P(z)$, Schottky's theorem states that

$$| P(z) | \leqq e^{\frac{C}{(1-|z|)}}$$

where C is a positive constant depending on $P(0)$ only, whereas (6.6) shows that a better estimate of a statistical nature can be given for $P(z)$.

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