

Remarks on differentiable structures on spheres

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J. Milnor [2] defined the invariant λ' for compact unbounded oriented differentiable $(4k-1)$ -manifolds which are homotopy spheres and boundaries of π -manifolds at the same time, and proved that the invariant λ' characterizes the J -equivalence classes of these $(4k-1)$ -manifolds for $k > 1$. Recently S. Smale [3] has shown that a compact unbounded (oriented) differentiable n -manifold ($n \geq 5$) having the homotopy type of S^n is homeomorphic to S^n and that two such manifolds belonging to the same J -equivalence class are diffeomorphic to each other if $n \neq 6$. Hence it turns out that the invariant λ' characterizes differentiable structures on S^{4k-1} which bound π -manifolds for $k > 1$.

In this note we shall compute the invariant λ' of $B_{m,1}^7$ (S^3 bundles over S^4 , see [4]) and show that every differentiable structure on S^7 can be expressed as a connected sum of $B_{m,1}^7$. We shall obtain also a similar result on S^{15} . Furthermore we shall show that $\bar{B}_{m,1}^8 \cup_i D^3$ such that $m(m+1) \equiv 0 \pmod{56}$ are 3-connected compact unbounded differentiable 8-manifolds with the 4th Betti number 1 and differentiable 8-manifolds of this type are exhausted by them, where $B_{m,1}^8$ are 4-cell bundles over S^4 ([4]). This will reveal that Pontrjagin numbers are not homotopy type invariants.

Notations and terminologies of this note are the same as in the previous paper [4]. We shall use them without a special reference.

1. The invariant λ' of $B_{m,1}^7$.

In the following $M_1^{n-1} \# M_2^{n-1}$ will denote the connected sum of two compact connected unbounded oriented differentiable $(n-1)$ -manifolds M_1^{n-1} and M_2^{n-1} (Milnor [2]). Let W_1^n and W_2^n be two compact connected oriented differentiable n -manifolds with non-vacuous boundaries; let $f_1: D^{n-1} \rightarrow \partial W_1^n$ be an orientation-preserving differentiable imbedding and $f_2: D^{n-1} \rightarrow \partial W_2^n$ be an orientation-reversing differentiable imbedding. Then $W_1^n + W_2^n$ denotes the compact connected oriented differentiable n -manifold with boundary obtained from the disjoint union of W_1^n and W_2^n by identifying $f_1(x)$ with $f_2(x)$ ($x \in D^{n-1}$), making use of the device of "straightening the angle".

We choose an orientation of $B_{m,1}^7$ (resp. $B_{m,1}^{15}$) and that of $\bar{B}_{m,1}^8$ (resp. $\bar{B}_{m,1}^{16}$)

in such a way that they are consistent and

$$(\alpha_4 \cup \alpha_4)[\bar{B}_{m,1}^8, B_{m,1}^7] = 1$$

(resp. $(\alpha_8 \cup \alpha_8)[\bar{B}_{m,1}^{16}, B_{m,1}^{15}] = 1$).

It is known that any differentiable structure on S^7 is the boundary of a π -manifold (Milnor [2, §6]). Let M_0^7 be the compact connected unbounded oriented differentiable 7-manifold which is homeomorphic to S^7 such that $\lambda'(M_0^7) = 1$, and let W_0^8 be the compact connected parallelizable oriented differentiable 8-manifold with the boundary $\partial W_0^8 = M_0^7$ such that $I(W_0^8) = 8$ (Milnor [2, §4]).

Suppose that $B_{m,1}^7$ is diffeomorphic to $M_0^7 \# M_0^7 \# \dots \# M_0^7$ (s -fold connected sum of M_0^7). Let $M^8 = \bar{B}_{m,1}^8 \cup ((-W_0^8) + (-W_0^8) + \dots + (-W_0^8))$ (s -fold sum of $-W_0^8$) be the compact connected unbounded oriented differentiable 8-manifold obtained from the disjoint union of $\bar{B}_{m,1}^8$ and $(-W_0^8) + (-W_0^8) + \dots + (-W_0^8)$ identifying $\partial \bar{B}_{m,1}^8 = B_{m,1}^7$ with $-\partial((-W_0^8) + (-W_0^8) + \dots + (-W_0^8)) = M_0^7 \# M_0^7 \# \dots \# M_0^7$ by the diffeomorphism.

Index theorem $I(M^8) = \frac{1}{45}(7p_2(M^8) - p_1^2(M^8))[M^8]$ yields

$$7p_2(M^8)[M^8] = 45(1 - 8s) + 4(2m + 1)^2. \tag{*}$$

Integrality of \hat{A} -genus $\hat{A}(M^8) = \frac{1}{2^7 \cdot 45}(-4p_2(M^8) + 7p_1^2(M^8))[M^8]$ implies

$$p_2(M^8)[M^8] \equiv 7(2m + 1)^2 \pmod{2^5 \cdot 45}. \tag{**}$$

By (*) and (**), we have

$$m(m + 1) \equiv -2s \pmod{8}.$$

Furthermore (*) implies

$$m(m + 1) \equiv -2s \pmod{7}.$$

Since there exist precisely 28 distinct differentiable structures on S^7 which form an abelian group under the connected sum (Smale [3]), we obtain therefore the following theorem.

THEOREM 1. *The invariant λ' of $B_{m,1}^7$ is equal to $-\frac{m(m+1)}{2}$.*

For example M_0^7 is diffeomorphic to $B_{10,1}^7$.

The following theorem is an immediate consequence of Theorem 1.

THEOREM 2. *$B_{m,1}^7$ and $B_{m',1}^7$ are diffeomorphic if and only if*

$$m(m + 1) \equiv m'(m' + 1) \pmod{56}.$$

In particular $B_{m,1}^7$ is diffeomorphic to the standard S^7 if and only if

$$m(m + 1) \equiv 0 \pmod{56}.$$

Theorem 1 also implies

THEOREM 3. *Every differentiable structures on S^7 can be expressed by means of connected sums of $B_{m,1}^7$.*

The following theorem follows from Theorem 3.

THEOREM 4. *For any C^∞ differentiable structure on S^7 , there exists a non-degenerate C^∞ function having one maximum, one minimum, and no other critical point.*

Now we consider differentiable structures on S^{15} . Since $\pi_{15+q}(S^q) \approx Z_2 + Z_{480}$ for large q , the order of the image of J -homomorphism $J_{15}: \pi_{15}(SO(q)) \rightarrow \pi_{15+q}(S^q)$ is equal to 480 and the greatest common divisor I_4 of $I(M)$ where M ranges over all almost parallelizable compact unbounded differentiable 16-manifolds is equal to 8×8128 (Milnor [2; Lemma 3.5]). Hence there exist precisely 8128 distinct differentiable structures on S^{15} which bound π -manifolds. Therefore by a similar argument as in the case of differentiable structures on S^7 , we obtain the following theorems.

THEOREM 5. *If $B_{m,1}^{15}$ bounds a π -manifold, the invariant λ' of $B_{m,1}^{15}$ is equal to $-\frac{m(m+1)}{2}$.*

THEOREM 6. *Suppose that both $B_{m,1}^{15}$ and $B_{m',1}^{15}$ bound π -manifolds. Then they are diffeomorphic if and only if*

$$m(m+1) \equiv m'(m'+1) \pmod{16256}.$$

In particular $B_{m,1}^{15}$ is diffeomorphic to the standard S^{15} if and only if it bounds a π -manifold and

$$m(m+1) \equiv 0 \pmod{16256}.$$

Since cokernel of J_{15} is Z_2 , $B_{m,1}^{15} \# B_{m,1}^{15}$ bounds a π -manifold (Milnor [2; Theorem 6.7]), and its invariant λ' is definable. We have

THEOREM 7. *The invariant λ' of $B_{m,1}^{15} \# B_{m,1}^{15}$ is equal to $-m(m+1)$.*

The proof is similar to that of Theorem 5.

For example $M_0^{15} \# M_0^{15}$ is diffeomorphic to $B_{1882,1}^{15} \# B_{1882,1}^{15}$.

Theorem 7 implies

THEOREM 8. *Every differentiable structure on S^{15} bounding a π -manifold for which the invariant λ' takes on even value can be expressed by a connected sum of $B_{m,1}^{15}$.*

2. 3-connected compact unbounded differentiable 8-manifolds with the 4 th Betti number 1.

Combining Theorem 2 and a result of the previous paper [4; Theorem 1], we have the following theorem.

THEOREM 9. *If $m(m+1) \equiv 0 \pmod{56}$, then $\bar{B}_{m,1}^8 \cup_i D^8$ is a 3-connected compact unbounded differentiable 8-manifold with the 4 th Betti number 1, and every*

such differentiable 8-manifold is diffeomorphic to $\bar{B}_{m,1}^8 \cup_i D^8$ with m satisfying $m(m+1) \equiv 0 \pmod{56}$.

Since the Euler-Poincaré characteristic of $\bar{B}_{m,1}^8 \cup_i D^8$ is 3, these manifolds cannot carry any (weak) almost complex structure (Hirzebruch [1]).

Theorem 9 yields

THEOREM 10. *Pontrjagin numbers are not homotopy type invariants.*

In fact, for example, $\bar{B}_{0,1}^8 \cup_i D^8$ and $\bar{B}_{48,1}^8 \cup_i D^8$ have the same homotopy type and their Pontrjagin numbers are given as follows ([4; Section 1]):

$$\begin{aligned} p_1^2(\bar{B}_{0,1}^8 \cup_i D^8)[\bar{B}_{0,1}^8 \cup_i D^8] &= 4, \\ p_2(\bar{B}_{0,1}^8 \cup_i D^8)[\bar{B}_{0,1}^8 \cup_i D^8] &= 7, \\ p_1^2(\bar{B}_{48,1}^8 \cup_i D^8)[\bar{B}_{48,1}^8 \cup_i D^8] &= 37636, \\ p_2(\bar{B}_{48,1}^8 \cup_i D^8)[\bar{B}_{48,1}^8 \cup_i D^8] &= 5383. \end{aligned}$$

This shows that L -genus (index theorem) is essentially the unique linear combination of Pontrjagin numbers which has the homotopy type invariance property. For example \hat{A} -genus is not homotopy type invariant.

Since $\bar{B}_{0,1}^8 \cup_i D^8$ is homeomorphic to the quaternion projective plane, also the following follows from Theorem 9.

THEOREM 11. *There exist infinitely many compact unbounded differentiable 8-manifolds having the same homotopy type as the quaternion projective plane which are not diffeomorphic to each other.*

Making use of this result we can construct compact unbounded differentiable 12-manifolds having the homotopy type of the quaternion projective space whose Pontrjagin numbers are different each other (Tamura [5]).

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