

Ordered idempotent semigroups

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1. By an *ordered semigroup*, we mean a system $S(\cdot, <)$, which satisfies the following conditions:

I. S is a semigroup with respect to the multiplication \cdot ;

II. S is simply ordered by $<$;

III. if a and b are elements of S such that $a < b$, then $ac \leq bc$ and $ca \leq cb$ for all $c \in S$.

Many authors, especially Alimov [1], Clifford [2], [4], [5], Hion [8] and Conrad [6], studied such semigroups with some restrictions. Alimov studied ordered semigroups which satisfy the conditions I, II and the stronger

III'. if a and b are elements of S such that $a < b$, then $ac < bc$ and $ca < cb$ for all $c \in S$.

But many ordered topological semigroups do not satisfy the condition III'. The remaining authors made rather artificial restrictions, and, as far as we know, none discussed ordered semigroups in our general sense. Indeed, the structure of such semigroups seems to be very complicated. However, it can be seen that, in any ordered semigroup, the set of all idempotents, if it is non-void, constitutes a subsemigroup. And, in this note, as the first step of the general study of ordered semigroups, we treat ordered idempotent semigroups i.e. ordered semigroups in which every element is idempotent.

In §§ 2-6 we discuss the structure of ordered idempotent semigroups, and in § 7 we show that some properties in previous sections, characterize ordered idempotent semigroups. As an appendix of this note, in the final § 8, we remark the characterizations of two special idempotent semigroups.

2. In this section, we give some preliminary properties of ordered idempotent semigroups. First we mention, without proof, an interesting following result of Clifford about idempotent semigroups.

LEMMA 1 (Theorem 3 of Clifford [3]). *An idempotent semigroup S can be decomposed into subsets $\{D_\alpha: \alpha \in A\}$ in such a way that*

(i) *for any pair of subsets D_α and D_β , the set $\{xy, yx; x \in D_\alpha, y \in D_\beta\}$ is contained in a third D_γ , and that*

(ii) *all the elements of every D_α can be arranged in a rectangular form, such*

that a and b are elements in the same row if and only if $ab = a$ and $ba = b$, and that a and b are elements in the same column if and only if $ab = b$ and $ba = a$.

Thus we can define, in $S^* = \{D_\alpha; \alpha \in A\}$, an operation \circ in a natural manner, i. e. $D_\alpha \circ D_\beta$ is the set D_γ determined from D_α and D_β by (i) in the above Lemma. Then S^* becomes a semilattice, and so we can define moreover, in S^* , a partial order, i. e. $D_\alpha \leq D_\beta$ if and only if $D_\alpha \circ D_\beta = D_\alpha$.

Each D_α of S^* is called a D -class in S . If two elements a and b of S belong to the same D -class, then they are called D -equivalent to each other, and are denoted by aDb . For an element a of S , we denote by $D(a)$ the D -class which contains a . Then it is clear that

$$D(a) \circ D(b) = D(b) \circ D(a) = D(ab) = D(ba).$$

If two elements a and b in the same D -class belong to the same row (column) in the arrangement of (ii) in the above Lemma, then they are called L -equivalent (R -equivalent) to each other, and are denoted by aLb (aRb). Each quotient set of S by the L -equivalence (R -equivalence) relation is called an L -class (R -class). Evidently the decomposition of S into L -classes (R -classes) is a subdivision of the decomposition into D -classes. These notions are only the specifications of those in the ideal theory of general semigroups (cf. for example, Miller and Clifford [9], Green [7]). Now we give a remark. Let a and b be elements of S such that $ab = a$ and $ba = b$. Then $D(a) = D(ab) = D(ba) = D(b)$, and so aDb . Hence

$$aLb \text{ if and only if } ab = a \text{ and } ba = b.$$

Similarly

$$aRb \text{ if and only if } ab = b \text{ and } ba = a.$$

In an ordered idempotent semigroup S , we use the following conventions. If $a \leq c \leq b$ or $b \leq c \leq a$, then we say that c lies between a and b . On the other hand, if $a < c < b$ or $b < c < a$, then we say that c lies between a and b in the strict sense.

Now we refer to two lemmas which are needed in this note. The proofs of these lemmas are derived easily, and they are omitted here.

LEMMA 2. For two elements a and b of S , both ab and ba lie between a and b .

LEMMA 3. In S , if an element c lies between a and b , then $ab = acb$.

As an immediate consequence of Lemma 3, we have

LEMMA 4. If c lies between a and b , then $D(c) \geq D(a) \circ D(b)$.

THEOREM 1. In an ordered idempotent semigroup, each D -class consists of either only one L -class or only one R -class.

PROOF. Let L_1 and L_2 be two L -classes in a D -class D , and let R_1 and R_2 be two R -classes in D . We denote the intersection element of L_1 and R_1 , of

L_1 and R_2 , of L_2 and R_1 and of L_2 and R_2 by a, b, c and d , respectively. Without loss of generality, we assume that $a \geq d$. Then we have

$$a = ab \geq db = b = ba \geq bd = d.$$

If $c \geq b$, then we have, in a similar way,

$$c \geq d \geq b,$$

and so $b = d$, from which it follows that $L_1 = L_2$. On the other hand, if $b \geq c$, then we obtain $R_1 = R_2$ similarly. This completes the proof of the theorem.

A D -class which consists of only one L -class (R -class) is called an L -typed (R -typed) D -class. Theorem 1 shows that every D -class is either L -typed or R -typed. Of course, a D -class is L -typed and at the same time R -typed, if and only if it consists of only one element.

THEOREM 2. *Let a and b be elements of an ordered idempotent semigroup such that $a \leq b$. If the D -class $D(a) \circ D(b)$ is L -typed, then*

$$ab = \min \{y; y \in D(a) \circ D(b) \text{ and } y \geq a\},$$

$$ba = \max \{y; y \in D(a) \circ D(b) \text{ and } y \leq b\}.$$

If $D(a) \circ D(b)$ is R -typed, then

$$ab = \max \{y; y \in D(a) \circ D(b) \text{ and } y \leq b\},$$

$$ba = \min \{y; y \in D(a) \circ D(b) \text{ and } y \geq a\}.$$

PROOF. Suppose that $D(a) \circ D(b)$ is L -typed. By Lemma 2, it is clear that ab belongs to the set $\{y; y \in D(a) \circ D(b) \text{ and } y \geq a\}$. Now we take any element y such that $y \in D(a) \circ D(b)$ and $y \geq a$. Then $abLy$ and so we have $y(ab) = y$. Therefore we have $ab \leq yb = y(ab)b = y$. Hence $ab = \min \{y; y \in D(a) \circ D(b) \text{ and } y \geq a\}$. The rest of assertions of this theorem can be proved similarly.

COROLLARY 1. *Let a and b be elements such that $a \leq b$. If $D(a) \circ D(b)$ is L -typed, then $ab \leq ba$. If $D(a) \circ D(b)$ is R -typed, then $ba \leq ab$. Moreover, in both cases, any element of $D(a) \circ D(b)$, which lies between a and b , lies between ab and ba .*

COROLLARY 2. *Suppose that $D(a) \leq D(b)$ in S^* . If $D(a)$ is L -typed, then $ab = a$. If $D(a)$ is R -typed, then $ba = a$.*

COROLLARY 3. *For an element a , if there exists an element b such that $D(a) < D(b)$ and $a < b$, then there exists $\max \{y; y \in D(a) \text{ and } y < b\}$. If there exists an element c such that $D(a) < D(c)$ and $c < a$, then there exists $\min \{y; y \in D(a) \text{ and } y > c\}$.*

3. In §2, we have seen that the set of all the D -classes of an ordered idempotent semigroup S forms a semilattice. This semilattice S^* is called the *associated semilattice* of S . In this section, we discuss the structure of the semilattice S^* .

We give a definition. A semilattice T is called a *tree semilattice* if it satisfies the following condition :

(T) If $\alpha, \alpha', \beta, \beta'$ are elements of T such that $\alpha \leq \alpha', \beta \leq \beta'$ and α and β are non-comparable, then α' and β' are non-comparable.

Evidently a semilattice T is not a tree semilattice, if and only if there exist three elements α, β and γ of T such that $\alpha \leq \gamma, \beta \leq \gamma$ and α and β are non-comparable. A simply ordered semilattice is a tree semilattice. If a lattice has a pair of non-comparable elements, then it is not a tree semilattice. In the following, the semilattice operation is denoted by \circ , in accordance with the consideration in §2.

THEOREM 3. *The associated semilattice S^* of an ordered idempotent semigroup S is a tree semilattice.*

PROOF. If S^* were not a tree semilattice, then there would exist three elements D_1, D_2 and D_3 of S^* such that $D_1 \leq D_3, D_2 \leq D_3$ and D_1 and D_2 are non-comparable. Therefore we would have $D_1 \circ D_2 < D_1$ and $D_1 \circ D_2 < D_2$. In S , we take arbitrarily elements a, b and c of D_1, D_2 and D_3 , respectively. Without loss of generality, we assume that $a \leq b$. Then, by Lemma 2, we would have $a \leq ab \leq b$. If $c \leq ab$, then ab would lie between c and b , and so

$$D(b) \circ D(c) = D_2 \circ D_3 = D_2 > D_1 \circ D_2 = D(ab),$$

which contradicts Lemma 4. Also, in the case when $c \geq ab$ we can deduce a contradiction in a similar way. This completes the proof of the theorem.

Let T be a tree semilattice. An element α of T is called a *maximal element* of T , if there is no element ξ such that $\alpha < \xi$. An element α of T is called a *branching element* of T , if there exist elements β and γ such that $\alpha < \beta, \alpha < \gamma$ and $\alpha = \beta \circ \gamma$. An element α of T which is neither a maximal element nor a branching element is called an *intermediate element*. For an element α of T which is not a maximal element, we associate a subset $T_\alpha = \{\xi; \xi \in T, \xi > \alpha\}$, which is called the *upper-set* of T determined by α . Now we introduce a relation in T_α as follows :

$$\beta \sim \gamma(\alpha) \text{ if and only if } \beta \circ \gamma > \alpha.$$

LEMMA 5. *The relation $\sim \text{ mod } \alpha$ is an equivalence relation in T_α .*

PROOF. Evidently it is reflexive and symmetric. Suppose that $\beta \sim \gamma(\alpha)$ and $\gamma \sim \delta(\alpha)$. Then

$$\alpha < \beta \circ \gamma \leq \gamma \text{ and } \alpha < \gamma \circ \delta \leq \gamma,$$

and so, since T is a tree semilattice, the elements $\beta \circ \gamma$ and $\gamma \circ \delta$ are comparable. Therefore

$$(\beta \circ \gamma) \circ (\gamma \circ \delta) = \beta \circ \gamma \text{ or } (\beta \circ \gamma) \circ (\gamma \circ \delta) = \gamma \circ \delta.$$

Hence $\alpha < (\beta \circ \gamma) \circ (\gamma \circ \delta) \leq \beta \circ \delta$, and so we obtain $\beta \sim \delta(\alpha)$.

For an element α of T which is not a maximal element, each equivalence class of T_α by the above relation is called a *branch* at α . The cardinal number of branches at α is called the *branch order* at α . The branch order at a maximal element is defined to be 0. If α is an intermediate element, then all elements of T_α belong to one branch at α , and hence the branch order at α is 1. If α is a branching element, then, by definition, there is a pair of elements β and γ of T_α such that $\alpha = \beta \circ \gamma$. Since β and γ belong to different branches at α , and so, in this case, the branch order at α is at least 2.

4. Now we return to the study of an ordered idempotent semigroup S . A subset K of S is called *S-convex*, if K contains with two of its elements all the elements of S between them. For an element d of a D -class D , we consider the set-union $K(d)$ of all those S -convex subsets of D which contain d . Then $K(d)$ is easily seen to be S -convex, and hence a maximal S -convex subset of D . Here the term 'maximal' is used with respect to the inclusion relation. A maximal S -convex subset of D is called a *component* of D -class D . If K and K' are two components of D , then it is easy to see that $K = K'$ or K and K' are mutually disjoint. Thus D can be decomposed into mutually disjoint components of D . For a D -class D of S , we denote the set of all the components of D by \mathfrak{R}_D .

Let K and K' be two mutually disjoint S -convex subsets of S . We write $K < K'$ if and only if $k < k'$ in S for some $k \in K$ and $k' \in K'$. As is easily seen, $K < K'$ if and only if $k < k'$ for all $k \in K$ and $k' \in K'$. It is also easy to see that the relation $<$ defined for S -convex subsets of S is transitive. In the case when K or K' consists of only one element, we write $k < K'$ or $K < k'$ in place of $\{k\} < K'$ or $K < \{k'\}$, respectively.

The set of all components in an ordered idempotent semigroup S i.e. the set-union of \mathfrak{R}_D when D goes through all the D -classes of S is denoted by \mathfrak{R} . For two distinct components, whether they belong to the same D -class or not, they are mutually disjoint, and so \mathfrak{R} can be simply ordered by the relation $<$ defined for S -convex subsets of S .

Let D_0 be a D -class of S and let \mathfrak{B} be a branch at D_0 in the associated semilattice S^* . The set-union of \mathfrak{R}_D when D goes through all the D -classes which belong to \mathfrak{B} in S^* , is called the *component-branch* at D_0 associated with \mathfrak{B} . Each component-branch is a subset of \mathfrak{R} .

LEMMA 6. *Every component-branch is \mathfrak{R} -convex, i.e. every component-branch contains with two of its elements all the elements of \mathfrak{R} between them.*

PROOF. Let $\mathfrak{R}(\mathfrak{B})$ be the component-branch associated with a branch \mathfrak{B} at D in S^* , let H and J be elements of $\mathfrak{R}(\mathfrak{B})$ and let K be an element of \mathfrak{R} which lies between H and J . We choose in S arbitrarily $h \in H$, $j \in J$ and $k \in K$. Since $D(h)$ and $D(j)$ belong to the same branch at D , we have $D(h) \circ D(j) > D$.

Therefore, by Lemma 4, we have

$$D(h) \circ D(k) \geq D(h) \circ (D(h) \circ D(j)) = D(h) \circ D(j) > D.$$

Hence, in S^* , $D(k)$ and $D(h)$ belong to the same branch at D , i. e. $D(k)$ belongs to \mathfrak{B} . Hence $K \in \mathfrak{R}(\mathfrak{B})$.

Let D be a D -class. By Lemma 6, every component-branch at D is \mathfrak{R} -convex, and it is clear that two distinct component-branches at D are mutually disjoint. Thus the set of all component-branches at D can be simply ordered naturally. For two distinct component-branches $\mathfrak{R}(\mathfrak{B})$ and $\mathfrak{R}(\mathfrak{B}')$ at D , if $K < K'$ for some $K \in \mathfrak{R}(\mathfrak{B})$ and $K' \in \mathfrak{R}(\mathfrak{B}')$, then $\mathfrak{R}(\mathfrak{B}) < \mathfrak{R}(\mathfrak{B}')$, and conversely if $\mathfrak{R}(\mathfrak{B}) < \mathfrak{R}(\mathfrak{B}')$, then $K < K'$ for all $K \in \mathfrak{R}(\mathfrak{B})$ and $K' \in \mathfrak{R}(\mathfrak{B}')$. The least component-branch and the greatest component-branch at D with respect to this order are simply called the *least component-branch* and the *greatest component-branch* at D , respectively.

LEMMA 7. *Let $\mathfrak{R}(\mathfrak{B})$ and $\mathfrak{R}(\mathfrak{B}')$ be two component-branches at D such that $\mathfrak{R}(\mathfrak{B}) < \mathfrak{R}(\mathfrak{B}')$. Then there exists a component K of D such that $\mathfrak{R}(\mathfrak{B}) < K < \mathfrak{R}(\mathfrak{B}')$.*

PROOF. We choose in \mathfrak{R} arbitrarily $H \in \mathfrak{R}(\mathfrak{B})$ and $H' \in \mathfrak{R}(\mathfrak{B}')$. Moreover, we choose in S arbitrarily $h \in H$ and $h' \in H'$. Then we have $h < h'$ and, by Lemma 2,

$$h \leq hh' \leq h'.$$

Furthermore, $D(h)$ and $D(h')$, in the associated semilattice S^* , belong to different branches at D , and so

$$D(hh') = D(h) \circ D(h') = D.$$

Now let K be the component of D which contains hh' . Then clearly $H < K < H'$. Since both $\mathfrak{R}(\mathfrak{B})$ and $\mathfrak{R}(\mathfrak{B}')$ are \mathfrak{R} -convex and do not contain K , we obtain $\mathfrak{R}(\mathfrak{B}) < K < \mathfrak{R}(\mathfrak{B}')$.

COROLLARY. *If $\mathfrak{R}(\mathfrak{B})$ is not the least component-branch at D , then there exists a component K of D such that $K < \mathfrak{R}(\mathfrak{B})$. If $\mathfrak{R}(\mathfrak{B})$ is not the greatest component-branch at D , then there exists a component K' of D such that $\mathfrak{R}(\mathfrak{B}) < K'$.*

5. Let $\mathfrak{R}(\mathfrak{B}_l)$ be the least component-branch at D . $\mathfrak{R}(\mathfrak{B}_l)$ may or may not have a component K of D such that $K < \mathfrak{R}(\mathfrak{B}_l)$. For a D -class D , if there exists the least component-branch $\mathfrak{R}(\mathfrak{B}_l)$ at D and if $\mathfrak{R}(\mathfrak{B}_l)$ has no component K of D such that $K < \mathfrak{R}(\mathfrak{B}_l)$, then we adjoin to the set of components of D an ideal component $G_l(D)$, which is called the *lower void component* of D . Also if there exists the greatest component-branch $\mathfrak{R}(\mathfrak{B}_u)$ at D and if $\mathfrak{R}(\mathfrak{B}_u)$ has no component K of D such that $\mathfrak{R}(\mathfrak{B}_u) < K$, then we adjoin to the set of components of D an ideal component $G_u(D)$, which is called the *upper void component* of D . Of course, in the case when both of the situations occur, we adjoin both the lower void component and the upper void component. Components of D previously defined and void components of D which may be adjoined are called

generalized components of D . The set of all generalized components of a fixed D -class D is denoted by \mathfrak{G}_D . The set of all generalized components of all D -classes is denoted by \mathfrak{G} . Each D -class D is non-void in S , and so there exists, in \mathfrak{G}_D , at least one non-void component of D , and it is evident that $\mathfrak{R}_D = \mathfrak{G}_D \cap \mathfrak{R}$.

Let D_0 be a D -class in S , and let \mathfrak{B} be a branch at D_0 in the associated semilattice S^* . The set-union of \mathfrak{G}_D when D goes through all the D -classes which belong to \mathfrak{B} in S^* , is called the *generalized component-branch* at D_0 associated with the branch \mathfrak{B} , and is denoted by $\mathfrak{G}(\mathfrak{B})$. It is clear that the number of generalized component-branches at D in \mathfrak{G} is equal to the number of component-branches at D in \mathfrak{R} and also to branch order at D in S^* . The generalized component-branch at D associated with the branch at D in S^* which determines the least component-branch at D in \mathfrak{R} , is called the *least generalized component-branch* at D , and is denoted by $\mathfrak{G}(\mathfrak{B}_l(D))$. The *greatest generalized component-branch* $\mathfrak{G}(\mathfrak{B}_u(D))$ at D is defined similarly.

THEOREM 4. *In an ordered idempotent semigroup S , the set \mathfrak{G} of all generalized components can be simply ordered to satisfy the following conditions:*

- (i) *for non-void components, the order coincides with the order defined in \mathfrak{R} ;*
- (ii) *if there exists the lower void component $G_l(D)$ of D , then $G_l(D) < G$ for every generalized component G which belongs to the least generalized component-branch $\mathfrak{G}(\mathfrak{B}_l(D))$ at D ;*
- (iii) *if there exists the upper void component $G_u(D)$ of D , then $G < G_u(D)$ for every generalized component G which belongs to the greatest generalized component-branch $\mathfrak{G}(\mathfrak{B}_u(D))$ at D ;*
- (iv) *every generalized component-branch $\mathfrak{G}(\mathfrak{B})$ is \mathfrak{G} -convex.*

PROOF. We well-order all void components. We adjoin to \mathfrak{R} each void component one by one according to the well-ordering. We shall show that, in each step, the extended set \mathfrak{G}_ξ can be simply ordered to satisfy the following conditions:

- (i $_\xi$) *for non-void components of \mathfrak{G}_ξ , the order coincides with the order in \mathfrak{R} ;*
- (ii $_\xi$) *if $G_l(D) \in \mathfrak{G}_\xi$, then $G_l(D) < G$ for every G of \mathfrak{G}_ξ which belongs to $\mathfrak{G}(\mathfrak{B}_l(D))$;*
- (iii $_\xi$) *if $G_u(D) \in \mathfrak{G}_\xi$, then $G < G_u(D)$ for every G of \mathfrak{G}_ξ which belongs to $\mathfrak{G}(\mathfrak{B}_u(D))$;*
- (iv $_\xi$) *for every branch \mathfrak{B} in S^* , $\mathfrak{G}(\mathfrak{B}) \cap \mathfrak{G}_\xi$ is \mathfrak{G}_ξ -convex.*

We prove this by transfinite induction. First, we suppose that \mathfrak{G}_α has been constructed and that the adjoined element to \mathfrak{G}_α is the lower void component $G_l(D_0)$ of D_0 . We order $\mathfrak{G}_{\alpha+1} = \mathfrak{G}_\alpha \cup \{G_l(D_0)\}$ as follows:

- (a) *for elements in \mathfrak{G}_α , we preserve the order in \mathfrak{G}_α ;*
- (b) *for $G \in \mathfrak{G}_\alpha$, we define $G_l(D_0) < G$ if there exists an element $G' \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_l(D_0))$ such that $G' \leq G$ in \mathfrak{G}_α ;*

(c) for $G \in \mathfrak{G}_\alpha$, we define $G < G_l(D_0)$ if $G < G'$ in \mathfrak{G}_α for every $G' \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_l(D_0))$.

It is not hard to verify that in this order $\mathfrak{G}_{\alpha+1}$ is simply ordered and moreover satisfies the conditions (i $_{\alpha+1}$), (ii $_{\alpha+1}$) and (iii $_{\alpha+1}$). We omit these verifications and only show that the ordered $\mathfrak{G}_{\alpha+1}$ satisfies (iv $_{\alpha+1}$). Let G_1, G_2 and G_3 be elements of $\mathfrak{G}_{\alpha+1}$ such that $G_1 < G_2 < G_3$ and $G_1, G_3 \in \mathfrak{G}(\mathfrak{B})$. We suppose that G_1, G_2 and G_3 are generalized components of D_1, D_2 and D_3 , respectively, and suppose that \mathfrak{B} is a branch at D in S^* . If $G_1, G_2, G_3 \in \mathfrak{G}_\alpha$, then $G_2 \in \mathfrak{G}(\mathfrak{B})$, by (iv $_\alpha$). If $G_1 = G_l(D_0)$, then $D_0 \in \mathfrak{B}$ in S^* , since $G_1 \in \mathfrak{G}(\mathfrak{B})$. Moreover, since $G_l(D_0) < G_2$, there exists an element $G_4 \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_l(D_0))$ such that $G_4 \leq G_2$. We suppose that G_4 is a generalized component of D_4 . Then $D_4 \succ D_0 \succ D$ and so D_4 and D_0 belong to the same branch at D . Hence $D_4 \in \mathfrak{B}$ and so $G_4 \in \mathfrak{G}(\mathfrak{B})$. Then, since $G_4 \leq G_2 < G_3$, we have $G_2 \in \mathfrak{G}(\mathfrak{B})$, by (iv $_\alpha$). If $G_3 = G_l(D_0)$, then we choose arbitrarily non-void component G_4 of $\mathfrak{G}(\mathfrak{B}_l(D_0))$. Clearly $G_4 \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_l(D_0))$. Hence, as above, we can prove $G_4 \in \mathfrak{G}(\mathfrak{B})$. Since $G_1 < G_2 < G_4$, we have $G_2 \in \mathfrak{G}(\mathfrak{B})$. Finally, if $G_2 = G_l(D_0)$, then there exists an element $G_4 \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_l(D_0))$ such that $G_4 \leq G_3$. We suppose that G_4 is a generalized component of D_4 . Now $G_1 < G_4 \leq G_3$, and, by (iv $_\alpha$), we have $G_4 \in \mathfrak{G}(\mathfrak{B})$. Hence $D_1, D_4 \in \mathfrak{B}$, and so $D_1 \circ (D_1 \circ D_4) = D_1 \circ D_4 \succ D$, from which it follows that $D_1 \circ D_4 \in \mathfrak{B}$. On the other hand, $D_1 \circ D_4 \leq D_4, D_0 \leq D_4$, and hence, since S^* is a tree semilattice, D_0 and $D_1 \circ D_4$ are comparable. But it is impossible that $D_0 \prec D_1 \circ D_4$. For, if $D_0 \prec D_1 \circ D_4$ were true, then D_1 and D_4 would lie, in S^* , in the same branch at D_0 . Therefore D_1 would lie in the branch $\mathfrak{B}_l(D_0)$, and so $G_1 \in \mathfrak{G}(\mathfrak{B}_l(D_0))$, which contradicts that $G_1 < G_2 = G_l(D_0)$. Thus $D_0 \geq D_1 \circ D_4$, and so, since $D_1 \circ D_4 \in \mathfrak{B}$, we have $D_0 \in \mathfrak{B}$. Hence $G_2 = G_l(D_0) \in \mathfrak{G}(\mathfrak{B})$. This completes the proof of (iv $_{\alpha+1}$). Next we suppose that $\mathfrak{G}_{\alpha+1}$ is obtained from \mathfrak{G}_α by adjoining the upper void component $G_u(D_0)$ of D_0 . We order $\mathfrak{G}_{\alpha+1}$ as follows:

(d) for elements in \mathfrak{G}_α , we preserve the order in \mathfrak{G}_α ;

(e) for $G \in \mathfrak{G}_\alpha$, we define $G < G_u(D_0)$ if there exists an element $G' \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_u(D_0))$ such that $G \leq G'$ in \mathfrak{G}_α ;

(f) for $G \in \mathfrak{G}_\alpha$, we define $G_u(D_0) < G$ if $G' < G$ in \mathfrak{G}_α for every $G' \in \mathfrak{G}_\alpha \cap \mathfrak{G}(\mathfrak{B}_u(D_0))$.

We can prove in a similar way that, with respect to this order, $\mathfrak{G}_{\alpha+1}$ is simply ordered to satisfy the conditions (i $_{\alpha+1}$)–(iv $_{\alpha+1}$). For a limit ordinal α , we order $\mathfrak{G}_\alpha = \bigcup_{\xi < \alpha} \mathfrak{G}_\xi$ as follows:

(g) for $G, G' \in G_\alpha$, we define $G < G'$ if $G < G'$ in \mathfrak{G}_ξ , where ξ is an ordinal number such that $\xi < \alpha$ and $G, G' \in \mathfrak{G}_\xi$.

As is easily seen, there exists such a ξ and the definition of the order in \mathfrak{G}_α is irrespective of the choice of ξ . Moreover, it is not hard to verify that, with respect to this order, \mathfrak{G}_α is simply ordered to satisfy the conditions

(i_α)–(iv_α). We omit these verifications. This completes the proof of the theorem.

In what follows, we always suppose that \mathfrak{G} is ordered as is shown in Theorem 4.

6. In this section, we discuss the structure of the set \mathfrak{G} of all generalized components.

LEMMA 8. *Let $\mathfrak{G}(\mathfrak{B})$ be a generalized component-branch at D and let G_0 be a generalized component of D . If there exists $G \in \mathfrak{G}(\mathfrak{B})$ such that $G < G_0$, then $\mathfrak{G}(\mathfrak{B}) < G_0$. If there exists $G' \in \mathfrak{G}(\mathfrak{B})$ such that $G_0 < G'$, then $G_0 < \mathfrak{G}(\mathfrak{B})$.*

PROOF. Suppose that $G < G_0$ for some $G \in \mathfrak{G}(\mathfrak{B})$. Since $\mathfrak{G}(\mathfrak{B})$ is \mathfrak{G} -convex and does not contain G_0 , we have $\mathfrak{G}(\mathfrak{B}) < G_0$. The second assertion can be proved similarly.

COROLLARY. *If there exists the lower void component $G_l(D)$ of D , then there exists the least generalized component-branch $\mathfrak{G}(\mathfrak{B}_l(D))$ at D and $\mathfrak{G}(\mathfrak{B}_l(D)) < K(D)$ for every non-void component $K(D)$ of D . If there exists the upper void component $G_u(D)$ of D , then there exists the greatest generalized component-branch $\mathfrak{G}(\mathfrak{B}_u(D))$ at D and $K(D) < \mathfrak{G}(\mathfrak{B}_u(D))$ for every non-void component $K(D)$ of D .*

THEOREM 5. *Let D be a D -class of an ordered idempotent semigroup S . If the lower void component $G_l(D)$ of D exists, then*

$$G_l(D) \leq G(D) \text{ for every } G(D) \in \mathfrak{G}_D.$$

If the upper void component $G_u(D)$ exists, then

$$G(D) \leq G_u(D) \text{ for every } G(D) \in \mathfrak{G}_D.$$

PROOF. We choose arbitrarily non-void component $K(D)$ of D . If $G_l(D)$ exists, then, by Corollary of Lemma 8 and Theorem 4, there exists the least generalized component-branch $\mathfrak{G}(\mathfrak{B}_l(D))$ at D and $G_l(D) < \mathfrak{G}(\mathfrak{B}_l(D)) < K(D)$. Hence we have $G_l(D) < K(D)$. If $G_u(D)$ exists, then by a similar way we can prove that $K(D) < G_u(D)$. This completes the proof of the theorem.

LEMMA 9. *Let G_1, G_2 and G_3 be generalized components of D_1, D_2 and D_3 , respectively. If $G_1 \leq G_2 \leq G_3$, then $D_2 \geq D_1 \circ D_3$.*

PROOF. We set $D = D_1 \circ D_2 \circ D_3$. If either $D_1 = D$ or $D_3 = D$, then it is clear that $D_2 \geq D_1 \circ D_3$. Hence we assume that $D_1 \succ D$ and $D_3 \succ D$. Thus G_1 and G_3 lie in generalized component-branches $\mathfrak{G}(\mathfrak{B}_1)$ and $\mathfrak{G}(\mathfrak{B}_3)$ at D , respectively. Now it is clear that $D_2 \geq D_1 \circ D_2$ and $D_2 \geq D_2 \circ D_3$. Therefore, since the associated semilattice S^* is a tree semilattice, $D_1 \circ D_2$ and $D_2 \circ D_3$ are comparable in S^* and so

$$D = D_1 \circ D_2 \quad \text{or} \quad D = D_2 \circ D_3.$$

This shows that either G_1 and G_2 or G_2 and G_3 lie in different generalized component-branches at D . Then, since every generalized component-branch is \mathfrak{G} -convex, $\mathfrak{G}(\mathfrak{B}_1)$ and $\mathfrak{G}(\mathfrak{B}_3)$ are distinct. Hence $D_1 \circ D_3 = D$, from which it

follows that $D_2 \geq D_1 \circ D_3$.

LEMMA 10. *Let $G_1(D)$ and $G_2(D)$ be two generalized components of D such that $G_1(D) < G_2(D)$. Then there exists a generalized component-branch $\mathfrak{G}(\mathfrak{B})$ at D such that $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$.*

PROOF. If $G_1(D)$ is the lower void component of D , then it is evident that $G_1(D) < \mathfrak{G}(\mathfrak{B}_l(D)) < G_2(D)$. If $G_2(D)$ is the upper void component of D , then $G_1(D) < \mathfrak{G}(\mathfrak{B}_u(D)) < G_2(D)$. If both $G_1(D)$ and $G_2(D)$ are non-void, then, since both $G_1(D)$ and $G_2(D)$ are maximal S -convex subsets of D , there exists in S an element a such that $G_1(D) < a < G_2(D)$ and $D(a) \neq D$. Let K be the component of $D(a)$ which contains a . Then we have $G_1(D) < K < G_2(D)$ and moreover, by Lemma 9, we have $D(a) \succ D$. Now let $\mathfrak{G}(\mathfrak{B})$ be the generalized component-branch at D which contains K . Then, by Lemma 8, we obtain $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$.

LEMMA 11. *Let $\mathfrak{G}(\mathfrak{B})$ and $\mathfrak{G}(\mathfrak{B}')$ be generalized component-branches at D such that $\mathfrak{G}(\mathfrak{B}) < \mathfrak{G}(\mathfrak{B}')$. Then there exists a non-void component $K(D)$ of D such that $\mathfrak{G}(\mathfrak{B}) < K(D) < \mathfrak{G}(\mathfrak{B}')$.*

PROOF. By Lemma 7, there exists a non-void component $K(D)$ of D such that $\mathfrak{G}(\mathfrak{B}) \cap \mathfrak{R} < K(D) < \mathfrak{G}(\mathfrak{B}') \cap \mathfrak{R}$ in \mathfrak{R} . Then, by Lemma 8, we have $\mathfrak{G}(\mathfrak{B}) < K(D) < \mathfrak{G}(\mathfrak{B}')$.

Two distinct generalized components $G_1(D)$ and $G_2(D)$ in \mathfrak{G}_D are said to be *consecutive*, if there exists no element of \mathfrak{G}_D which lies between $G_1(D)$ and $G_2(D)$ in the strict sense.

THEOREM 6. *Let D be a D -class of an ordered idempotent semigroup S and let $G_1(D)$ and $G_2(D)$ be consecutive generalized components of D such that $G_1(D) < G_2(D)$. Then*

$$\mathfrak{A} = \{G; G \in \mathfrak{G} \text{ and } G_1(D) < G < G_2(D)\}$$

is a generalized component-branch at D .

PROOF. By Lemma 10, there exists a generalized component-branch $\mathfrak{G}(\mathfrak{B})$ at D such that $\mathfrak{G}(\mathfrak{B}) \subset \mathfrak{A}$. Now we take any element G' of \mathfrak{A} . Then, by Lemma 9 and the consecutivity of $G_1(D)$ and $G_2(D)$, G' is a generalized component of D' such that $D' \succ D$. Thus G' lies in a generalized component-branch $\mathfrak{G}(\mathfrak{B}')$ at D . If $\mathfrak{G}(\mathfrak{B}')$ were different from $\mathfrak{G}(\mathfrak{B})$, then, by Lemma 11, there would exist a component $K(D)$ of D which lies between $\mathfrak{G}(\mathfrak{B})$ and $\mathfrak{G}(\mathfrak{B}')$ in the strict sense. This contradicts the fact that $G_1(D)$ and $G_2(D)$ are consecutive. Hence $\mathfrak{G}(\mathfrak{B}) = \mathfrak{G}(\mathfrak{B}')$, and so G' belongs to $\mathfrak{G}(\mathfrak{B})$. This completes the proof of the theorem.

LEMMA 12. *Let $\mathfrak{R}(\mathfrak{B})$ be a component-branch at D . If there exists, in \mathfrak{R} , a non-void component $K(D)$ of D such that $K(D) < \mathfrak{R}(\mathfrak{B})$, then there exists, in S ,*

$$a = \max \{y; y \in D \text{ and } y < K \text{ for every } K \in \mathfrak{R}(\mathfrak{B})\}.$$

The component $K_1(D)$ of D which contains the element a satisfies the following

conditions :

(i) $K_1(D) < \mathfrak{R}(\mathfrak{B})$;

(ii) if $K(D)$ is a component of D such that $K(D) < \mathfrak{R}(\mathfrak{B})$, then $K(D) \leq K_1(D)$.

Also if there exists, in \mathfrak{R} , a component $K'(D)$ of D such that $\mathfrak{R}(\mathfrak{B}) < K'(D)$, then there exists, in S ,

$$b = \min \{y; y \in D \text{ and } K < y \text{ for every } K \in \mathfrak{R}(\mathfrak{B})\} .$$

The component $K_2(D)$ of D which contains b satisfies the following conditions :

(iii) $\mathfrak{R}(\mathfrak{B}) < K_2(D)$;

(iv) if $K(D)$ is a component of D such that $\mathfrak{R}(\mathfrak{B}) < K(D)$, then $K_2(D) \leq K(D)$.

PROOF. Suppose that $K(D) < \mathfrak{R}(\mathfrak{B})$ for some component $K(D)$ of D . We choose arbitrarily $K' \in \mathfrak{R}(\mathfrak{B})$. Moreover we choose in S arbitrarily $k \in K(D)$ and $k' \in K'$. Then, by Corollary 3 of Theorem 2, there exists

$$a^* = \max \{y; y \in D \text{ and } y < k'\} .$$

Let K^* be the component of D which contains a^* . Then, we have $K^* < K'$ and hence $K^* < \mathfrak{R}(\mathfrak{B})$. Therefore in S

$$a^* = \max \{y; y \in D \text{ and } y < K \text{ for every } K \in \mathfrak{R}(\mathfrak{B})\} .$$

Hence the element a of the lemma exists and $a = a^*$. Therefore $K_1(D) = K^* < \mathfrak{R}(\mathfrak{B})$, which proves (i). Moreover, for an element k of a component $K(D)$ of D such that $K(D) < \mathfrak{R}(\mathfrak{B})$, we have clearly $k \in D$ and $k < k'$. Hence $k \leq a^* = a$, and so $K(D) \leq K_1(D)$, which proves (ii). The rest of the assertions can be proved similarly.

THEOREM 7. *Let D be a D -class of an ordered idempotent semigroup S . If the lower void component $G_l(D)$ of D exists, then there exists a non-void component $G_1(D)$ of D , with which $G_l(D)$ forms a consecutive pair. If the upper void component $G_u(D)$ of D exists, then there exists a non-void component $G_2(D)$ of D , with which $G_u(D)$ forms a consecutive pair.*

PROOF. We choose arbitrarily non-void component $K_0(D)$ of D . If $G_l(D)$ exists, then, by Corollary of Lemma 8, there exists the least generalized component-branch $\mathfrak{G}(\mathfrak{B}_l(D))$ at D , and $\mathfrak{G}(\mathfrak{B}_l(D)) < K_0(D)$. Hence, by Lemma 12, there exists a non-void component $G_1(D)$ of D satisfying the condition that if $K(D)$ is a non-void component of D such that $K(D) > K$ for every non-void $K \in \mathfrak{G}(\mathfrak{B}_l(D))$, then $G_1(D) \leq K(D)$. But every non-void component of D satisfies the condition given for $K(D)$. Hence $G_1(D) \leq K(D)$ for every non-void component $K(D)$ of D . Even if $G_u(D)$ exists, it is clear that $G_1(D) < G_u(D)$. Hence $G_l(D)$ and $G_1(D)$ are consecutive generalized components of D . The second assertion can be proved similarly.

THEOREM 8. *Let D be a D -class of an ordered idempotent semigroup S , and let $G_1(D)$ and $G_2(D)$ be consecutive generalized components of D such that $G_1(D)$*

$< G_2(D)$. If $G_1(D)$ is a non-void component, then there exists, in S , $\max G_1(D)$. Also if $G_2(D)$ is a non-void component, then there exists, in S , $\min G_2(D)$.

PROOF. By Theorem 6,

$$\mathfrak{G}(\mathfrak{B}) = \{G; G \in \mathfrak{G} \text{ and } G_1(D) < G < G_2(D)\}$$

is a generalized component-branch at D . If $G_1(D)$ is non-void, then, by Lemma 12, there exists in S

$$a = \max \{y; y \in D \text{ and } y < K \text{ for every non-void } K \in \mathfrak{G}(\mathfrak{B})\}.$$

We take the component $K_1(D)$ of D which contains the element a . Then, by (i) in Lemma 12, we have $K_1(D) < K < G_2(D)$ for every non-void $K \in \mathfrak{G}(\mathfrak{B})$. But $G_1(D)$ and $G_2(D)$ are consecutive, and hence $K_1(D) \leq G_1(D)$. On the other hand, by (ii) in Lemma 12, we have $G_1(D) \leq K_1(D)$. Hence $G_1(D) = K_1(D)$, and so the element a belongs to $G_1(D)$. Moreover, since

$$G_1(D) \subset \{y; y \in D \text{ and } y < K \text{ for every non-void } K \in \mathfrak{G}(\mathfrak{B})\},$$

we have $a = \max G_1(D)$. The second assertion of this theorem can be proved similarly.

THEOREM 9. Let D be a D -class of an ordered idempotent semigroup S , and let $\mathfrak{G}(\mathfrak{B})$ be a generalized component-branch at D . Then there exist consecutive generalized components $G_1(D)$ and $G_2(D)$ of D such that

$$\mathfrak{G}(\mathfrak{B}) = \{G; G \in \mathfrak{G} \text{ and } G_1(D) < G < G_2(D)\}.$$

PROOF. We first show that, for $\mathfrak{G}(\mathfrak{B})$, there exist consecutive generalized components $G_1(D)$ and $G_2(D)$ of D such that $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$. In the case when there exists no non-void component $K(D)$ of D such that $K(D) < \mathfrak{G}(\mathfrak{B})$, we have, by Lemma 11, $\mathfrak{G}(\mathfrak{B}) = \mathfrak{G}(\mathfrak{B}_l(D))$. Moreover there exists the lower void component $G_l(D)$. Therefore, by Theorem 7, there exists a non-void component $G_2(D)$ of D such that $G_l(D)$ and $G_2(D)$ are consecutive generalized components of D . Then it is clear that $G_l(D) < \mathfrak{G}(\mathfrak{B}_l(D)) = \mathfrak{G}(\mathfrak{B}) < G_2(D)$. In the case when there exists no non-void component $K(D)$ of D such that $\mathfrak{G}(\mathfrak{B}) < K(D)$, we can prove in a similar way that there exist consecutive generalized components $G_1(D)$ and $G_u(D)$ of D such that $G_1(D) < \mathfrak{G}(\mathfrak{B}) = \mathfrak{G}(\mathfrak{B}_u(D)) < G_u(D)$. In the case when there exist non-void components $K_1(D)$ and $K_2(D)$ of D such that $K_1(D) < \mathfrak{G}(\mathfrak{B}) < K_2(D)$, then, by Lemma 12, there exist non-void components $G_1(D)$ and $G_2(D)$ of D satisfying the following conditions:

- (i) $G_1(D) < K < G_2(D)$ for every non-void $K \in \mathfrak{G}(\mathfrak{B})$;
- (ii) if $K(D)$ is a component of D such that $K(D) < K$ for every non-void $K \in \mathfrak{G}(\mathfrak{B})$, then $K(D) \leq G_1(D)$;
- (iii) if $K'(D)$ is a component of D such that $K < K'(D)$ for every non-void $K \in \mathfrak{G}(\mathfrak{B})$, then $G_2(D) \leq K'(D)$.

By Lemma 8, (i) implies that $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$. Now we take any gener-

alized component $G(D)$ of D which lies between $G_1(D)$ and $G_2(D)$. Then clearly $G(D)$ is a non-void component. We choose arbitrarily non-void $K \in \mathfrak{G}(\mathfrak{B})$. If $G(D) < K$, then $G(D) < \mathfrak{G}(\mathfrak{B})$, and so, by (ii), we have $G(D) = G_1(D)$. If $K < G(D)$, then we have $G(D) = G_2(D)$ similarly. Hence $G_1(D)$ and $G_2(D)$ are consecutive components of D . Thus, in all cases, we have shown that there exist consecutive generalized components $G_1(D)$ and $G_2(D)$ such that $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$. Now the set

$$\mathfrak{G}(\mathfrak{B}') = \{G; G \in \mathfrak{G} \text{ and } G_1(D) < G < G_2(D)\}$$

is, by Theorem 6, a generalized component-branch at D . $\mathfrak{G}(\mathfrak{B})$ and $\mathfrak{G}(\mathfrak{B}')$ are generalized component-branches at the same D -class D and have common element. Hence $\mathfrak{G}(\mathfrak{B}) = \mathfrak{G}(\mathfrak{B}')$. This completes the proof of this theorem.

COROLLARY 1. *There exists one-to-one correspondence between branches at D in S^* and consecutive pairs of generalized components of D .*

COROLLARY 2. *The number of consecutive pairs of generalized components of D is equal to the branch order at D in S^* .*

THEOREM 10. *Let D be a D -class of an ordered idempotent semigroup S , and let $G_1(D)$ and $G_2(D)$ be generalized components of D such that $G_1(D) < G_2(D)$. Then there exist consecutive generalized components $G_3(D)$ and $G_4(D)$ of D such that $G_1(D) \leq G_3(D) < G_4(D) \leq G_2(D)$.*

PROOF. By Lemma 10, there exists a generalized component-branch $\mathfrak{G}(\mathfrak{B})$ at D such that $G_1(D) < \mathfrak{G}(\mathfrak{B}) < G_2(D)$. Then, by Theorem 9, there exist consecutive generalized components $G_3(D)$ and $G_4(D)$ of D such that $G_3(D) < \mathfrak{G}(\mathfrak{B}) < G_4(D)$. Therefore, since $G_3(D)$ and $G_4(D)$ are consecutive, we have $G_1(D) \leq G_3(D) < G_4(D) \leq G_2(D)$.

7. In this section, we consider conversely, and show that some of the theorems previously given characterize ordered idempotent semigroups. More precisely

THEOREM 11. *Let S^* be a tree semilattice. Suppose that, for each element D of S^* , there associates a collection of sets*

$$\mathfrak{G}_D = \{G_\alpha(D); \alpha \in I(D)\}$$

such that every pair of sets, whether it belongs to the same \mathfrak{G}_D or not, is mutually disjoint. Moreover suppose that each collection \mathfrak{G}_D is simply ordered to satisfy the following conditions:

- (i) *every $G_\alpha(D)$ in \mathfrak{G}_D , if it is neither the least nor the greatest member of \mathfrak{G}_D , is non-void;*
- (ii) *there exists one-to-one correspondence between consecutive pairs of members in \mathfrak{G}_D and branches at D in S^* .*

Furthermore, suppose that each set $G_\alpha(D)$, if it is non-void, is simply ordered to satisfy the following conditions:

(iii) if $G_\alpha(D)$ is non-void and has the immediate successor in \mathfrak{S}_D , then there exists $\max G_\alpha(D)$ in $G_\alpha(D)$;

(iv) if $G_\alpha(D)$ is non-void and has the immediate predecessor in \mathfrak{S}_D , then there exists $\min G_\alpha(D)$ in $G_\alpha(D)$.

For an element D of S^* , we denote the set-union $\bigcup_{\alpha \in I(D)} G_\alpha(D)$ of all the sets in \mathfrak{S}_D by $S(D)$. Moreover we denote $\bigcup_{D \in S^*} S(D)$ by S . Besides, for two elements D and D' in S^* such that $D' \prec D$, there exists, by the above condition (ii), a consecutive pair of members which corresponds to the branch at D' which contains D . We denote the lesser and the greater of these members by $L_{D'}(D)$ and $M_{D'}(D)$, respectively. We introduce an order in S . For $a \in G(D)$ and $b \in G'(D')$, let us define $a < b$ in S if one of the following conditions is satisfied:

- (a) $G(D) = G'(D')$ and $a < b$ in $G(D)$;
- (b) $D = D'$ and $G(D) < G'(D')$ in \mathfrak{S}_D ;
- (c) $D \prec D'$ and $G(D) \leq L_{D'}(D)$;
- (d) $D' \prec D$ and $M_{D'}(D) \leq G'(D')$;
- (e) $D \circ D' \prec D$, $D \circ D' \prec D'$ and $M_{D'}(D \circ D') \leq L_{D'}(D \circ D')$.

Moreover, for each element D of S^* , we assign arbitrarily one of two types which we call *L-type* and *R-type*. We introduce a multiplication in S . For $a \in G(D)$ and $b \in G'(D')$, let us define the product ab as follows:

- (f) if $D \circ D'$ is *L-typed* and $a \leq b$, then

$$ab = \min \{y; y \in S(D \circ D') \text{ and } a \leq y\};$$
- (g) if $D \circ D'$ is *L-typed* and $b \leq a$, then

$$ab = \max \{y; y \in S(D \circ D') \text{ and } y \leq a\};$$
- (h) if $D \circ D'$ is *R-typed* and $a \leq b$, then

$$ab = \max \{y; y \in S(D \circ D') \text{ and } y \leq b\};$$
- (i) if $D \circ D'$ is *R-typed* and $b \leq a$, then

$$ab = \min \{y; y \in S(D \circ D') \text{ and } b \leq y\}.$$

Then, with respect to the order and the product given above, S is an ordered idempotent semigroup.

PROOF. We prove this theorem by dividing in several steps.

1°. For $a, b \in S$, one and only one of the relations $a < b$, $a = b$ and $a > b$ holds.

In fact, suppose that $a \in G(D)$, $b \in G'(D')$ and $a \not\leq b$. Then it is easy to show that $a > b$ in the case when D and D' are comparable. In this paper, we show it only in the case when D and D' are non-comparable. In this case, since $a \not\leq b$, we have

$$M_{D'}(D \circ D') > L_{D'}(D \circ D'), \text{ and so } M_{D'}(D \circ D') \geq M_D(D \circ D').$$

But, since D and D' belong to different branches at $D \circ D'$, we have

$$M_D(D \circ D') > M_{D'}(D \circ D').$$

Hence

$$L_D(D \circ D') \geq M_{D'}(D \circ D'), \text{ and so } a > b \text{ in } S.$$

This proves the ‘one’ part of the assertion. It is easy to prove the ‘only one’ part of the assertion and we omit it.

2°. *Suppose that $a \in G(D)$, $b \in G'(D')$, $D'' < D$, $D'' < D'$ and $M_D(D'') \leq L_D(D'')$ in $\mathfrak{G}_{D''}$. Then $D \circ D' = D''$ and $a < b$ in S .*

In fact, under these assumptions, D and D' belong to different branches at D'' , for otherwise we would have

$$L_{D'}(D'') = L_D(D'') < M_D(D'').$$

Therefore $D \circ D' = D''$. Hence we have $M_D(D \circ D') \leq L_{D'}(D \circ D')$, and so $a < b$ in S .

3°. *S is simply ordered.*

In fact, by 1°, it suffices to prove the transitivity. Suppose that $a \in G(D)$, $b \in G'(D')$ and $c \in G''(D'')$, and that $a < b$ and $b < c$. Now we remark that $D \circ D'$ and $D' \circ D''$ are always comparable, since $D \circ D' \leq D'$, $D' \circ D'' \leq D'$ and S^* is a tree semilattice. First we suppose that $D \circ D' \leq D' \circ D''$. In this paper, we prove that $a < c$ only in the case when D and D' are non-comparable. If D and D' are non-comparable and $D \circ D' \neq D' \circ D''$, then $D \circ D' < D' \circ D''$, and so, in S^* , D' and D'' belong to the same branch at $D \circ D'$. Therefore

$$M_D(D \circ D') \leq L_{D'}(D \circ D') = L_{D''}(D \circ D').$$

Hence, by 2°, we have $a < c$ in S . If D and D' are non-comparable and $D \circ D' = D''$, then

$$M_D(D'') = M_D(D \circ D') \leq L_{D'}(D \circ D') \leq G''(D''),$$

and so $a < c$ in S . Finally if D and D' are non-comparable, $D \circ D' = D' \circ D''$ and $D \circ D' \neq D''$, then D' and D'' belong to different branches at $D \circ D'$. Hence

$$M_D(D \circ D') \leq L_{D'}(D \circ D') < M_{D'}(D \circ D') \leq L_{D''}(D \circ D'),$$

and so, by 2°, we have $a < c$ in S . In the case when $D \circ D' > D' \circ D''$, we can prove the assertion in a similar way.

4°. *For every $a, b \in S$, the product ab is well-defined.*

In fact, suppose that $a \in G(D)$ and $b \in G'(D')$. First we suppose that $D \circ D'$ is L -typed and $a \leq b$. If $D \circ D' = D$, then it is clear that $\min \{y; y \in S(D \circ D') \text{ and } a \leq y\}$ exists and is equal to a . If $D \circ D' < D$, then $a < b$ and there exist both $L_D(D \circ D')$ and $M_D(D \circ D')$. Clearly $L_D(D \circ D') < M_D(D \circ D')$, and so, in $\mathfrak{G}_{D \circ D'}$, $M_D(D \circ D')$ is not the least member. Now $M_D(D \circ D')$ is non-void. For, if $b \in M_D(D \circ D')$, then $M_D(D \circ D')$ is evidently non-void. If $D' = D \circ D'$ and $b \in M_D(D \circ D')$, then

$$M_D(D \circ D') < G'(D \circ D') = G'(D'),$$

and so $M_D(D \circ D')$ is not the greatest member in $\mathfrak{G}_{D \circ D'}$. Hence, by (i), $M_D(D \circ D')$ is non-void. Finally, if $D \circ D' \prec D'$, then, since $a < b$, we have

$$M_D(D \circ D') \leq L_{D'}(D \circ D') < M_{D'}(D \circ D').$$

Hence, also in this case, $M_D(D \circ D')$ is not the greatest member in $\mathfrak{G}_{D \circ D'}$ and so is non-void. Clearly $M_D(D \circ D')$ has the immediate predecessor $L_D(D \circ D')$ in $\mathfrak{G}_{D \circ D'}$, and so, by (iv), there exists

$$p = \min M_D(D \circ D').$$

It is clear that

$$p \in \{y; y \in S(D \circ D') \text{ and } a \leq y\}.$$

Now we take any $y \in S(D \circ D')$ such that $a \leq y$. We suppose that $y \in G''(D \circ D')$. Since $D \circ D' \prec D$, we have $a < y$ and $M_D(D \circ D') \leq G''(D \circ D')$, and therefore $p = \min M_D(D \circ D') \leq y$. Hence

$$p = \min \{y; y \in S(D \circ D') \text{ and } a \leq y\}.$$

In all other cases, we can prove the assertion in a similar way.

5°. For every $a, b \in S$, the element ab lies between a and b . In particular, we have $a^2 = a$.

In fact, suppose that $a \in G(D)$ and $b \in G'(D')$. First we suppose that $D \circ D'$ is L -typed and $a \leq b$. If $D \circ D' = D$, then, by the proof of 4°, we have $ab = a$. Hence, in this case, ab evidently lies between a and b . If $D \circ D' \prec D$ and $D \circ D' = D'$, then, by the proof of 4°, $ab \in M_D(D \circ D')$ and $M_D(D \circ D') \leq G'(D \circ D') = G'(D')$. Hence $a < ab \leq b$. If $D \circ D' \prec D$ and $D \circ D' \prec D'$, then, by the proof of 4°, $ab \in M_D(D \circ D')$ and $M_D(D \circ D') \leq L_{D'}(D \circ D')$. Hence $a < ab < b$. In all other cases we can prove the assertion in a similar way.

6°. If $a \in G(D)$ and $b \in G'(D')$, then both ab and ba belong to $S(D \circ D')$ and every element of $S(D \circ D')$ which lies between a and b , lies between ab and ba .

In fact, if $a \leq b$ and $D \circ D'$ is L -typed, then, by 5°, we have $a \leq ab \leq b$ and $a \leq ba \leq b$, and so

$$\begin{aligned} ab &= \min \{y; y \in S(D \circ D') \text{ and } a \leq y \leq b\} \\ &\leq \max \{y; y \in S(D \circ D') \text{ and } a \leq y \leq b\} = ba, \end{aligned}$$

from which the assertion is clear. In the other cases, we can prove the assertion in a similar way.

7°. If $a \in G(D)$, $b \in G'(D')$, $c \in G''(D'')$ and $a \leq b \leq c$, then $D \circ D'' \leq D'$.

In fact, by 5° and 6°, we have $a \leq ac \leq c$ and $ac \in S(D \circ D'')$. If $D \circ D'' \leq D'$ were false, then we would have $D \circ D' \circ D'' \prec D \circ D''$. Now $(ac)b \in S(D \circ D' \circ D'')$, and so $(ac)b \neq ac$. Moreover $(ac)b$ would lie between ac and b , and so lie between a and c . If $(ac)b < ac$, then $(ac)b$ would lie between a and ac . On the other hand, we have $D \circ D' \circ D'' \prec D \circ D'' = D \circ (D \circ D'')$, and so D and $D \circ D''$ would lie in the same branch at $D \circ D' \circ D''$. Hence $(ac)b$

$< ac$ would imply that $(ac)b < a$, which is a contradiction. Also in the case when $ac < (ac)b$, we can deduce a contradiction in a similar way.

8°. *If $a \leq b$, then $ac \leq bc$ and $ca \leq cb$.*

In fact, suppose that $a \in G(D)$, $b \in G'(D')$ and $c \in G''(D'')$. If $a \leq c \leq b$, then, by 5°, we have

$$a \leq ac \leq c \leq bc \leq b \quad \text{and} \quad a \leq ca \leq c \leq cb \leq b.$$

If $c \leq a \leq b$, then, by 7°, we have $D' \circ D'' \leq D$, and so $D' \circ D'' \leq D \circ D''$. Now we suppose that $bc \leq ac$. Then clearly $c \leq bc \leq ac$, and so, by 7°, we have $D \circ D'' = (D \circ D'') \circ D'' \leq D' \circ D''$. Hence we have $D \circ D'' = D' \circ D''$. Therefore, if $D \circ D'' = D' \circ D''$ is L -typed, then

$$\begin{aligned} ac &= \max \{y; y \in S(D \circ D'') \text{ and } c \leq y \leq a\} \\ &\leq \max \{y; y \in S(D' \circ D'') \text{ and } c \leq y \leq b\} = bc, \end{aligned}$$

and so $ac = bc$. If $D \circ D'' = D' \circ D''$ is R -typed, then we can deduce $ac = bc$ in a similar way. This evidently proves that $ac \leq bc$. We can prove that $ca \leq cb$ similarly. In all the remaining cases, we can prove the assertion in a similar way.

9°. *For every, $a, b \in S$, we have $a(ab) = (ab)b = ab$.*

In fact, suppose that $a \in G(D)$ and $b \in G'(D')$. If $D \circ D'$ is L -typed and $a \leq b$, then $a \leq ab \leq b$ and so $a \leq a(ab) \leq ab$. On the other hand, $a(ab) \in S(D \circ D \circ D') = S(D \circ D')$. Therefore

$$ab = \min \{y; y \in S(D \circ D') \text{ and } a \leq y\} \leq a(ab).$$

Hence we have $ab = a(ab)$. Moreover, since $ab \leq (ab)b \leq b$ and $(ab)b \in S(D \circ D')$, it is clear that

$$(ab)b = \min \{y; y \in S(D \circ D') \text{ and } ab \leq y\} = ab.$$

In all other cases, we can prove the assertion in a similar way.

10°. *If b lies between a and c , then $ac = ab$ or $ac = bc$.*

In fact, first we suppose that $a \leq b \leq c$. Then $a \leq ac \leq c$. If $ac \leq b$, then, by 9° and 8°, we have $ac = a(ac) \leq ab$. On the other hand, since $b \leq c$, we have $ab \leq ac$. Therefore we have $ac = ab$. If $b \leq ac$, then we can prove similarly that $ac = bc$. In the case when $c \leq b \leq a$, we can prove the assertion in a similar way.

11°. *S is an ordered idempotent semigroup.*

In fact, by 3°, 5° and 8°, it suffices to prove that, for every $a, b, c \in S$, we have $(ab)c = a(bc)$. Let a, b and c be elements of $S(D)$, $S(D')$ and $S(D'')$, respectively. First we suppose that $a \leq b \leq c$. Then, by 10°, we have $ac = ab$ or $ac = bc$. If $ac = ab$, then

$$(ab)c = (ac)c = ac \quad \text{by } 9^\circ$$

and $ac = ab \leq a(bc) \leq ac$ since $b \leq bc \leq c$.

Hence $(ab)c = ac = a(bc)$. If $ac = bc$, then we can prove that $(ab)c = a(bc)$ in a similar way. Next, we suppose that $a \leq c \leq b$. Then, by 10°, we have $ab = ac$ or $ab = cb$. If $ab = ac$, then we can prove that $(ab)c = a(bc)$ in the same argument as above. If $ab = cb$, then $D \circ D' = D' \circ D''$, and so $D \circ D' \circ D'' = D \circ D' = D' \circ D''$. The element $(ab)c = (cb)c$ lies between c and b and is an element of $S(D \circ D' \circ D'') = S(D' \circ D'')$. Therefore, by 6°, $\min(bc, cb) \leq (ab)c$. On the other hand,

$$(ab)c \leq (bb)c = bc \quad \text{and} \quad (ab)c \leq (cb)b = cb.$$

Hence we have $(ab)c = \min(bc, cb)$. Now, if $D \circ D' \circ D''$ is L -typed, then

$$\begin{aligned} a(bc) &= \min \{y; y \in S(D \circ D' \circ D'') \text{ and } a \leq y\} \\ &= \min \{y; y \in S(D \circ D') \text{ and } a \leq y\} = ab = cb. \end{aligned}$$

But, by the proof of 6°, we have $cb \leq bc$, and so

$$a(bc) = \min(bc, cb) = (ab)c.$$

If $D \circ D' \circ D''$ is R -typed, then

$$\begin{aligned} a(bc) &= \max \{y; y \in S(D \circ D' \circ D'') \text{ and } y \leq bc\} \\ &= \max \{y; y \in S(D' \circ D'') \text{ and } y \leq bc\} = bc. \end{aligned}$$

Since, in this case, $bc \leq cb$, we have the same result as above. Thus we obtain the associativity in the case when $a \leq c \leq b$. In the remaining cases, we can similarly prove that $(ab)c = a(bc)$. This completes the proof of Theorem 11.

THEOREM 12. *In addition to the assumptions of Theorem 11, we set the following assumptions:*

- (v) *for each D of S^* , there exists a non-void set $G_\alpha(D)$ which belongs to \mathfrak{G}_D ;*
- (vi) *for two members $G_\alpha(D)$ and $G_\beta(D)$ of \mathfrak{G}_D such that $G_\alpha(D) < G_\beta(D)$, there exist consecutive members $G_\gamma(D)$ and $G_\delta(D)$ of \mathfrak{G}_D such that $G_\alpha(D) \leq G_\gamma(D) < G_\delta(D) \leq G_\beta(D)$ in \mathfrak{G}_D ;*
- (vii) *if $G_\gamma(D)$ is the least member of \mathfrak{G}_D which is a void set, then $G_\gamma(D)$ has the immediate successor in \mathfrak{G}_D ;*
- (viii) *if $G_\alpha(D)$ is the greatest member of \mathfrak{G}_D which is a void set, then $G_\alpha(D)$ has the immediate predecessor in \mathfrak{G}_D .*

Under these assumptions, for each D of S^ , $S(D)$ is a D -class in the ordered idempotent semigroup S constructed in Theorem 11. Moreover, the semilattice associated with S is isomorphic to S^* and \mathfrak{G}_D is the set of all the generalized components of D -class $S(D)$ in S .*

PROOF. By (v), $S(D)$ is non-void. It $a, b \in S(D)$, then it is clear that

$$\begin{aligned} ab = a \quad \text{and} \quad ba = b & \quad \text{if } D \text{ is } L\text{-typed,} \\ ab = b \quad \text{and} \quad ba = a & \quad \text{if } D \text{ is } R\text{-typed.} \end{aligned}$$

Hence a and b belong to the same D -class in S . Conversely suppose that a

and c belong to the same D -class in S and that $a \in S(D)$ and $c \in S(D')$. Then, by Theorem 1, we have

$$ac = a \quad \text{and} \quad ca = c,$$

or

$$ac = c \quad \text{and} \quad ca = a.$$

In both cases, we have $S(D') = S(D \circ D') = S(D)$, and so $c \in S(D)$. Therefore $S(D)$ is a D -class in S . For two D -classes $S(D)$ and $S(D')$ of S , we take arbitrarily $a \in S(D)$ and $b \in S(D')$. Then $ab \in S(D \circ D')$, and so

$$S(D) \circ S(D') = S(D \circ D').$$

Here, in the left hand side, \circ represents the operation defined in §2 for the associated semilattice of S . Hence the associated semilattice of S is isomorphic to S^* . In particular, the associated semilattice of S can be considered to coincide with S^* , by identifying $S(D)$ with D . Now we show that each member $G(D)$ of \mathfrak{G}_D is a generalized component of D -class $S(D)$ in S . First we suppose that $G(D)$ is non-void. Let a and b be elements of $G(D)$ such that $a \leq b$ and let c be an element of S between them. We suppose that $c \in G'(D')$. Then, by Lemma 4, we have $D \leq D'$. If $D < D'$ were true, then $a \leq c$ would imply that $a < c$ and $G(D) < L_{D'}(D)$, and so we would have $b < c$ which is a contradiction. Therefore $D = D'$, and so, since $a \leq c \leq b$, we have $G(D) = G'(D')$. Hence $c \in G(D)$. Thus $G(D)$ is S -convex. Now let A be a subset of $S(D)$ which properly contains $G(D)$. Let a be an element of A which does not belong to $G(D)$ and let $G''(D)$ be a member of \mathfrak{G}_D which contains a . Then, since $G(D) \neq G''(D)$, there exists, by (vi), a consecutive pair of members in \mathfrak{G}_D which lies between $G(D)$ and $G''(D)$, and moreover, by (ii), there exists a branch \mathfrak{B} at D which corresponds to this consecutive pair. We take arbitrarily $D' \in \mathfrak{B}$, and moreover take arbitrarily $b \in S(D')$. Then it is easy to see that b lies between $G(D)$ and a in the strict sense. Therefore A is not S -convex. Hence $G(D)$ is a component of D -class $S(D)$ in S . Since the set-union of all non-void members of \mathfrak{G}_D is the D -class $S(D)$, it is clear that conversely every non-void component of $S(D)$ is a non-void member of \mathfrak{G}_D . Now we suppose that there exists the least member $G_i(D)$ of \mathfrak{G}_D which is a void set. Then, by (vii), $G_i(D)$ has the immediate successor $G_1(D)$, and moreover, by (ii), we can consider the branch \mathfrak{B}_i at D which corresponds to the consecutive members $G_i(D)$ and $G_1(D)$. Then it is easy to see that the component-branch $\mathfrak{R}(\mathfrak{B}_i)$ at $S(D)$ associated with \mathfrak{B}_i is the least component-branch at $S(D)$, and moreover $\mathfrak{R}(\mathfrak{B}_i) < K(D)$ for every non-void component $K(D)$ of $S(D)$. Hence there exists the lower void component of $S(D)$. Conversely, it can easily be proved that if the lower void component of $S(D)$ exists, then there exists the least member of \mathfrak{G}_D which is a void set. Thus we can identify the least member $G_i(D)$ of \mathfrak{G}_D which is a void set with the lower void component of D -class $S(D)$. Similarly we can prove

that we are able to identify the greatest member $G_u(D)$ of \mathfrak{G}_D which is a void set with the upper void component of $S(D)$. This completes the proof of Theorem 12.

8. In this final section, we mention criteria of two special sorts of ordered idempotent semigroups. The proofs of these theorems are easy and we omit them here.

THEOREM 13. *In order that an ordered idempotent semigroup S satisfies the strict monotone condition III' in § 1, it is necessary and sufficient that S consists of only one element.*

COROLLARY. *If an ordered semigroup S satisfies the strict monotone condition III', then it has at most one idempotent.*

THEOREM 14. *In order that an ordered idempotent semigroup S is commutative, it is necessary and sufficient that every D -class of S consists of only one element.*

COROLLARY. *In the associated semilattice S^* of an ordered idempotent commutative semigroup S , the branch order at every D -class is at most 2.*

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