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The index of coset spaces of compact Lie groups

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§ 1. Introduction.

Let G be a compact connected Lie group and H a closed connected subgroup of G. We shall denote by $r(G)$ and $r(H)$ the ranks of G and H respectively. In the present note, we shall prove that, if $r(H)<r(G)$, then the index $\tau(G/H)$ of G/H (in the sense of Thom) vanishes; and that, if $r(H)=r(G)$, then the index can be expressed as the integral of some central function on H over the group manifold H. The precise statement will be given by Theorem 1 which we shall obtain at the end of § 3.

In the latter case, Borel-Hirzebruch [1] gave a formula which expresses the index $\tau(G/H)$ in terms of roots of G and those of H. They computed actually the L-genus which, as the index theorem of Thom-Hirzebruch asserts, coincides with the index. In § 4 we evaluate the integral in Theorem 1 to derive the formula of Borel-Hirzebruch. Here we do not make use of the index theorem. Thus our result can be regarded as providing a new proof of the index theorem for the space G/H with $r(H)=r(G)$.

§ 2. The index $\tau(G/H)$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the analytic subgroup H. There exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$ and is invariant under the adjoint representation of H. We shall denote by Λ the exterior algebra of $\mathfrak{m}$ and by Λ^* the exterior algebra of the dual $\mathfrak{m}^*$ of the vector space $\mathfrak{m}$. The adjoint representation of H on $\mathfrak{m}$ extends to representations of H on Λ and on Λ^* in the standard fashion. Let us denote by Λ^H and Λ^{*H} the subalgebras of Λ and Λ^* respectively consisting of elements fixed under all operations of H. The algebra Λ^{*H} may be canonically identified with the algebra of G-invariant differential forms on G/H, and, as such, carries a differential operator d. The real cohomology ring $H^*(G/H, \boldsymbol{R})$ is then the derived ring of Λ^{*H} with respect to d.

Let e be a non zero element of Λ^n where Λ^n denotes the n-th exterior product of $\mathfrak{m}$, and $n=\dim \mathfrak{m}=\dim G/H$. Since H is compact and connected, e

is invariant under H and therefore belongs to Λ^H. Since the vector spaces Λ^H and Λ^{*H} are duals of each other, e determines uniquely an element e^* of $\Lambda^{*n} \subset \Lambda^{*H}$ such that $\langle e, e^* \rangle = 1$. The cohomology class $\bar{e}$ of e^* is also non zero. The class $\bar{e}$ determines an orientation of the manifold G/H. The element e determines an orientation of the vector space $\mathfrak{m}$ which is identified with the tangent space at the coset H to the coset space G/H. Translating this orientation by $g \in G$ on the tangent space at gH, we define an orientation of G/H, which is just the orientation determined by $\bar{e}$.

Since Λ^{*H} is a Poincaré ring with a differentiation, its index relative to e^* is equal, in virtue of Lemma 4 of [2], to the index of the derived ring $H^*(G/H, \boldsymbol{R})$ relative to $\bar{e}$, that is, to the index of the manifold G/H relative to the orientation determined by $\bar{e}$. (We refer to [2] for the notions of Poincaré ring and its index.)

The algebra Λ^H is isomorphic (but not canonically) to the algebra Λ^{*H} by an isomorphism which sends e to e^*. Thus we have proved the following

PROPOSITION 2.1. *Let an orientation of G/H be determined by a non zero element e of Λ^n as above. Then the index $\tau(G/H)$ relative to this orientation is equal to the index $\tau(\Lambda^H)$ relative to e of the Poincaré ring Λ^H.*

The index $\tau(\Lambda^H)$ is obtained as follows. If $\dim \mathfrak{m} \not\equiv 0 \bmod 4$, then $\tau(\Lambda^H)$ is zero. If $\dim \mathfrak{m} = 4k$, and if $\{x_1, \cdots, x_N\}$ is a basis of $(\Lambda^{2k})^H$ such that

$$x_i \wedge x_j = \varepsilon_i \delta_{ij} e\,, \qquad \varepsilon_i = \pm 1\,, \qquad (1 \leqq i,\ j \leqq N)$$

then $\tau(\Lambda^H) = \sum_{i=1}^{N} \varepsilon_i$.

Introduce an H-invariant inner product (,) on $\mathfrak{m}$. This inner product extends to an H-invariant inner product (,) on Λ^p, $p = 1, \cdots, n$. If $X_1 \wedge \cdots \wedge X_p$, $Y_1 \wedge \cdots \wedge Y_p \in \Lambda^p$, X_i, $Y_j \in \mathfrak{m}$, then we have by definition

$$(X_1 \wedge \cdots \wedge X_p,\ Y_1 \wedge \cdots \wedge Y_p) = \det((X_i, Y_j))\,.$$

Orient $\mathfrak{m}$ by a base $e \in \Lambda^n$ such that $(e, e) = 1$. Denote by $\omega_p\colon \Lambda^p \to \Lambda^{n-p}$ the star operation of Hodge with respect to this inner product and to e. The linear map ω_p is characterized by the formula

$$(\omega_p(x), y) = (x \wedge y, e), \qquad x \in \Lambda^p,\ y \in \Lambda^{n-p}\,.$$

Let $\{X_1, \cdots, X_n\}$ be an orthonormal basis of $\mathfrak{m}$. Then we have $X_1 \wedge \cdots \wedge X_n = \alpha e$ with $\alpha = \pm 1$. If $(i_1, \cdots, i_p, j_1, \cdots, j_{n-p})$ is a permutation of $(1, \cdots, n)$ with $i_1 < \cdots < i_p$, $j_1 < \cdots < j_{n-p}$, then ω_p is also given by the formula

$$\omega_p(X_{i_1} \wedge \cdots \wedge X_{i_p}) = \alpha \operatorname{sgn}(i_1, \cdots, i_p, j_1, \cdots, j_{n-p}) X_{j_1} \wedge \cdots \wedge X_{j_{n-p}}\,.$$

We have the following identities:

(2.2) $$\omega_{n-p} \circ \omega_p = (-1)^{p(n-p)}\,,$$

(2.3) $$(\omega_p(x), \omega_p(y)) = (x, y), \qquad x, y \in \Lambda^p\,,$$

where the right hand side of (2.2) means the scalar multiplication by $(-1)^{p(n-p)}$ on Λ^p.

Hereafter we assume that $n=\dim \mathfrak{m}$ is a multiple of 4; $n=4k$.

To abbreviate write ω for the endomorphism ω_{2k} which is an involutive orthogonal transformation by (2.2) and (2.3). Let V_+ be the eigenspace of ω with eigenvalue $+1$ and V_- be the eigenspace of ω with eigenvalue -1. The vector space Λ^{2k} decomposes into direct sum of V_+ and V_-. V_+ and V_- are orthogonal to each other. Since the inner product is H-invariant, operations of H commute with ω. In particular V_+ and V_- are H-invariant subspaces of Λ^{2k}, and we have

(2.4) $$(\Lambda^{2k})^H = V_+{}^H + V_-{}^H \qquad \text{(direct sum)},$$

where $V_+{}^H = (\Lambda^{2k})^H \cap V_+$, $V_-{}^H = (\Lambda^{2k})^H \cap V_-$.

LEMMA 2.5. *Let $\{x_i\}$ be an orthonormal basis of V_+ and $\{y_j\}$ be an orthonormal basis of V_-. Then we have*

i) $x_i \wedge x_j = \delta_{ij} e$,

ii) $y_i \wedge y_j = -\delta_{ij} e$,

iii) $x_i \wedge y_j = 0$.

In fact we have

$$\delta_{ij} = (x_i, x_j) = (\omega(x_i), x_j) = (x_i \wedge x_j, e).$$

Hence $x_i \wedge x_j = \delta_{ij} e$.

Similarly we have

$$\delta_{ij} = (y_i, y_j) = (-\omega(y_i), y_j) = -(y_i \wedge y_j, e).$$

Hence $y_i \wedge y_j = -\delta_{ij} e$.

$x_i \wedge y_j = 0$ follows from

$$0 = (x_i, y_j) = (\omega(x_i), y_j) = (x_i \wedge y_j, e).$$

The following proposition is an immediate consequence of (2.4) and Lemma 2.5.

PROPOSITION 2.6. *We have*

$$\tau(\Lambda^H) = \dim V_+{}^H - \dim V_-{}^H.$$

We shall denote by U_C the complexification of a real vector space U.

The representation of H on V_+ (respectively on V_-) extends in an obvious way to a complex representation of H on $(V_+)_C$ (respectively on $(V_-)_C$), which we call also the adjoint representation of H on $(V_+)_C$ (respectively on $(V_-)_C$). The complex vector space $(V_+{}^H)_C$ (respectively $(V_-{}^H)_C$) is then canonically identified with $(V_+)_C{}^H$ (respectively $(V_-)_C{}^H$). We have

(2.7) $$\text{complex } \dim (V_+)_C{}^H = \dim V_+{}^H, \qquad \text{complex } \dim (V_-)_C{}^H = \dim V_-{}^H.$$

Let χ_+ and χ_- be the characters of the repesentations of H on $(V_+)_C$ and $(V_-)_C$ respectively. By a formula of Weyl [3] we have

$$\text{(2.8)}\qquad \begin{aligned} \text{complex dim}\,(V_+)_C{}^H &= \int_H \chi_+\omega_H\,, \\ \text{complex dim}\,(V_-)_C{}^H &= \int_H \chi_-\omega_H\,, \end{aligned}$$

where ω_H denotes the Haar measure on H with total measure $\int_H \omega_H = 1$.

Combining (2.6), (2.7) and (2.8), we get

PROPOSITION 2.9.

$$\tau(\Lambda^H) = \int_H (\chi_+ - \chi_-)\omega_H\,.$$

§ 3. Calculation of $\chi_+ - \chi_-$.

We identify Λ_C with the exterior algebra $\tilde{\Lambda}$ over the complex vector space $\mathfrak{m}_C$, and denote by $\tilde{\omega}: \tilde{\Lambda}^{2k} \to \tilde{\Lambda}^{2k}$ the complexification of ω. The spaces $(V_+)_C$ and $(V_-)_C$ are identified with eigenspaces U_+ and U_- of $\tilde{\omega}$ with eigenvalues $+1$ and -1 respectively. We extend the inner product on Λ to a hermitian inner product on $\tilde{\Lambda}$; if $x, y \in \Lambda^p$ and a, b are complex numbers, then the product (ax, by) on $\tilde{\Lambda}^p$ is given by the formula

$$(ax, by) = a\bar{b}(x, y)\,.$$

The map $\tilde{\omega}$ is characterised by the formula

$$(\tilde{\omega}(x), y) = (x \wedge \bar{y}, e), \qquad x, y \in \tilde{\Lambda}^{2k}\,,$$

where $\bar{y}$ denotes the conjugate of y in $\tilde{\Lambda} = \Lambda_C$.

If $\{X_1, \cdots, X_{4k}\}$ is an orthonormal basis of $\mathfrak{m}_C$ with $X_1 \wedge \cdots \wedge X_{4k} = \alpha e$, $|\alpha| = 1$, and if $(i_1, \cdots, i_{2k}, j_1, \cdots, j_{2k})$ is a permulation of $(1, \cdots, 4k)$ with $i_1 < \cdots < i_{2k}$, $j_1 < \cdots < j_{2k}$, then we have

$$\text{(3.1)}\qquad \tilde{\omega}(X_{i_1} \wedge \cdots \wedge X_{i_{2k}}) = \alpha\,\text{sgn}\,(i_1, \cdots, i_{2k}, j_1, \cdots, j_{2k})\bar{X}_{j_1} \wedge \cdots \wedge \bar{X}_{j_{2k}}\,.$$

Let T be a maximal torus of H. The adjoint representation of T on $\mathfrak{m}$ decomposes $\mathfrak{m}$ into a direct sum of T-invariant subspaces $\mathfrak{m}_0$ and $\mathfrak{m}_i$, $i = 1, \cdots, n_1$, orthogonal to each other, such that $\mathfrak{m}_0$ is the largest subspace on which T acts trivially and $\dim \mathfrak{m}_i = 2$, $1 \leqq i \leqq n_1$.

Note that $n_0 = \dim \mathfrak{m}_0$ vanishes if and only if $r(H) = r(G)$, that is, if and only if T is also a maximal tours of G.

Let $\{F_1, \cdots, F_{n_0}\}$ be an orthonormal basis of $\mathfrak{m}_0$, and $\{X_i, Y_i\}$ be an orthonormal basis of $\mathfrak{m}_i$, $1 \leqq i \leqq n_1$. Then we have

$$\text{(3.2)}\qquad \begin{aligned} \text{Ad}\,(g)X_i &= \cos 2\pi\lambda_i(g)X_i - \sin 2\pi\lambda_i(g)Y_i\,, \\ \text{Ad}\,(g)Y_i &= \sin 2\pi\lambda_i(g)X_i + \cos 2\pi\lambda_i(g)Y_i\,, \end{aligned}$$

for $g\in T$, where $\lambda_i: T\to \boldsymbol{R}/\boldsymbol{Z}$ is a continuous homomorphism.

Let $\mathfrak{t}$ be the Lie algebra of T. To a continuous homomorphism $\lambda: T\to\boldsymbol{R}/\boldsymbol{Z}$ there corresponds a unique integral linear form α on $\mathfrak{t}$ such that

$$e^{2\pi\sqrt{-1}\,\lambda(\exp X)}=e^{2\pi\sqrt{-1}\,\alpha(X)},\qquad X\in\mathfrak{t}\,.$$

We write $e^{\alpha}(g)=e^{2\pi\sqrt{-1}\,\lambda(g)}$. If α and β are integral forms on $\mathfrak{t}$, then $e^{\alpha+\beta}(g)=e^{\alpha}(g)e^{\beta}(g)$ and $e^{-\alpha}(g)=\overline{e^{\alpha}(g)}$. Let α_i be the form corresponding to λ_i. If $r(H)=r(G)$, then the linear forms $\pm\alpha_i$ are the roots of G complementary to those of H [**1**].

We put

$$E_i=\frac{1}{\sqrt{2}}(X_i+\sqrt{-1}\;Y_i)\,,\qquad i=1,\cdots,n_1\,.$$

E_i and $\bar{E}_i=\frac{1}{\sqrt{2}}(X_i-\sqrt{-1}\;Y_i)$ form a basis of $\mathfrak{m}_{iC}$. We have by (3.2) that

$$\text{(3.3)}\qquad \begin{aligned}\operatorname{Ad}(g)E_i&=e^{2\pi\sqrt{-1}\,\lambda_i(g)}E_i=e^{\alpha_i}(g)E_i\,,\\ \operatorname{Ad}(g)\bar{E}_i&=e^{-2\pi\sqrt{-1}\,\lambda_i(g)}\bar{E}_i=e^{-\alpha_i}(g)\bar{E}_i\,,\qquad g\in T\,.\end{aligned}$$

We orient $\mathfrak{m}$ by $e\in\Lambda^{4k}$ defined by

$$\begin{aligned}e&=F_1\wedge\cdots\wedge F_{n_0}\wedge X_1\wedge Y_1\wedge\cdots\wedge X_{n_1}\wedge Y_{n_1}\\ &=(\sqrt{-1})^{n_1}(-1)^{\frac{1}{2}n_1(n_1-1)}F_1\wedge\cdots\wedge F_{n_0}\wedge E_1\wedge\cdots\wedge E_{n_1}\wedge\bar{E}_1\wedge\cdots\wedge\bar{E}_{n_1}\,.\end{aligned}$$

If $r(H)=r(G)$, then we have $n_0=0$, $n_1=n/2=2k$, so that

$$\text{(3.4)}\qquad e=E_1\wedge\cdots\wedge E_{2k}\wedge\bar{E}_1\wedge\cdots\wedge\bar{E}_{2k}\,.$$

Consider the basis of $\tilde{\Lambda}^{2k}$ consisting of elements of the form

$$F_{i_1}\wedge\cdots\wedge F_{i_r}\wedge E_{j_1}\wedge\cdots\wedge E_{j_s}\wedge\bar{E}_{k_1}\wedge\cdots\wedge\bar{E}_{k_t}\,,\qquad (r+s+t=2k)\,,$$

with $i_1<\cdots<i_r$, $j_1<\cdots<j_s$ and $k_1<\cdots<k_t$. We put

$$\begin{aligned}&\{i'_1,\cdots,i'_{n_0-r}\}=\{1,\cdots,n_0\}-\{i_1,\cdots,i_r\}\,,\qquad i'_1<\cdots<i'_{n_0-r}\,;\\ &\{j'_1,\cdots,j'_{n_1-s}\}=\{1,\cdots,n_1\}-\{j_1,\cdots,j_s\}\,,\qquad j'_1<\cdots<j'_{n_1-s}\,;\\ &\{k'_1,\cdots,k'_{n_1-t}\}=\{1,\cdots,n_1\}-\{k_1,\cdots,k_t\}\,,\qquad k'_1<\cdots<k'_{n_1-t}\,.\end{aligned}$$

We put also

$$\begin{aligned}&\{\mu_1,\cdots,\mu_c\}=\{j_1,\cdots,j_s\}\cap\{k_1,\cdots,k_t\}\,,\qquad \mu_1<\cdots<\mu_c\,;\\ &\{\nu_1,\cdots,\nu_d\}=\{j'_1,\cdots,j'_{n_1-s}\}\cap\{k'_1,\cdots,k'_{n_1-t}\}\,,\qquad \nu_1<\cdots<\nu_d\,;\\ &\{\bar{j}_1,\cdots,\bar{j}_{s-c}\}=\{j_1,\cdots,j_s\}-\{\mu_1,\cdots,\mu_c\}\,,\qquad \bar{j}_1<\cdots<\bar{j}_{s-c}\,;\\ &\{\bar{k}_1,\cdots,\bar{k}_{t-c}\}=\{k_1,\cdots,k_t\}-\{\mu_1,\cdots,\mu_c\}\,,\qquad \bar{k}_1<\cdots<\bar{k}_{t-c}\,.\end{aligned}$$

Note that we have

$$\{k'_1,\cdots,k'_{n_1-t}\}-\{\nu_1,\cdots,\nu_d\}=\{\bar{j}_1,\cdots,\bar{j}_{s-c}\}\,;$$

$$\{j'_1, \cdots, j'_{n_1-s}\} - \{\nu_1, \cdots, \nu_d\} = \{\bar{k}_1, \cdots, \bar{k}_{t-c}\};$$

and $n_1 = s + t - c + d$.

Using (3.1), it is easily checked that that effect of $\tilde{\omega}$ on elements of the basis above is given by

$$(3.5) \qquad \tilde{\omega}(F_{i_1} \wedge \cdots \wedge F_{i_r} \wedge E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \bar{E}_{k_t}) = \varepsilon F_{i'_1} \wedge \cdots \wedge F_{i'_{n_0-r}} \wedge E_{k'_1} \wedge \cdots \wedge E_{k'_{n_1-t}} \wedge \bar{E}_{j'_1} \wedge \cdots \wedge \bar{E}_{j'_{n_1-s}},$$

where $\varepsilon = (\sqrt{-1})^{n_1}(-1)^{(n_0-r)(s+t)+(n_1-s)n_1+\frac{1}{2}n_1(n_1-1)}\varepsilon(i)\varepsilon(j)\varepsilon(k)$, $\varepsilon(i)$, $\varepsilon(j)$ and $\varepsilon(k)$ denoting the signs of permutations $(i_1, \cdots, i_r, i'_1, \cdots, i'_{n_0-r})$, $(j_1, \cdots, j_s, j'_1, \cdots, j'_{n_1-s})$ and $(k_1, \cdots, k_t, k'_1, \cdots, k'_{n_1-t})$ respectively.

But we have

$$(3.6) \qquad F_{i_1} \wedge \cdots \wedge F_{i_r} \wedge E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_1 \wedge \cdots \wedge \bar{E}_{k_t} = \pm F_{i_1} \wedge \cdots \wedge F_{i_r} \wedge E_{\bar{j}_1} \wedge \cdots \wedge E_{\bar{j}_{s-c}} \wedge \bar{E}_{\bar{k}_1} \wedge \cdots \wedge \bar{E}_{\bar{k}_{t-c}} \wedge E_{\mu_1} \wedge \bar{E}_{\mu_1} \wedge \cdots \wedge E_{\mu_c} \wedge \bar{E}_{\mu_c},$$

and

$$(3.7) \qquad F_{i'_1} \wedge \cdots \wedge F_{i'_{n_0-r}} \wedge E_{k'_1} \wedge \cdots \wedge E_{k'_{n_1-t}} \wedge \bar{E}_{j'_1} \wedge \cdots \wedge \bar{E}_{j'_{n_1-s}} = \pm F_{i'_1} \wedge \cdots \wedge F_{i'_{n_0-r}} \wedge E_{\bar{j}_1} \wedge \cdots \wedge E_{\bar{j}_{s-c}} \wedge \bar{E}_{\bar{k}_1} \wedge \cdots \wedge \bar{E}_{\bar{k}_{t-c}} \wedge E_{\nu_1} \wedge \bar{E}_{\nu_1} \wedge \cdots \wedge E_{\nu_d} \wedge \bar{E}_{\nu_d}.$$

It follows from (3.3) and (3.6) that, for the element

$$x = F_{i_1} \wedge \cdots \wedge F_{i_r} \wedge E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \wedge \bar{E}_{k_t},$$

we have

$$\mathrm{Ad}(g)x = e^{\gamma}(g)x, \qquad g \in T,$$

where

$$(3.8) \qquad \gamma = \alpha_{\bar{j}_1} + \cdots + \alpha_{\bar{j}_{s-c}} - \alpha_{\bar{k}} - \cdots - \alpha_{\bar{k}_{t-c}}.$$

It follows also from (3.3), (3.5) and (3.7) that, for the same x, we have

$$\mathrm{Ad}(g)\tilde{\omega}(x) = e^{\gamma}(g)\tilde{\omega}(x), \qquad g \in T.$$

We note also that x and $\tilde{\omega}(x)$ is linearly dependent if and only if $n_0 = 0$ (i.e., $r(H) = r(G)$) and $\{j_1, \cdots, j_s\} \cap \{k_1, \cdots, k_t\} = \phi$. If $n_0 = 0$ and $\{j_1, \cdots, j_s\} \cap \{k_1, \cdots, k_t\} = \phi$, then we have, by (3.5) and (3.7), that

$$\tilde{\omega}(E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \wedge \bar{E}_{k_t}) = \pm E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \wedge \bar{E}_{k_t}.$$

It is easily checked that the sign ± 1 is given by $(-1)^s$. Thus, in this case,

$$\tilde{\omega}(E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \wedge \bar{E}_{k_t}) = (-1)^s E_{j_1} \wedge \cdots \wedge E_{j_s} \wedge \bar{E}_{k_1} \wedge \cdots \wedge \bar{E}_{k_t}.$$

We have thereby proved the following facts.

Suppose that $r(H) = r(G)$. Then, there exists a set of linearly independent elements $x_1, \cdots, x_N$ of $\tilde{\Lambda}^{2k}$ such that

i) U_+ has a basis consisting of elements $x_i + \tilde{\omega}(x_i)$, $1 \leqq i \leqq N$, together with

$E_{j_1}\wedge\cdots\wedge E_{j_s}\wedge\bar{E}_{k_1}\wedge\cdots\wedge\bar{E}_{k_t}$, $\{j_1,\cdots,j_s\}\cap\{k_1,\cdots,k_t\}=\phi$, s being even;

ii) U_- has a basis consisting of elements $x_i-\tilde{\omega}(x_i)$, $1\leqq i\leqq N$, together with $E_{j_1}\wedge\cdots\wedge E_{j_s}\wedge\bar{E}_{k_1}\wedge\cdots\wedge\bar{E}_{k_t}$, $\{j_1,\cdots,j_s\}\cap\{k_1,\cdots,k_t\}=\phi$, s being odd;

iii) for $g\in T$, we have

$$\mathrm{Ad}(g)(x_i+\tilde{\omega}(x_i))=e^{\gamma_i}(g)(x_i+\tilde{\omega}(x_i)),$$
$$\mathrm{Ad}(g)(x_i-\tilde{\omega}(x_i))=e^{\gamma_i}(g)(x_i-\tilde{\omega}(x_i)),$$

where γ_i is a linear form on the Lie algebra of T.

Suppose that $r(H)<r(G)$. Then, there exists a set of linearly independent elements $x_1,\cdots,x_N$ of $\tilde{\Lambda}^{2k}$ such that

i) U_+ has a basis consisting of elements $x_i+\tilde{\omega}(x_i)$, $1\leqq i\leqq N$,

ii) U_- has a basis consisting of elements $x_i-\tilde{\omega}(x_i)$, $1\leqq i\leqq N$,

iii) $\mathrm{Ad}(g)(x_i+\tilde{\omega}(x_i))=e^{\gamma_i}(g)(x_i+\tilde{\omega}(x_i))$,
$\mathrm{Ad}(g)(x_i-\tilde{\omega}(x_i))=e^{\gamma_i}(g)(x_i-\tilde{\omega}(x_i))$, $\quad g\in T$.

Taking the trace of $\mathrm{Ad}(g)$ with respect to the basis given above of U_+ and U_- respectively we get the values of χ_+ and χ_- on g. By subtracting and using (3.8) we find the following formulas.

If $r(H)=r(G)$, then

$$(\chi_+-\chi_-)(g)=\sum(-1)^s e^{\alpha_{j_1}+\cdots+\alpha_{j_s}-\alpha_{k_1}-\cdots-\alpha_{k_t}}(g),\qquad g\in T,$$

where the summation is extended over all permutations $(j_1,\cdots,j_s,k_1,\cdots,k_t)$ of $(1,\cdots,2k)$ with $j_1<\cdots<j_s$, $k_1<\cdots<k_t$ and $0\leqq s\leqq 2k$. Or

$$(\chi_+-\chi_-)(g)=\prod_{1\leqq i\leqq 2k}(e^{\alpha_i}-e^{-\alpha_i})(g),\qquad g\in T. \tag{3.9}$$

If $r(H)<r(G)$, then

$$(\chi_+-\chi_-)(g)=0. \tag{3.10}$$

Since a central function on H is determined completely by its values on T, formulas (3.9) and (3.10) determine $\chi_+-\chi_-$.

Combining with (2.1) and (2.9), we have

THEOREM 1. *If $r(H)=r(G)$, then we have*

$$\tau(G/H)=\int_H\prod_{1\leqq i\leqq 2k}(e^{\alpha_i}-e^{-\alpha_i})\omega_H,$$

where $\prod(e^{\alpha_i}-e^{-\alpha_i})$ denotes the central function on H whose value on $g\in T$ is given by $\prod(e^{\alpha_i}-e^{-\alpha_i})(g)$. If $r(H)<r(G)$, then we have

$$\tau(G/H)=0.$$

REMARK. Let H be a compact connected Lie group and ρ a real representation of H on a real vector space $\mathfrak{m}$ of dimension $2k'$. We may suppose that $\mathfrak{m}$ is endowed with an H-invariant inner product. Let $\omega_p: \Lambda^p(\mathfrak{m})\to\Lambda^{2k'-p}(\mathfrak{m})$ be the Hodge operation with respect to the inner product and to an orientation of $\mathfrak{m}$. Let $\tilde{\omega}: \Lambda^{k'}(\mathfrak{m}_C)\to\Lambda^{k'}(\mathfrak{m}_C)$ be the complexification of $\omega_{k'}$, multiplied by $\sqrt{-1}$

if k' is odd. We have a direct sum decomposition $\Lambda^{k'}(\mathfrak{m}_C)=V_+ + V_-$ such that $\tilde{\omega}\,|\,V_+=1$ and $\tilde{\omega}\,|\,V_-=-1$. We denote by χ_+ (respectively by χ_-) the character of the representation λ_+ (respectively λ_-) of H on V_+ (respectively on V_-). Let $\pm\alpha_1, \cdots, \pm\alpha_{k'}$ be the weights of the complexification of the representation ρ (possibly some of α_i may be zero). From the argument in this section it follows that

$$\chi_+ - \chi_- = \pm \prod_{1\leqq i\leqq k'} (e^{\alpha_i} - e^{-\alpha_i}).$$

To get this fact more quickly we may proceed as follows. Via inner product on $\mathfrak{m}$ we may regard ρ as a homomorphism $H\to SO(2k')$, so that we have only to prove (3.11) when H coincides with $SO(2k')$ and ρ is the natural representation of $SO(2k')$ on $R^{2k'}$. But in this case (3.11) holds since $\lambda_+ - \lambda_- = \pm(\Delta_+\otimes\Delta_+ - \Delta_-\otimes\Delta_-)$ where Δ_+ and Δ_- are half spinor representations of Spin$(2k')$ (cf. M. Atiyah-F. Hirzebruch, Bull. Soc. math. France, **87** (1959), 383-396; § 4.1, Formula (5)).

§ 4. Formula of Borel-Hirzebruch.

In this section we assume $r(H)=r(G)$, $\dim G/H = 2n_1$ not being assumed to be a multiple of 4.

We will denote by Σ_H (respectively by Σ_G) the set of all roots of H (respectively of G) with respect to T; $\Sigma_H \subset \Sigma_G$. Elements of $\Sigma_G - \Sigma_H$ are roots of G complementary to those of H. A subset Θ of Σ_H (respectively of Σ_G) is called a system of positve roots of H (respectively of G) if there exists an ordering of the Lie algebra of T such that Θ consists of all roots of H (respectively of G) which are positive relative to the ordering [**1**]. We denote by $\mathfrak{P}_H$ (respectively by $\mathfrak{P}_G$) the set of all systems of positive roots of H (respectively of G).

Let $\Psi=\{\alpha_i\}$ be a subset of $\Sigma_G - \Sigma_H$ which contains for each complementary root α exactly one of the roots α, $-\alpha$. Let $\Theta \in \mathfrak{P}_H$. According to Borel-Hirzebruch we define $k^p(G/H; \Psi, \Theta)$ as the number of those elements Φ of $\mathfrak{P}_G$ such that 1) $\Theta \subset \Phi$ and 2) $\Phi \cap \Psi$ consists of $(n_1 - p)$ roots (or equivalently $\Phi \cap (-\Psi)$ consists of p roots).

THEOREM 2.

$$\int_H \prod_{\alpha_i\in\Psi} (e^{-\alpha_i} - e^{+\alpha_i})\omega_H = \sum_{0\leqq p\leqq n_1} (-1)^p k^p(G/H;\ \Psi, \Theta).$$

We denote by W_H and W_G the Weyl groups of H and G with respect to T. O_H and O_G denote the orders of W_H and W_G respectively.

Theorem 2 is a special case of

THEOREM 2′.

$$\int_H \sum_{\sigma\in W_H} \sigma\Big(\prod_{\alpha_i\in\Psi} (1+ye^{-\alpha_i})(1-e^{\alpha_i})\Big)\omega_H$$

$$=\sum_{0\leqq p\leqq n_1}(-y)^p\sum_{\Theta'\in\mathfrak{P}_H}k^p(G/H;\Psi,\Theta'),$$

where y is an independent variable.

Let Θ_1 and Θ_2 be any two systems of positive roots of H. There exists a unique element τ of W_H such that $\tau(\Theta_1)=\Theta_2$. τ transforms the set of systems of positive roots of G which contain Θ_1 onto the set of systems of positive roots of G which contain Θ_2 in one to one fashion. The transformation τ is induced by an automorphism $g\to hgh^{-1}$ of T where h is an appropriate element of H. Then $\mathrm{Ad}(h)$ permutes among themselves the vector spaces $\mathfrak{m}_i$, $1\leqq i\leqq n_1$. If $\tau(\alpha_i)=\varepsilon_i\alpha_j$, then the determinant of $\mathrm{Ad}(h)$ considered as an automorphism of $\mathfrak{m}$ is equal to $\prod\varepsilon_i$. But h is an element of a connected group H which operates on $\mathfrak{m}$ via adjoint operation. Therefore $\prod\varepsilon_i$ must be equal to 1.

Let $\Psi'\cup\Theta_1$ be a system of positive roots of G which contains p roots of $-\Psi$. Suppose that the transformed system $\tau(\Psi'\cup\Theta_1)=\tau(\Psi')\cup\Theta_2$ contains q roots of $-\Psi$, then we have $(-1)^{p-q}=\prod\varepsilon_i=1$.

It follows that

$$\sum_{0\leqq p\leqq n_1}(-1)^pk^p(G/H;\Psi,\Theta_1)=\sum_{0\leqq p\leqq n_1}(-1)^pk^p(G/H;\Psi,\Theta_2).$$

Hence Theorem 2′ reduces to Theorem 2 for $y=1$.

PROOP OF THEOREM 2′. Fix an ordering of the Lie algebra of T and let $\beta_1,\cdots,\beta_m$ (respectively $\beta_1,\cdots,\beta_m,\delta_1,\cdots,\delta_{n_1}$) be the system of positive roots of H (respectively of G) with respect to the ordering. Let Q_H and Q_G be the operators defined by

$$Q_H=\sum_{\sigma\in W_H}(\mathrm{sgn}\,\sigma)\cdot\sigma,\qquad Q_G=\sum_{\sigma\in W_G}(\mathrm{sgn}\,\sigma)\cdot\sigma.$$

Set $\Delta_H=\prod_{1\leqq i\leqq m}(e^{\frac{1}{2}\beta_i}-e^{-\frac{1}{2}\beta_i})$ and $\Delta_G=\prod_{1\leqq i\leqq m}(e^{\frac{1}{2}\beta_i}-e^{-\frac{1}{2}\beta_i})\prod_{1\leqq i\leqq n_1}(e^{\frac{1}{2}\delta_i}-e^{-\frac{1}{2}\delta_i})$. We have [3]

$$\Delta_H=Q_H(e^{\frac{1}{2}(\beta_1+\cdots+\beta_m)}),\qquad \Delta_G=Q_G(e^{\frac{1}{2}(\beta_1+\cdots+\beta_m+\delta_1+\cdots+\delta_{n_1})}).$$

Note that we have

$$\bar{\Delta}_H=\prod(e^{-\frac{1}{2}\beta_i}-e^{+\frac{1}{2}\beta_i}),$$

$$\bar{\Delta}_G=\prod(e^{-\frac{1}{2}\beta_i}-e^{+\frac{1}{2}\beta_i})\prod(e^{-\frac{1}{2}\delta_i}-e^{+\frac{1}{2}\delta_i}),$$

where $^-$ denotes the conjugate operation.

Let ω_T be the Haar measure on T with $\int_T\omega_T=1$.

By a formula of Weyl [3] we have

$$I=\int_H\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i}))\omega_H$$

$$=\frac{1}{O_H}\int_T\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i}))\Delta_H\bar{\Delta}_H\omega_T.$$

Hence

$$I=\frac{1}{O_G O_H}\int_T \sum_{\tau\in W_G}\tau\{\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i})\Delta_H\bar{\Delta}_H)\}\omega_T\,. \tag{4.1}$$

Since the coefficient of y^p in $\tau\sigma\{\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i})\Delta_H\bar{\Delta}_H\}$ is divisible by $\bar{\Delta}_G$, and since

$$\sum_{\tau\in W_G}\tau\{\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i})\Delta_H\bar{\Delta}_H)\}/\bar{\Delta}_G$$

is W_G-antisymmetric, we may write [3]

$$\sum_{\tau\in W_G}\tau\{\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i})\Delta_H\bar{\Delta}_H)\}=\sum_{\Lambda,p}a_{\Lambda,p}y^p Q_G e^{\Lambda+\rho}\bar{\Delta}_G\,, \tag{4.2}$$

where $\rho=\frac{1}{2}(\delta_1+\cdots+\delta_{n_1}+\beta_1+\cdots+\beta_m)$ and the sum is extended over a finite number of dominant integral forms Λ.

Compare in (4.2) the coefficients of y^p. At the left hand side, the coefficient of y^p is

$$\sum_{\tau\in W_G}\tau\{\sum_{\sigma\in W_H}\sigma(\sum e^{-\alpha_{i_1}-\cdots-\alpha_{i_p}}(1-e^{\alpha_1})\cdots(1-e^{\alpha_{n_1}})\Delta_H\bar{\Delta}_H)\}\,.$$

The highest term in this expression is

$$(-1)^m(-1)^{n_1-p}b_p e^{\delta_1+\cdots+\delta_{n_1}+\beta_1+\cdots+\beta_m} \tag{4.3}$$

where b_p is equal to the number of $(\tau,\sigma)\in W_G\times W_H$ such that there exists a system $\Phi=\{-\alpha_{i_1},\cdots,-\alpha_{i_p},\alpha_{j_1},\cdots,\alpha_{j_{n_1-p}},\varepsilon_1\beta_1,\cdots,\varepsilon_m\beta_m\}$ $(\varepsilon_i=\pm1)$ of positive roots of G such that $\tau\sigma$ transforms Φ in $\{\delta_1,\cdots,\delta_{n_1},\beta_1,\cdots,\beta_m\}$. Now if $\Phi\in\mathfrak{P}_G$ then there is a unique element τ' of W_G such that $\tau'\Phi=\{\delta_1,\cdots,\delta_{n_1},\beta_1,\cdots,\beta_m\}$. Therefore the number of elements (τ,σ) of $W_G\times W_H$ such that $\tau\sigma$ sends Φ to $\{\delta_1,\cdots,\delta_{n_1},\beta_1,\cdots,\beta_m\}$ is equal to O_H. Thus

$$b_p=O_H\sum_{\Theta'\in\mathfrak{P}_H}k^p(G/H;\Psi,\Theta')\,. \tag{4.4}$$

On the other hand, at the right hand side of (4.2), the highest term in the coefficient of y^p is

$$(-1)^{m+n_1}a_{\Lambda_p,p}e^{\Lambda_p+\rho}e^{\rho}=(-1)^{m+n_1}a_{\Lambda_p,p}e^{\Lambda_p+\delta_1+\cdots+\delta_{n_1}+\beta_1+\cdots+\beta_m} \tag{4.5}$$

where Λ_p is the highest form among Λ's for which $a_{\Lambda,p}\neq 0$.

Since (4.3) and (4.5) must be equal, we have $\Lambda_p=0$,

$$a_{\Lambda_p,p}=(-1)^p b_p=(-1)^p O_H\cdot\sum_{\Theta'\in\mathfrak{P}_H}k^p(G/H;\Psi,\Theta')\text{ and } a_{\Lambda,p}=0\quad\text{for}\quad \Lambda\neq 0\,.$$

Therefore from (4.2) we get the following formula

$$\begin{aligned}\sum_{\tau\in W_G}\tau\{\sum_{\sigma\in W_H}\sigma(\prod(1+ye^{-\alpha_i})(1-e^{\alpha_i})\Delta_H\bar{\Delta}_H)\}\\ =O_H\cdot\sum_p(-y)^p\sum_{\Theta'\in\mathfrak{P}_H}k^p(G/H;\Psi,\Theta')\Delta_G\bar{\Delta}_G\,.\end{aligned} \tag{4.6}$$

Consequently we have

$$I=\frac{1}{O_G}\int_T \Delta_G\bar{\Delta}_G\omega_T\cdot\sum_p(-y)^p\sum_{\Theta'}k^p(G/H;\Psi,\Theta').$$

Since $\frac{1}{O_G}\int_T \Delta_G\bar{\Delta}_G\omega_T=1$ by Wely's formula, we have

$$I=\sum_p(-y)^p\sum_{\Theta'\in\mathfrak{P}_H}k^p(G/H;\Psi,\Theta').$$

REMARK. If we assume that G/H admits an invariant almost complex structure and that Ψ is the set of roots of an invariant almost complex structure on G/H [1], then we have

$$k^p(G/H;\Psi,\Theta_1)=k^p(G/H;\Psi,\Theta_2)$$

for any systems Θ_1, Θ_2 of positive roots of H. Thus, under the above assumption, Theorem 2′ reduces to the formula

$$\frac{1}{O_H}\int_H\sum_{\sigma\in W_H}\sigma(\prod_{\alpha_i\in\Psi}(1+ye^{-\alpha_i})(1-e^{\alpha_i}))\omega_H=\sum_{0\leqq p\leqq n_1}(-y)^p k^p(G/H;\Psi,\Theta).$$

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