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On automorphisms of conformally flat K-spaces

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Introduction. It is known that in a compact almost-Kählerian space an infinitesimal isometry is almost-analytic and hence an automorphism.[1] On the other hand, in a compact K-space an infinitesimal isometry is not necessarily an automorphism.[2] In the 6-dimensional unit sphere with the structure given by Fukami-Ishihara, which is an example of a compact K-space, an almost-analytic transformation is an isometry and hence is an automorphism.[3]

In this paper we shall give some theorems on the automorphisms of conformally flat K-spaces.

In §1 we shall give definitions and well known identities. In §2 we shall deal with a conformally flat K-space and prove that the scalar curvature of such a space is non-negative constant. In §3 we shall obtain a theorem on automorphisms of compact conformally flat K-spaces. The last section will be devoted to discussions on automorphisms of K-spaces of positive constant curvature.

1. Preliminaries. Let us consider an n-dimensional K-space M.[4] By definition, M admits a tensor field $\varphi_i{}^h$ and a positive definite Riemannian metric tensor g_{ji} such that

$$\varphi_i{}^r\varphi_r{}^h = -\delta_i^h, \tag{1.1}$$

$$g_{rs}\varphi_j{}^r\varphi_i{}^s = g_{ji}, \tag{1.2}$$

$$\nabla_j\varphi_i{}^h = -\nabla_i\varphi_j{}^h, \tag{1.3}$$

where ∇ denotes the operator of Riemannian covariant derivation.

(1.1) and (1.2) mean that M is an almost-Hermitian space and hence is even dimensional and orientable.

The tensor $\varphi_{ji} = \varphi_j{}^r g_{ri}$ is skew-symmetric by virtue of (1.1) and (1.2) and so is $\nabla_j\varphi_{ih}$ by (1.3). φ_{ji} is a Killing tensor of order 2 in the sense of Yano-Bochner [**6**].

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1) Tachibana, S., [**2**]. The number in brackets refers to Bibliography at the end of the paper.

2) Tachibana, S., [**3**].

3) Fukami, T. and S. Ishihara., [**1**].

4) As to the notations we follow Tachibana, S., [**3**]. Indices run over $1, 2, \cdots, n$. Throughout the paper we assume that $n > 2$.

From (1.1), (1.2) and (1.3) we see that tensors $\varphi_i^{\ h}$ and $\nabla_j\varphi_i^{\ h}$ are pure while φ_{ji} and g_{ji} are hybrid.[5]

As $\nabla_j\varphi_{ih}$ is pure, we have

(1.4) $$\nabla_r\varphi_i^{\ r}=0,$$

(1.5) $$\varphi_j^{\ r}\nabla_r\varphi_{ih}=\varphi_i^{\ r}\nabla_j\varphi_{rh}.$$

Let $R_{kji}^{\ \ \ h}$ and $R_{ji}=R_{rji}^{\ \ \ r}$ be Riemannian curvature tensor and Ricci tensor respectively and put

$$R^*_{kj}=(1/2)\varphi^{rs}R_{rstj}\varphi_k^{\ t},$$

then the following identities hold good[6]

(1.6) $$\nabla^r\nabla_r\varphi_j^{\ h}=(R_j^{*r}-R_j^{\ r})\varphi_r^{\ h},$$

(1.7) $$R^*_{ji}=R^*_{ij},$$

(1.8) $$(\nabla_j\varphi_{rs})\nabla_i\varphi^{rs}=R_{ji}-R^*_{ji},$$

where $\varphi^{rs}=\varphi_i^{\ s}g^{ir}$.

We know that R_{ji} and R^*_{ji} are hybrid, i.e. the following relations hold

$$R_{jr}\varphi_i^{\ r}=-R_{ri}\varphi_j^{\ r},\qquad R^*_{jr}\varphi_i^{\ r}=-R^*_{ri}\varphi_j^{\ r}.$$

By Ricci's identity we have

(1.9) $$\nabla_r\nabla_s\varphi_{ji}-\nabla_s\nabla_r\varphi_{ji}=-R_{rsj}^{\ \ \ t}\varphi_{ti}-R_{rsi}^{\ \ \ t}\varphi_{jt},$$

from which we have, taking account of (1.7),

(1.10) $$\varphi^{rs}\nabla_r\nabla_s\varphi_{ji}=0.$$

For any vector field v^i we define a vector $N(v)_h$ by

$$N(v)_h=(\nabla^r v^s)(\nabla_t\varphi_{rs})\varphi_h^{\ t},$$

where $\nabla^r=g^{ri}\nabla_i$.

A vector field or an infinitesimal transformation v^i is called almost-analytic if it satisfies $\underset{v}{\pounds}\varphi_i^{\ h}=0$, where $\underset{v}{\pounds}$ denotes the operator of Lie derivation with respect to v^i. A vector field v^i is called an (infinitesimal) isometry or a Killing vector if it satisfies $\underset{v}{\pounds}g_{ji}=0$. If an isometry is almost-analytic, then it is called an (infinitesimal) automorphism.

we know the following

LEMMA 1. *In a* K-*space an almost-analytic vector field v^i satisfies the following equations*

$$\nabla^r\nabla_r v^i+R_r^{\ i}v^r=0,$$

$$2N(v)_h=(R^*_{hr}-R_{hr})v^r.$$

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5) Tachibana, S., [4].

6) Tachibana, S., [3].

2. A conformally flat K-space. In the rest of the parer we assume that our K-space M is conformally flat. Thus the conformal curvature tensor vanishes and we have

(2.1) $$(n-2)R_{kjih}=g_{kh}R_{ji}-g_{jh}R_{ki}+R_{kh}g_{ji}-R_{jh}g_{ki} \\ -b(g_{kh}g_{ji}-g_{jh}g_{ki}),$$

where

$$b=R/(n-1), \qquad R=R_{ji}g^{ji}.$$

From (2.1) we have

(2.2) $$(n-2)R^*_{ji}=2R_{ji}-bg_{ji},$$

(2.3) $$(n-2)(R_{ji}-R^*_{ji})=(n-4)R_{ji}+bg_{ji}.$$

If we put $R^*=R^*_{ji}g^{ji}$, then from (2.2) we get $R^*=b$ and hence

(2.4) $$R-R^*=(n-2)b.$$

On the other hand we have, by virtue of (1.8),

$$R-R^*=(\nabla_i\varphi_{rs})\nabla^i\varphi^{rs}.$$

From (2.4) and the last equation we get

(2.5) $$(n-2)b=(\nabla_i\varphi_{rs})\nabla^i\varphi^{rs}.$$

Since the right hand member of (2.5) is non-negative, we see that $R\geqq 0$.

Now let P^{kji} be an arbitrary pure tensor, then we have from (2.1)

$$P^{kji}R_{kjih}=0$$

because of the fact that g_{ji} and R_{ji} are both hybrid.[7] As the tensors $\nabla^k\varphi^{ji}$ and $\varphi_s{}^i\nabla^k\varphi^{js}$ are pure, we have

(2.6) $$(\nabla^k\varphi^{ji})R_{kjih}=0,$$

(2.7) $$\varphi_s{}^i(\nabla^k\varphi^{js})R_{kjih}=0.$$

We shall now prove a theorem which will play an essential role in the next section.

THEOREM 1. *In an n (>2) dimensional conformally flat* K-*space, the scalar curvature R is a non-negative constant. Especially if the space is non-Kählerian, then R is a positive constant.*

PROOF. On account of (2.5), it is sufficient to prove that the following vector u_j vanishes,

$$u_j=(\nabla^i\varphi^{rs})\nabla_j\nabla_i\varphi_{rs}.$$

Since we have by Ricci's identity

7) Tachibana, S., [4].

$$\nabla_j\nabla_i\varphi_{rs}=\nabla_i\nabla_j\varphi_{rs}-R_{jir}{}^t\varphi_{ts}-R_{jis}{}^t\varphi_{rt}\,,$$

we get by virtue of (2.7)

$$\begin{aligned} u_j &= (\nabla^i\varphi^{rs})\nabla_i\nabla_j\varphi_{rs}+\varphi_s{}^t(\nabla^i\varphi^{rs})R_{jirt}-\varphi_r{}^t(\nabla^i\varphi^{rs})R_{jist} \\ &= (\nabla^i\varphi^{rs})\nabla_i\nabla_j\varphi_{rs}\,. \end{aligned}$$

As $\nabla^i\varphi^{rs}$ is skew-symmetric, we have

$$\begin{aligned} u_j &= -(\nabla^i\varphi^{rs})\nabla_i\nabla_r\varphi_{js}=-(1/2)(\nabla^i\varphi^{rs})(\nabla_i\nabla_r\varphi_{js}-\nabla_r\nabla_i\varphi_{js}) \\ &= (1/2)(\nabla^i\varphi^{rs})(R_{irj}{}^t\varphi_{ts}+R_{irs}{}^t\varphi_{jt})=0 \end{aligned}$$

by virtue of (2.6) and (2.7). Thus u_j vanishes and hence R is a constant. If $R=0$, then from (2.5) we have $\nabla_i\varphi_{rs}=0$ which means that the space is Kählerian. Thus Theorem 1 is proved.

3. Automorphisms of a conformally flat K-space. Let us consider a vector field v^h in an n-dimensional conformally flat K-space M. If we operate $\nabla^h=g^{hi}\nabla_i$ to

$$N(v)_h=(\nabla^r v^s)(\nabla_t\varphi_{rs})\varphi_h{}^t\,,$$

we have

$$\nabla^h N(v)_h=(\nabla^h\nabla^r v^s)(\nabla_t\varphi_{rs})\varphi_h{}^t+(\nabla^r v^s)(\nabla^h\nabla_t\varphi_{rs})\varphi_h{}^t\,.$$

In the right hand side, the last term vanishes because of (1.10) and the first term vanishes too because we have

$$\begin{aligned} \varphi_h{}^t(\nabla_t\varphi_{rs})\nabla^h\nabla^r v^s &= \varphi_t{}^h(\nabla^t\varphi_s{}^r)\nabla_h\nabla_r v^s \\ &= (1/2)\varphi_t{}^h(\nabla^t\varphi_s{}^r)(\nabla_h\nabla_r v^s-\nabla_r\nabla_h v^s) \\ &= (1/2)\varphi_t{}^h(\nabla^t\varphi_s{}^r)R_{hri}{}^s v^i=0\,. \end{aligned}$$

Thus we get the following

LEMMA 2. *In a conformally flat* K-*space, any vector field* v^i *satisfies* $\nabla^i N(v)_i=0$.

In the rest of this section we prove the following

THEOREM 2. *In a compact* n (>4) *dimensional conformally flat* K-*space, an almost-analytic transformation is an automorphism.*

Let v^i be almost-analytic, then from Lemma 1 and (2.3) we have

$$\nabla^r\nabla_r v^i+R_r{}^i v^r=0\,, \tag{3.1}$$

$$(n-4)R_i{}^r v_r+bv_i=-2(n-2)N(v)_i\,. \tag{3.2}$$

As R is a constant by virtue of Theorem 1, we have $\nabla_i R=0$ and hence $\nabla_i R_r{}^i=0$. Taking account of this fact and of Lemma 2 we have from (3.2)

$$(n-4)R_{ri}\nabla^r v^i+bf=0\,, \tag{3.3}$$

where

$$f=\nabla_i v^i .$$

On the other hand we have from (3.1)

$$\nabla^r \nabla_r f + 2R_{ri}\nabla^r v^i = 0 \tag{3.4}$$

because of

$$\nabla_i \nabla^r \nabla_r v^i = \nabla_i \nabla_r \nabla^r v^i = \nabla_r \nabla_i \nabla^r v^i$$
$$= \nabla^r \nabla_r \nabla_i v^i + R_{ri}\nabla^r v^i .$$

Thus from (3.3) and (3.4) we get $\nabla^r \nabla_r f = 2bf/(n-4), b \geqq 0$. Hence the theorem is proved.

REMARK. In case $n=4$, Theorem 2 is also true for compact conformally flat non-Kählerian K-spaces, for, in this case, we have $f=0$ by virtue of (3.3).

4. Automorphisms of a K-space of constant curvature. In this section we consider a K-space of positive constant curvature. In this case the Riemannian curvature tensor takes the form

$$R_{kjih} = a(g_{kh}g_{ji} - g_{jh}g_{ki}) , \tag{4.1}$$

where

$$a = R/n(n-1) .$$

Transvecting (4.1) with g^{kh} we have

$$R_{ji} = cg_{ji} , \qquad c = R/n .$$

From (4.1) we have also

$$\varphi^{rs}R_{rsih} = -2a\varphi_{ih} , \qquad R^*_{ji} = ag_{ji} , \tag{4.2}$$

$$R_{ji} - R^*_{ji} = (c-a)g_{ji} . \tag{4.3}$$

Next we suppose that our space admits a non-trivial automorphism v^i. Then as v^i is a Killing vector, it satisfies

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + R_{rji}{}^h v^r = 0 . \tag{4.4}$$

Now we define a scalar function g by

$$g = \varphi_s{}^r \nabla_r v^s , \tag{4.5}$$

so we have

$$\nabla_i g = -(\nabla^r v^s)\nabla_i \varphi_{rs} + 2a\varphi_i{}^r v_r$$

by virtue of (4.2) and (4.4). Let us put

$$u_j = \varphi_j{}^i \nabla_i g ,$$

then the vector u_j thus defined satisfies

$$u_j = -N(v)_j - 2av_j . \tag{4.6}$$

On the other hand, as v^i is almost-analytic, we have

(4.7) $$2N(v)_j = -(c-a)v_j$$

by virtue of Lemma 1 and of (4.3). From (4.6) and (4.7) we get

(4.8) $$2u_j = (c-5a)v_j .$$

From (4.8) and the definition of $N(u)_h$ we have

$$2N(u)_h = (c-5a)N(v)_h .$$

Substituting (4.7) into the right hand side and taking account of (4.8), we have

(4.9) $$2N(u)_h = -(c-a)u_h .$$

On the other hand we have from the definition of $N(u)_h$

(4.10) $$N(u)_h = (\nabla^r u^s)(\nabla_t \varphi_{rs})\varphi_h{}^t .$$

From (4.5) we have

$$\nabla^r u^s = \nabla^r(\varphi^{si}\nabla_i g) = (\nabla^r \varphi^{si})\nabla_i g + \varphi^{si}\nabla^r\nabla_i g .$$

Substituting the last equation into (4.10) we have

$$N(u)_h = (\nabla^i \varphi^{rs})(\nabla_t \varphi_{rs})\varphi_h{}^t\nabla_i g$$
$$= (R_t{}^i - R_t^{*i})\varphi_h{}^t \nabla_i g$$

by virtue of (1.3), (1.5) and (1.8). Thus we have

(4.11) $$N(u)_h = (c-a)u_h .$$

From (4.9) and (4.11) we have $(c-a)u_h = 0$, where $c-a = (n-2)R/n(n-1) > 0$. Thus we get $u_h = 0$ and from (4.8) and the non-trivialness of v^i we have $c-5a = (n-6)R/n(n-1) = 0$. Hence we obtain

THEOREM 3. *An n (>2) dimensional* K-*space of positive constant curvature cannot admits a non-trivial automorphism provided that $n \neq 6$.*

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