J. Math. Soc. Japan
Vol. 13, No. 1, 1961

On the Dirichlet problem for quasi-linear elliptic differential equations of the second order

By Kiyoshi AKÔ

(Received April 21, 1960)
(Revised May 25, 1960)

§ 1. Introduction.

1. The purpose of this note is to discuss the Dirichlet problem for the general quasi-linear elliptic differential equation of the second order

$$a_{ij}(x, u, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, u, \operatorname{grad} u)\partial u/\partial x_k = f(x, u, \operatorname{grad} u)^{1)} \tag{1}$$

with the bounded coefficients b_k and with the free term f which is bounded if the argument u is bounded[2].

2. In solving the Dirichlet problem of second order elliptic equations it is well-known that there are three kinds of important estimation for linear elliptic equations (see, e. g., L. Nirenberg [**12**]). One of the three is Schauder's interior estimates which will not be used in this note. Two others of the three concern the estimations of solutions belonging to $C^2(\bar{G})$ in the Banach spaces $C^{1,\alpha}(\bar{G})$ and $C^{2,\alpha}(\bar{G})$[3]. The latter, namely the estimation in $C^{2,\alpha}(\bar{G})$ first derived by J. Schauder ([**13**], [**14**]) and called by the name of Schauder's boundary estimates, is complete. But the estimation in $C^{1,\alpha}(\bar{G})$ is, even though considerably satisfactory by efforts of many eminent scholars (here we quote, in particular, L. Nirenberg [**11**] and H. O. Cordes [**3**]), still incomplete in the present situation.

The author is sorry to say that the results of the present note are considerably general but will become complete when the estimation in $C^{1,\alpha}(\bar{G})$ is completely accomplished. For this reason the author also hopes to make better the estimation in $C^{1,\alpha}(\bar{G})$ in the near future.

3. The results of this note are formulated in the sections 4, 6 and 7. There we shall give four theorems; namely, *the Main theorem, the envelope*

1) Here $x=(x_1, \cdots, x_n)$ and grad $u=(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. Throughout this note we use the summation convention for doubly appeared indices.

2) Actually we shall concern the equation (1) under more general assumptions. But for the sake of brevity we shall state the results of this note under rather simpler assumptions in the introduction.

3) See § 2 and Lemmas A, C of § 3.

theorem, a theorem of Perron type, and *a theorem of Peano type.* In this introduction we only refer to the main theorem under rather simpler assumptions.

Let G be a smooth, bounded domain in the Euclidean n-space ($n \geqq 2$) and let the function $\varphi(x)$ be in $C^3(\bar{G})$[4]. Let the functions a_{ij}, b_k and f be defined in $\mathfrak{D}: x \in \bar{G}$, $|u| < \infty$, $|q| < \infty$ and Hölder-continuous (with some exponent τ, $0 < \tau < 1$) in every compact subset of $\mathfrak{D}$. We assume that there exist a finite quasi-supersolution $\bar{\omega}(x)$ and a finite quasi-subsolution $\underline{\omega}(x)$[5] such that $\underline{\omega}(x) \leqq \bar{\omega}(x)$ in $\bar{G}$ and that $\underline{\omega}(x) \leqq \varphi(x) \leqq \bar{\omega}(x)$ on the boundary Γ of G. We assume further that the family of differential operators

$$\mathcal{L}^{(u)} \equiv a_{ij}(x, u(x), \operatorname{grad} u(x))\partial^2/\partial x_i \partial x_j \quad \text{for all} \quad u(x) \in C^1(\bar{G}) \tag{2}$$

is a suitable set of elliptic operators of complete type[6].

Then we obtain

The Main Theorem. *The Dirichlet problem* $[D_\varphi]$: *to seek functions* $u(x)$ *in* $C^2(\bar{G})$ *which satisfy*

the differential equation

$$a_{ij}(x, u, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, u, \operatorname{grad} u)\partial u \partial x_k = f(x, u, \operatorname{grad} u) \tag{1}$$

in the domain G *as well as*

the boundary condition

$$u(x) = \varphi(x) \qquad \text{on the boundary } \Gamma \text{ of } G \tag{3}$$

has at least one solution $u(x)$ *such that*

$$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x) \tag{4}$$

in the closure $\bar{G}$ *of* G.

4. At the end of the introduction the author of this note wishes to express his deepest gratitude to Prof. M. Hukuhara who has constantly inspired and stimulated him.

§ 2. The Banach spaces $C^r(\bar{G})$ and $C^{r,\tau}(\bar{G})$.

1. *The domain* G: Hereafter we shall denote by G a fixed bounded domain in the Euclidean n-space ($n \geqq 2$) whose boundary Γ is of type Bh[7].

2. *The Banach space* $C^r(\bar{G})$: It is well-known that all functions $g(x)$ con-

---

4) $\bar{G}$ is the closure of the domain G. As for the precise assumptions imposed on G and the function $\varphi(x)$ see the first paragraph of § 2 and Assumption II of § 4.

5) See Definition 3 of § 3 and a remark below it. These notions are a generalization of those of superharmonic functions and subharmonic functions.

6) See Definitions 2 and 3 in § 3.

7) i.e., G is bounded by a h-Hölder continuously twice differentiable hypersurface Γ ($0 < h < 1$) which does not intersect itself.

tinuous in the closure $\bar{G}$ of G together with their derivatives up to order r[8] form a Banach space by the norm

$$(5)_0 \qquad \|g\| = \|g\|_0 = \max_{x\in\bar{G}} |g(x)| \qquad \text{provided } r=0\,,$$

or by the norm

$$(5)_r \qquad \|g\|_r = \sum_{k=1}^{r} \sum_{1\leqq i_s\leqq n} \|\partial^k g/\partial x_{i_1}\cdots\partial x_{i_k}\| + \|g\| \qquad \text{provided } r\geqq 1\,.$$

We denote this Banach space by $C^r(\bar{G})$.

3. *The Banach space* $C^{r,\tau}(\bar{G})$: Let r be a non-negative integer and let τ be a positive number less than 1. Then we can define the quantity $H_\tau(g)$ by

$$(6) \qquad H_\tau(g) = \sup_{\substack{x,y\in\bar{G}\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|^\tau} \qquad (\leqq\infty)\,.$$

It is well-known that all functions $g(x)$ in $C^r(\bar{G})$ such that the "norm"

$$(7) \qquad \|g\|_{r,\tau} = \|g\|_r + \sum_{1\leqq i_s\leqq n} H_\tau(\partial^r g/\partial x_{i_1}\cdots\partial x_{i_r})$$

are finite form a Banach space. We denote it by $C^{r,\tau}(\bar{G})$. We note here that every bounded, closed sphere in the space $C^{r,\tau}(\bar{G})$ is compact in $C^r(\bar{G})$.

§ 3. Preliminary definitions and lemmas.

1. *Quasi-supersolutions and quasi-subsolutions*: As in the paper [**10**] of M. Nagumo we shall start with the following

Definition 1. A function $\bar{\omega}(x)$ is said to be *a quasi-supersolution in* G if the following requirements are satisfied:

i) The function $\bar{\omega}(x)$ is in $C^{0,\tau}(\bar{G})$ for some $0<\tau<1$.

ii) For any point $x_0\in G$ there exists a neighborhood U of x_0 in which the function $\bar{\omega}(x)$ is of the form

$$(8)^* \qquad \min_{1\leqq\nu\leqq k} \bar{\omega}_\nu(x)$$

(k is any positive integer which may depend on U), where $\bar{\omega}_\nu(x)$ are in $C^2(U)$ and satisfy in U the differential inequalities

$$(1)^* \qquad a_{ij}(x, \bar{\omega}_\nu(x), \operatorname{grad}\bar{\omega}_\nu(x))\partial^2\bar{\omega}_\nu/\partial x_i\partial x_j + b_k(x, \bar{\omega}_\nu(x), \operatorname{grad}\bar{\omega}_\nu(x))\partial\bar{\omega}_\nu/\partial x_k$$
$$\leqq f(x, \bar{\omega}_\nu(x), \operatorname{grad}\bar{\omega}_\nu(x))\,.$$

8) i.e., that the function $g(x)$ is continuous in the closure $\bar{G}$ of G and that there exists a set of functions $g_{i_1\cdots i_k}(x)$ $(1\leqq i_s\leqq n, 1\leqq k\leqq r)$ all continuous in the closure $\bar{G}$ such that

$$\partial^k g/\partial x_{i_1}\cdots\partial x_{i_k} = g_{i_1\cdots i_k}(x)$$

in the domain G.

A quasi-subsolution $\underline{\omega}(x)$ *in* G is defined analogously by the following two properties:

i)′ $\underline{\omega}(x)$ is in $C^{0,\tau}(\bar{G})$ for some $0<\tau<1$.

ii)′ For any point x_0 of G there exists a neighborhood U of x_0 in which the function $\underline{\omega}(x)$ is of the form

$$(8)_* \qquad \max_{1\leqq\nu\leqq k} \underline{\omega}_\nu(x)$$

(k is any positive integer which may depend on U), where $\underline{\omega}_\nu(x)$ are in $C^2(U)$ and satisfy in U the differential inequalities

$$(1)_* \qquad a_{ij}(x, \underline{\omega}_\nu(x), \operatorname{grad} \underline{\omega}_\nu(x))\partial^2\underline{\omega}_\nu/\partial x_i\partial x_j + b_k(x, \underline{\omega}_\nu(x), \operatorname{grad} \underline{\omega}_\nu(x))\partial\underline{\omega}_\nu/\partial x_k \geqq f(x, \underline{\omega}_\nu(x), \operatorname{grad} \underline{\omega}_\nu(x)).$$

Remark. The extended real numbers ∞ and $-\infty$ are *defined to be* a quasi-supersolution and a quasi-subsolution respectively.

The notions of such functions as given above are a generalization of those of superharmonic functions and subharmonic functions as mentioned in the footnote 5) in Introduction.

2. *Elliptic operators of complete type*[9]: We call a differential operator

$$(9) \qquad \mathcal{L} = a_{ij}(x)\partial^2/\partial x_i\partial x_j$$

an *elliptic operator in* G if the symmetric matrix $\|a_{ij}\|$ is continuous and positive-definite in the closure $\bar{G}$ of G. Then if $\mathcal{L}$ is an elliptic operator in G there exists a constant $A\geqq 1$ such that

$$(10) \qquad A^{-1}|\xi|^2 \leqq a_{ij}(x)\xi_i\xi_j \leqq A|\xi|^2$$

for any n-tuple $\xi=(\xi_1, \cdots, \xi_n)$ ($|\xi|^2=\xi_i\xi_i$) and for $x\in\bar{G}$.

Definition 2. We say an elliptic operator $\mathcal{L}$ of the form (9) in G is *of complete type* if $\mathcal{L}$ has the following property:

For all functions $u(x)$ in $C^2(\bar{G})$ which satisfy in G the linear differential equation

$$(11) \qquad a_{ij}(x)\partial^2 u/\partial x_i\partial x_j + b_k(x)\partial u/\partial x_k = f(x)$$

with the functions $b_k(x)$ and $f(x)$ in $C(\bar{G})$, there exists positive constants K and α $(0<\alpha<1)$, depending only on the constant A, the shape of the domain G and the least upper bound Q of $|b_k|$ $(k=1, \cdots, n)$, and the following estimation lemma A is assumed to hold.

Lemma A. $\|u\|_{1,\alpha} \leqq K(\|\varphi\|_2 + \|f\|)$, *where* $\varphi(x)$ *is any function in* $C^2(\bar{G})$ *coinciding with the function* $u(x)$ *on the boundary* Γ *of* G.

Definition 3. A family of elliptic operators

$$(12) \qquad \mathcal{L}^{(\lambda)} = a_{ij}^{(\lambda)}(x)\partial^2/\partial x_i\partial x_j \qquad \text{(with indices } \lambda)$$

9) This term is not an official one, but a private one used for the sake of author's convenience.

is called *a suitable set of elliptic operators of complete type* if the following two conditions are satisfied:

i) There exists a positive constant $A \geqq 1$ such that

$$A^{-1}|\xi|^2 \leqq a_{ij}^{(\lambda)}\xi_i\xi_j \leqq A|\xi|^2 \tag{13}$$

for any n-tuple $\xi=(\xi_1, \cdots, \xi_n)$ and for any index λ.

ii) Let $\mathfrak{U}$ be the family of all functions $u_\lambda(x)$ in $C^2(\bar{G})$ satisfying in G differential equations

$$a_{ij}^{(\lambda)}(x)\partial^2 u_\lambda/\partial x_i \partial x_j + b_k^{(\lambda)}(x)\partial u_\lambda/\partial x_k = f^{(\lambda)}(x) \tag{14}$$

with the indices λ. Here the functions $b_k^{(\lambda)}$ and $f^{(\lambda)}$ are assumed to be continuous in $\bar{G}$ with the $|b_k^{(\lambda)}|$ bounded by a constant Q and with the $|f^{(\lambda)}|$ bounded by a constant F. Then the following estimation Lemma A$'$ is assumed to hold.

Lemma A$'$. *For every $u_\lambda(x) \in \mathfrak{U}$, the estimates*

$$\|u_\lambda\|_{1,\alpha} \leqq K(\|\varphi_\lambda\|_2 + F) \tag{15}$$

hold with the positive constants K and α $(0<\alpha<1)$ depending only on the constants A, Q and on the shape of G. Here the functions $\varphi_\lambda(x)$ is any function in $C^2(\bar{G})$ coinciding with the function $u_\lambda(x)$ on the boundary Γ of G.

Remark 1. It is known that any elliptic operator in G in the plane (i.e., $n=2$) is of complete type. (See, e.g., L. Nirenberg [**11**][10]. Also see L. Bers and L. Nirenberg [**2**] and L. Nirenberg [**12**].) Hence a set of elliptic operators in G in the plane is a suitable set of elliptic operators of complete type provided the inequalities (13) are valid with a constant $A \geqq 1$.

Remark 2. In the case $n \geqq 3$, H. O. Cordes [**3**][11] recently proved that any elliptic opearator in G is of complete type provided that the coefficient-matrix $\|a_{ij}(x)\|$ satisfies a cone-condition (K$'_\varepsilon$) (i.e., the condition

$$(n-1)\Big(1+\frac{n(n-2)}{(n+1)(n-1)}\Big)\sum_{i<j}(\lambda_i-\lambda_j)^2 \leqq (1-\varepsilon)(\sum_{i=1}^{n}\lambda_i)^2, \qquad 0<\varepsilon<1, \tag{K$'_\varepsilon$}$$

imposed on the eigenvalues $\{\lambda_i\}_{i=1}^n$ of the matrix $\|a_{ij}(x)\|$).

Hence we see that any set of elliptic operators in G is a suitable set of elliptic operators of complete type provided that the conditions (13) and (K$'_\varepsilon$) with a fixed ε are valid.

3. *A maximum principle of E. Hopf.*

---

10) See Theorem V of L. Nirenberg [**11**, p. 126]. He has not given his Theorem V in the form of Lemma A, but it is clear that we can improve his Theorem V in the form of Lemma A.

11) See H. O. Cordes [**3**, Satz 9, p. 311]. Since the coefficient of the dependent variable u vanishes in our case his Satz 9 takes the form of Lemma A.

Lemma B. *Let $v(x)$ be in $C^2(G)\frown C(\bar{G})$ and satisfy the elliptic differential inequality*

$$a_{ij}(x)\partial^2 v/\partial x_i\partial x_j+b_k(x)\partial v/\partial x_k\geqq 0$$

in G, where the $a_{ij}(x)$, $b_k(x)$ are continuous in the closure $\bar{G}$ of G. Then we get

$$v(x)\leqq\max_{y\in\Gamma}(v(y))$$

in $\bar{G}$.

For a proof see E. Hopf [**6**] or C. Miranda [**9**, Chap. I].

4. *Theorems of J. Schauder*: In this paragraph we shall state the theorems of J. Schauder ([**13**], [**14**], see also C. Miranda [**9**, Chap. V]) as the lemmas C and D.

Let us consider the elliptic differential equation

$$a_{ij}(x)\partial^2 u/\partial x_i\partial x_j+b_k(x)\partial u/\partial x_k=f(x) \tag{11}$$

in the domain G under the assumptions that

i) $\sum_{i,j=1}^{n}\|a_{ij}\|_{0,\tau}\leqq A_0$

for some positive constants A_0 and τ $(0<\tau\leqq h)$,

ii) for some $A\geqq 1$

$$A^{-1}|\xi|^2\leqq a_{ij}(x)\xi_i\xi_j\leqq A|\xi|^2, \tag{10}$$

where $\xi=(\xi_1,\cdots,\xi_n)$ is any n-tuple,

iii) $\|b_k\|_{0,\tau}\leqq Q$ $(k=1,\cdots,n)$ for some constant Q, and finally

iv) $\|f\|_{0,\tau}<\infty$.

Lemma C. *There exists a constant K_0, depending only on the quantities A_0, A, Q, τ and on the shape of G, and the following estimate is valid:*

$$\|u\|_{2,\tau}\leqq K_0(\|\varphi\|_{2,\tau}+\|f\|_{0,\tau}) \tag{16}$$

for any function $u(x)$ in $C^{2,\tau}(\bar{G})$ which satisfies in G the linear differential equation (11). *Here $\varphi(x)$ is any function in $C^{2,\tau}(\bar{G})$ which coincides with $u(x)$ on the boundary Γ of G.*

Lemma D. *The Dirichlet problem [to seek function $u(x)$ in $C^2(G)\frown C(\bar{G})$ satisfying the differential equation* (11) *in the domain G as well as the boundary condition $u(x)=\varphi(x)$ on the boundary Γ of G] has one and only one solution $u(x)$ provided that the boundary data $\varphi(x)$ is such that $\|\varphi\|_{2,\tau}<\infty$. Furthermore, the unique solution $u(x)$ is in $C^{2,\tau}(\bar{G})$.*

Remark. J. Schauder did not formulate his results in this form. But using his existence theorem of solutions in $C^{2,\tau}(\bar{G})$ and the lemma B, we see the unique existence of solutions of the above Dirichlet problem. Hence the unique solution must be in $C^{2,\tau}(\bar{G})$.

5. *A comparison theorem*: In this paragraph we consider the differential equation

(1) $$a_{ij}(x, u, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, u, \operatorname{grad} u)\partial u/\partial x_k = f(x, u, \operatorname{grad} u)$$

under the following assumptions:

Assumptions:

i) There exist a finite quasi-supersolution $\bar{\omega}(x)$ and a finite quasi-subsolution $\underline{\omega}(x)$ such that $\underline{\omega}(x) \leqq \bar{\omega}(x)$ in the closure $\bar{G}$ of the domain G.

ii) The coefficients $a_{ij}(x, u, q)$, $b_k(x, u, q)$ and the free term $f(x, u, q)$ are defined and continuous in $\mathfrak{D}: x \in \bar{G}$, $|u| < \infty$, $|q| = (q_1^2 + \cdots + q_n^2)^{\frac{1}{2}} < \infty$.

iii) If the argument u is greater than $\bar{\omega}(x)$ at the point x, then

(17)* $$a_{ij}(x, u, q) = a_{ij}(x, \bar{\omega}(x), q), \qquad b_k(x, u, q) = b_k(x, \bar{\omega}(x), q)$$

and

(18)* $$f(x, u, q) > f(x, \bar{\omega}(x), q).$$

If the argument u is less than $\underline{\omega}(x)$ at the point x, then

$(17)_*$ $$a_{ij}(x, u, q) = a_{ij}(x, \underline{\omega}(x), q), \qquad b_k(x, u, q) = b_k(x, \underline{\omega}(x), q)$$

and

$(18)_*$ $$f(x, u, q) < f(x, \underline{\omega}(x), q).$$

Lemma E. *Let $u(x)$ be a function in $C^2(\bar{G})$ satisfying the differential equation (1) in the domain G as well as the boundary condition: $\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$ on the boundary Γ. Then in the closure $\bar{G}$ of G the inequalities*

$$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$$

are valid.

Proof. We shall only prove the inequality $u(x) \leqq \bar{\omega}(x)$. Another inequality will be proved similarly.

Suppose $c > 0$ be the maximum of the function $u(x) - \bar{\omega}(x)$ in $\bar{G}$. There would be a point x_0 in G at which $u(x_0) - \bar{\omega}(x_0) = c$. Since $\bar{\omega}(x)$ is of the form

$$\bar{\omega}(x) = \min_{1 \leqq \nu \leqq k} \bar{\omega}_\nu(x)$$

in a suitable neighborhood U of x_0, if we pick up a function $\bar{\omega}_\nu(x)$ such that $\bar{\omega}_\nu(x_0) = \bar{\omega}(x_0)$, c is the maximum value of the function $u(x) - \bar{\omega}_\nu(x)$ in U.

Let us denote by $\bar{\mathcal{L}}$ the elliptic operator with constant coefficients

(19) $$\bar{\mathcal{L}} \equiv a_{ij}(x_0, \bar{\omega}_\nu(x_0), \operatorname{grad} \bar{\omega}_\nu(x_0))\partial^2/\partial x_i \partial x_j + b_k(x_0, \bar{\omega}_\nu(x_0), \operatorname{grad} \bar{\omega}_\nu(x))\partial/\partial x_k.$$

Then by virtue of well-known criterion the inequality

(20) $$\bar{\mathcal{L}}(u - \bar{\omega}_\nu) \leqq 0$$

holds at the point x_0.

On the other hand, since

$$\text{(21)} \qquad \operatorname{grad} u(x_0) = \operatorname{grad} \bar{\omega}_\nu(x_0)\,, \qquad u(x_0) > \bar{\omega}_\nu(x_0) = \bar{\omega}(x_0)\,,$$

the direct calculation would show that at the point x_0

$$\text{(22)} \qquad \bar{\mathcal{L}}(u-\bar{\omega}_\nu) \geqq f(x_0, u(x_0), \operatorname{grad} u(x_0)) - f(x_0, \bar{\omega}_\nu(x_0), \operatorname{grad} \bar{\omega}_\nu(x_0)) > 0\,,$$

which contradicts (20). Thus we have verified the inequality $u(x) \leqq \bar{\omega}(x)$ in $\bar{G}$.

§ 4. The Main Theorem.

Hereafter we shall consider the differential equation

$$\text{(1)} \qquad a_{ij}(x, u, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, u, \operatorname{grad} u)\partial u/\partial x_k = f(x, u, \operatorname{grad} u)$$

under the following assumptions.

Assumptions:

I. The coefficients $a_{ij}(x, u, q)$, $b_k(x, u, q)$ and the free term $f(x, u, q)$ are defined in $\mathcal{D}: x \in \bar{G},\ |u| < \infty,\ |q| < \infty$, and Hölder-continuous in every compact subset of $\mathcal{D}$ (with some exponent $\tau, 0 < \tau < 1$).

II. The function $\varphi(x)$ is assumed to be in $C^{2,\sigma}(\bar{G})$ for some σ $(0 < \sigma < 1)$.

III. Let $\bar{\omega}(x)$ and $\underline{\omega}(x)$ (possibly $\pm\infty$) be respectively a quasi-supersolution and a quasi-subsolution in G such that $\underline{\omega}(x) \leqq \bar{\omega}(x)$ on $\bar{G}$ and $\underline{\omega}(x) \leqq \varphi(x) \leqq \bar{\omega}(x)$ on Γ. We define the function $a_{ij}^{(u)}(x)$, $b_k^{(u)}(x)$, $f^{(u)}(x)$ as follows: Let $g(x, u, q)$ be any one of the a_{ij}, b_k, f. Then the function $g^{(u)}(x)$ is defined by

$$\text{(23)} \qquad g^{(u)}(x) = \begin{cases} g(x, \bar{\omega}(x), \operatorname{grad} u(x)) & \text{provided } u(x) \geqq \bar{\omega}(x)\,, \\ g(x, u(x), \operatorname{grad} u(x)) & \text{provided } \underline{\omega}(x) < u(x) < \bar{\omega}(x)\,, \\ g(x, \underline{\omega}(x), \operatorname{grad} u(x)) & \text{provided } u(x) \leqq \underline{\omega}(x)\,, \end{cases}$$

where $u(x)$ is any element of $C^1(\bar{G})$.

Then the assumption III is characterized by the following a) and b):

a) The functions $|b_k^{(u)}(x)|$ are bounded by a positive constant Q, while for the functions $f^{(u)}(x)$ there exist three non-negative constants B, F and θ $(0 < \theta < 1)$ such that

$$|f^{(u)}(x)| \leqq B|\operatorname{grad} u|^\theta + F$$

for every $u(x) \in C^1(\bar{G})$.

b) The family of differential operators

$$\text{(24)} \qquad \mathcal{L}^{(u)} \equiv a_{ij}^{(u)}(x)\partial^2/\partial x_i \partial x_j$$

with the index set $u \in C^1(\bar{G})$ is a suitable set of elliptic operators of complete type (see Definition 3 in § 3).

The Main Theorem. *Under the assumptions* I-III *there exists at least one solution $u(x)$ in $C^2(\bar{G})$ of the Dirichlet problem* $[D_\varphi]$ [*to seek functions $u(x)$ in $C^2(\bar{G})$ which satisfy the differential equation* (1) *in the domain G as well as the boundary* *condition* $u(x)=\varphi(x)$ *on the boundary* Γ *of* G] *such that* $\underline{\omega}(x)\leqq u(x)\leqq\overline{\omega}(x)$ *in* $\overline{G}$.

REMARK 1. The most simple case is one that $f(x, u, 0)=0$. In this case we can set

$$\underline{\omega}(x)\equiv\min_{y\in\Gamma}\varphi(y)\ \text{(a const.)}, \quad \text{and} \quad \overline{\omega}(x)=\max_{y\in\Gamma}\varphi(y)\ \text{(a const.)}. \tag{25}$$

The main theorem assures that there exists at least one solution $u(x)$ in $C^2(\overline{G})$ of the Dirichlet problem [D_φ] such that

$$\min_{y\in\Gamma}\varphi(y)\leqq u(x)\leqq\max_{y\in\Gamma}\varphi(y)$$

for $x\in\overline{G}$.

REMARK 2. In this remark we shall impose following two assumptions on the differential equation (1):

i) The coefficients $a_{ij}(x, u, q)$ and $b_k(x, u, q)$ are bounded in $\mathscr{D}: x\in\overline{G}$, $|u|<\infty$, $|q|<\infty$ and the matrix $\|a_{ij}(x, u, q)\|$ satisfies the condition

$$A^{-1}|\xi|^2\leqq a_{ij}(x, u, q)\xi_i\xi_j\leqq A|\xi|^2 \qquad (A\geqq 1), \tag{26}$$

which holds for any n-tuple $\xi=(\xi_1,\cdots,\xi_n)$ and for every point (x, u, q) in $\mathscr{D}$.

ii) The free term $f(x, u, q)$ is of the form

$$c(x, u, q)u+f_1(x, u, q)+f_2(x, u, q), \tag{27}$$

where

a) $c(x, u, q)\geqq 0$,

b) $f_1(x, u, q)$ is bounded,

c) $f_2(x, u, q)$ is non-decreasing with respect to u and for every $M>0$ there exist two constants $B(M)$ and $F(M)$ such that

$$|f_2(x, u, q)|\leqq B(M)|q|+F(M)$$

in $\mathscr{D}_M: x\in\overline{G}$, $|u|\leqq M$, $|q|<\infty$.

Then, *we can always construct an arbitrarily large quasi-supersolution* $\overline{\omega}(x)$ *in the form*

$$C(C-e^{-\gamma x_1}) \tag{28}$$

with two suitably chosen positive constants C and γ. Furthermore,

$$\underline{\omega}(x)\equiv-\overline{\omega}(x)=-C(C-e^{-\gamma x_1}) \tag{29}$$

is quasi-subsolution which is arbitrarily large in the absolute value if C and γ are sufficiently large.

§ 5. Proof of the main theorem.

1. *The Banach space* $\mathfrak{B}\equiv C^1(\overline{G})$ *and the closed sphere* $\mathfrak{S}_N$: Let α $(0<\alpha<1)$ be the constant in the lemma A′ for the family of elliptic differential operators

$$\mathscr{L}^{(u)}\equiv a_{ij}^{(u)}(x)\partial^2/\partial x_i\partial x_j \tag{24}$$

for any $u(x) \in C^1(\bar{G})$ as an index. Such an α certainly exists by virtue of Assumption III b).

Let N be a sufficiently large positive number which will be determined later on. We denote by $\mathfrak{S}_N$ the closed sphere: $\| u \|_{1,\alpha} \leqq N$. It is clear that the sphere $\mathfrak{S}_N$ is a convex, compact subset of $\mathfrak{B}$.

2. *We define the function* $h^{(u)}(x)$ by

$$h^{(u)}(x) - f^{(u)}(x) = \begin{cases} \dfrac{u(x)-\bar{\omega}(x)}{1+|u(x)|+|\bar{\omega}(x)|} & \text{provided } u(x) \geqq \bar{\omega}(x)\,, \\ 0 & \text{provided } \underline{\omega}(x) < u(x) < \bar{\omega}(x)\,, \\ \dfrac{u(x)-\underline{\omega}(x)}{1+|u(x)|+|\underline{\omega}(x)|} & \text{provided } u(x) \leqq \underline{\omega}(x)\,, \end{cases} \tag{30}$$

for any function $u(x)$ in $\mathfrak{S}_N$. Then we easily see that

$$|h^{(u)}(x) - f^{(u)}(x)| \leqq 1 \quad \text{and hence} \quad |h^{(u)}(x)| \leqq F+1+BN^{\theta}\,. \tag{31}$$

3. *The mapping* $\mathfrak{T}$: We define a mapping $\mathfrak{T}$ from $\mathfrak{S}_N$ into $\mathfrak{B}$ in the following way:

Let $u(x)$ be in $\mathfrak{S}_N$. Then the Dirichlet problem $[D_{\varphi}(u)]$—to seek functions $U(x)$ in $C^2(\bar{G})$ satisfying

the differential equation

$$a_{ij}^{(u)}(x)\partial^2 U/\partial x_i \partial x_j + b_k^{(u)}(x)\partial U/\partial x_k = h^{(u)}(x) \tag{32}$$

in the domain G, as well as

the boundary condition

$$U(x) = \varphi(x) \quad \text{on the boundary } \Gamma \text{ of } G \tag{33}$$

—has one and only one solution $U(x)$ in $C^2(\bar{G})$ by virtue of Lemma C. We define the mapping $\mathfrak{T}$ by

$$U = \mathfrak{T}(u)\,. \tag{34}$$

4. *We shall show that* $\mathfrak{T}(\mathfrak{S}_N) \subseteqq \mathfrak{S}_N$ *for a sufficiently large* N. By virtue of Lemma A′ for every $u(x) \in \mathfrak{S}_N$ (and $U = \mathfrak{T}(u)$) we obtain

$$\begin{aligned} \| U \|_{1,\alpha} = \| \mathfrak{T}(u) \|_{1,\alpha} &\leqq K(\| \varphi \|_2 + \| h^{(u)} \|) \\ &\leqq K(\| \varphi \|_2 + F + 1 + BN^{\theta}) \leqq N \end{aligned} \tag{35}$$

if we choose N sufficiently large.

5. *We insist that* $\mathfrak{T}$ *is continuous in* $\mathfrak{S}_N$: To prove this, let $u_1(x)$ be a fixed element of $\mathfrak{S}_N$ and let $u(x)$ by any in $\mathfrak{S}_N$. And let us denote by $\bar{\mathfrak{L}}^{(u)}$ the differential operator

$$a_{ij}^{(u)}(x)\partial^2/\partial x_i \partial x_j + b_k^{(u)}(x)\partial/\partial x_k\,. \tag{36}$$

Then the function $V \equiv U_1 - U = \mathfrak{T}(u_1) - \mathfrak{T}(u)$ $(\in C^2(\bar{G}))$ satisfies the differential equation

(37) $$\bar{\mathcal{L}}^{(u)}(V)=(\bar{\mathcal{L}}^{(u)}-\bar{\mathcal{L}}^{(u_1)})(U_1)+h^{(u_1)}(x)-h^{(u)}(x)$$

in G as well as the boundary condition

(38) $$V(x)=0 \qquad \text{on the boundary } \Gamma .$$

Since U_1 is in $C^2(\bar{G})$, the absolute value of the right side of (37) is less than an arbitrary constant $\varepsilon>0$ if $\| u-u_1 \|_1 \leqq \kappa$ for some $\kappa=\kappa(\varepsilon, u_1)>0$. Hence, by virtue of Lemma A′ of § 3 we get

(39) $$\| V \|_{1,\alpha} \leqq K(\| 0 \|_2+\varepsilon)=K\varepsilon$$

for $u \in \mathfrak{S}_N$ such that $\| u-u_1 \|_1 \leqq \kappa$, which is just to be proved.

6. *As we have already seen* in 4 and 5 $\mathcal{T}(\mathfrak{S}_N) \subseteqq \mathfrak{S}_N$ and $\mathcal{T}$ is continuous in $\mathfrak{S}_N$ if we choose N sufficiently large. Therefore by the well-known fixed point theorem of Schauder-Tychonoff (see, e. g., Dunford-Schwartz [**4**, p. 456]) there is at least a fixed element $u \in \mathfrak{S}_N$ under the mapping $\mathcal{T}$, i. e.,

(40) $$u=\mathcal{T}(u) .$$

Let $u(x)$ be one of the fixed points under the mapping $\mathcal{T}$. Then $u(x)$ satisfies the differential equation

(41) $$a_{ij}^{(u)}(x)\partial^2 u/\partial x_i \partial x_j+b_k^{(u)}(x)\partial u/\partial x_k=h^{(u)}(x)$$

in G as well as the boundary condition

(42) $$u(x)=\varphi(x) \qquad \text{on } \Gamma .$$

By virtue of Lemma E, the function $u(x)$ satisfies the inequalities

(43) $$\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$$

in the closure $\bar{G}$. Hence we see that

(44) $$h^{(u)}(x)=f^{(u)}(x)$$

in G. Therefore, the function $u(x) \in C^2(\bar{G})$ is a required solution of the Dirichlet problem $[D_\varphi]$:

$$a_{ij}(x, u, \text{grad } u)\partial^2 u/\partial x_i \partial x_j+b_k(x, u, \text{grad } u)\partial u/\partial x_k=f(x, u, \text{grad } u)$$

in the domain G and

$$u(x)=\varphi(x) \qquad \text{on the boundary } \Gamma \text{ of } G .$$

Thus we have established the theorem.

§ 6. The envelopes of solutions.

1. *The envelope theorem*: Under the same assumptions as in the Main Theorem we obtain

THE ENVELOPE THEOREM. *Let $\mathfrak{U}$ be the family of all solutions $u(x)$ in $C^2(\bar{G})$*

of the Dirichlet problem $[D_\varphi]$ *such that* $\underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$. *Then*

$$u_{\sup}(x) \equiv \sup u(x) \qquad (u(x) \in \mathfrak{U}) \tag{45}^*$$

and

$$u_{\inf}(x) \equiv \inf u(x) \qquad (u(x) \in \mathfrak{U}) \tag{45}_*$$

are solutions of the Dirichlet problem $[D_\varphi]$.

Proof. 1°. According to Lemma A′ there exist positive constants K and α $(0<\alpha<1)$, depending only on the quantity A of (26), the shape of G and the least upper bound Q of $|b_k^{(u)}(x)|$, such that

$$\| u \|_{1,\alpha} \leqq K(\| \varphi \|_2 + \| f^{(u)}(x) \|) \leqq K(\| \varphi \|_2 + F + B \| u \|_1{}^\theta) \tag{46}$$

for every $u(x) \in \mathfrak{U}$. Hence there exists a constant K_1 such that

$$\| u \|_{1,\alpha} \leqq K_1\,, \tag{47}$$

for every $u(x) \in \mathfrak{U}$. Hence by Lemma C we get

$$\| u \|_{2,\beta} \leqq K_0(\| \varphi \|_{2,\sigma} + \| f^{(u)}(x) \|_{0,\alpha'}) \leqq K_* \tag{48}$$

with positive constants K_0, K_* and β independent of any special choice of $u(x) \in \mathfrak{U}$. Here $\alpha' = \min(\alpha, \tau)$ and β is a positive constant less than α' and σ. Hence the functions $u_{\sup}(x)$ and $u_{\inf}(x)$ are well-defined and continuous in $\bar{G}$.

2°. Let $\{u_k(x)\}_{k=1}^\infty$ be an increasing sequence of functions in $\mathfrak{U}$, and let us set $u(x) = \lim_{k\to\infty} u_k(x)$. Then, according to (48) there exists a subsequence $\{u_{k'}(x)\}$ such that the sequence $\{u_{k'}(x)\}$ converges uniformly in $\bar{G}$ to the function $u(x)$ with their derivatives up to the second order.

Hence we see that the limit function $u(x)$ is also in $\mathfrak{U}$, i. e., a solution in $C^2(\bar{G})$ of the Dirichlet problem $[D_\varphi]$.

3°. Let $\{r_l\}_{l=1}^\infty$ be the set of all rational points of G. Then we can choose a sequence $\{u_k^{(l)}(x)\}_{k=1}^\infty$ in $\mathfrak{U}$ such that $u_k^{(l)}(r_l) \uparrow u_{\sup}(r_l)$ $(l=1,2,3,\cdots)$. First we set $\tilde{u}_1(x) = u_1^{(1)}(x)$. Next we set

$$\underline{\lambda}_2(x) = \max(\tilde{u}_1(x), u_2^{(1)}(x), u_2^{(2)}(x))\,. \tag{49}$$

Then $\underline{\lambda}_2(x)$ is a quasi-supersolution in G and satisfies the condition $\underline{\omega}(x) \leqq \underline{\lambda}_2(x) \leqq \bar{\omega}(x)$ in $\bar{G}$. According to the main theorem there exists a solution $\tilde{u}_2(x)$ in $\mathfrak{U}$ such that

$$\underline{\lambda}_2(x) \leqq \tilde{u}_2(x) \leqq \bar{\omega}(x)\,. \tag{50}$$

By induction we can choose an increasing sequence $\{\tilde{u}_k(x)\}_{k=1}^\infty$ in $\mathfrak{U}$ such that

$$\underline{\lambda}_k(x) \leqq \tilde{u}_k(x) \leqq \bar{\omega}(x)\,, \tag{51}$$

where

$$\underline{\lambda}_k(x) = \max(\tilde{u}_{k-1}(x), u_k^{(1)}(x), \cdots, u_k^{(k)}(x))\,. \tag{52}$$

By the construction we see that

$$\lim_{k\to\infty} \tilde{u}_k(r_l) = u_{\sup}(r_l) \qquad (l=1,2,3,\cdots)\,. \tag{53}$$

The limit function $u(x)$ of the sequence $\{\tilde{u}_k(x)\}_{k=1}^{\infty}$ is also in $\mathfrak{U}$ according to the 2°. Since $u(x)$ and $u_{\sup}(x)$ are both continuous in $\bar{G}$ and coincide at all rational points of $\bar{G}$, we see that

$$u(x) \equiv u_{\sup}(x)$$

in the closure $\bar{G}$. Thus we have shown that $u_{\sup}(x)$ is in $\mathfrak{U}$.

Simialarly the function $u_{\inf}(x)$ will be verified to be in $\mathfrak{U}$.

2. *A theorem of Perron type:* We shall show an analogous theorem to Perron's theorem concerning harmonic functions.

A Theorem of Perron Type. *Let $\bar{\Lambda}$ be the family of all quasi-supersolutions $\bar{\lambda}(x)$ such that $\bar{\lambda}(x) \leqq \bar{\omega}(x)$ in $\bar{G}$ and $\bar{\lambda}(x) \geqq \varphi(x)$ on the boundary Γ of G, and let $\underline{\Lambda}$ be the family of all quasi-subsolutions $\underline{\lambda}(x)$ such that $\underline{\lambda}(x) \geqq \underline{\omega}(x)$ in $\bar{G}$ ond $\underline{\lambda}(x) \leqq \varphi(x)$ on Γ. Then for any $\bar{\lambda}_0(x) \in \bar{\Lambda}$*

$$u_*(x) \equiv \sup \underline{\lambda}(x) \qquad [\underline{\lambda}(x) \in \underline{\Lambda}\,;\, \underline{\lambda}(x) \leqq \bar{\lambda}_0(x)]$$

is a solution of the Dirichlet problem $[D_\varphi]$. And the similar statement for the function

$$u^*(x) \equiv \inf \bar{\lambda}(x) \qquad [\bar{\lambda}(x) \in \bar{\Lambda}\,;\, \bar{\lambda}(x) \geqq \underline{\lambda}_0(x)]$$

is valid.

Proof. For each $\underline{\lambda}(x) \in \underline{\Lambda}$ such that $\underline{\lambda}(x) \leqq \bar{\lambda}_0(x)$ in $\bar{G}$ there exists a solution $u_{\underline{\lambda}}(x)$ in $\mathfrak{U}$ satisfying the inequalities

$$\underline{\lambda}(x) \leqq u_{\underline{\lambda}}(x) \leqq \bar{\lambda}_0(x)$$

in G, by virtue of the main theorem. Hence the function $u_*(x)$ can be written in the form

$$\sup u_{\underline{\lambda}}(x) \qquad [\underline{\lambda}(x) \in \underline{\Lambda}\,;\, \underline{\lambda}(x) \leqq \bar{\lambda}_0(x)]\,.$$

According to the envelope theorem we see that $u_*(x)$ is in $\mathfrak{U}$. Thus we have established the theorem.

§ 7. A Theorem of Peano Type.

1. In the Cauchy problem (the initial value problem) of the general ordinary differential equation

$$\begin{cases} y' = f(t, y)\,, & 0 \leqq t \leqq t_0\,, \\ y(0) = y_0\,, \end{cases} \tag{54}$$

it is well-known that there exist two solutions $y_{\max}(t)$ and $y_{\min}(t)$ of the problem such that for any solution $y(t)$ of the problem the inequalities

$$y_{\max}(t) \geqq y(t) \geqq y_{\min}(t) \tag{55}$$

are valid in $0 \leqq t \leqq t_0$, provided that $f(t, y)$ is continuous and bounded in $0 \leqq t \leqq t_0$, $|y| < \infty$. Furthermore all other solutions than $y_{\max}(t)$ and $y_{\min}(t)$ fill the gap between $y_{\max}(t)$ and $y_{\min}(t)$. See, e. g., M. Hukuhara [7, p. 59, Theorem 21.1], or E. Kamke [8, p. 67].

2. Similar circumstances to paragraph 1 occur in the Dirichlet problem for second order elliptic differential equations. (I. Hirai and the author [5] have already proved generalized Peano's theorem for the Dirichlet problem concerning semilinear elliptic equation.) In this paragraph we shall prove a theorem of this type for quasi-linear elliptic equations.

Now, we return to the Dirichlet problem $[D_\varphi]$. In this section we require the following assumptions which are more restricted than the ones given in § 4.

We shall consider the differential equation

$$a_{ij}(x, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, \operatorname{grad} u)\partial u/\partial x_k = f(u, u, \operatorname{grad} u) \tag{56}$$

in the domain G. We set the following

Assumptions:

I′. The coefficients $a_{ij}(x, q)$, $b_k(x, q)$ are defined and bounded in $\mathfrak{D}^*: x \in \bar{G}$, $|q| < \infty$, while the free term f is defined in $\mathfrak{D}: x \in \bar{G}$, $|u| < \infty$, $|q| < \infty$. And the a_{ij}, b_k (the f) are Hölder-continuous with an exponent τ $(0 < \tau < 1)$ in every compact subset of $\mathfrak{D}^*$ (of $\mathfrak{D}$).

II′. The function $\varphi(x)$ is assumed to be in $C^{2,\sigma}(\bar{G})$ with a certain σ $(0 < \sigma < 1)$.

III′. The family of differential operators defined by

$$\mathcal{L}^{(u)} \equiv a_{ij}(x, \operatorname{grad} u(x))\partial^2/\partial x_i \partial x_j \tag{57}$$

for all $u(x) \in C^1(\bar{G})$ is assumed to be a suitable set of elliptic operators of complete type.

IV′. a) The free term $f(x, u, q)$ is non-decreasing with respect to u, i. e.,

$$f(x, u, q) \leqq f(x, \bar{u}, q) \qquad \text{provided } u < \bar{u}. \tag{58}$$

b) For every $M > 0$ there are three non-negative constants B_M, F_M and θ $(0 < \theta < 1)$ such that

$$|f(x, u, q)| \leqq B_M |q|^\theta + F_M$$

provided that $x \in \bar{G}$, $|u| \leqq M$, $|q| < \infty$.

Further, without loss of generality, we may assume that the domain G is in the half space $x_1 > 0$. Let C and γ be sufficiently large positive numbers such that $C\gamma e^{-\gamma x_1} \geqq 1$ in $\bar{G}$. Then the functions

$$\bar{\omega}(x) = C(2 - e^{-\gamma x_1}) \tag{59)*}$$

and

$$\underline{\omega}(x) = -C(2 - e^{-\gamma x_1}) \tag{59)_*}$$

satisfy in G the differential inequalities

$$a_{ij}^{(u)}(x)\partial^2 \bar{\omega}/\partial x_i \partial x_j + b_k^{(u)}(x)\partial \bar{\omega}/\partial x_k < f(x, \bar{\omega}, \operatorname{grad} \bar{\omega}) \tag{60)*}$$

and

$$(60)_* \qquad a_{ij}^{(u)}(x)\partial^2\underline{\omega}/\partial x_i\partial x_j + b_k^{(u)}(x)\partial\underline{\omega}/\partial x_k > f(x, \underline{\omega}, \operatorname{grad}\underline{\omega})$$

as well as the inequalites

$$(61) \qquad \underline{\omega}(x) \leqq -|\varphi(x)| \leqq |\varphi(x)| \leqq \bar{\omega}(x)$$

on the boundary Γ provided that the positive constants C and γ are suitably chosen.

Under Assumptions I′-IV′, we get

A Theorem of Peano Type. *There exist two solutions $u_{\max}(x)$ and $u_{\min}(x)$ of the Dirichlet problem $[D_\varphi]$ such that for any solution $u(x)$ in $C^2(\bar{G})$ of the problem $[D_\varphi]$ the inequalities*

$$u_{\min}(x) \leqq u(x) \leqq u_{\max}(x)$$

are valid in the closure $\bar{G}$ of G. Further, all solutions other than $u_{\min}(x)$ and $u_{\max}(x)$ fill the gap between $u_{\min}(x)$ and $u_{\max}(x)$.

Remark. In the paragraph 4 of this section the author will give a concrete example for this theorem.

Proof. 1°. Let $\mathfrak{U}^*$ be the family of all solutions in $C^2(\bar{G})$ of the Dirichlet problem $[D_\varphi]$. Then for any $u(x) \in \mathfrak{U}^*$ the inequalities

$$(62) \qquad \underline{\omega}(x) \leqq u(x) \leqq \bar{\omega}(x)$$

are valid in G, where $\underline{\omega}(x)$ and $\bar{\omega}(x)$ are functions given by $(59)^*$ and $(59)_*$ with suitably chosen C and γ[12]. The existence of the solutions $u_{\max}(x)$ and $u_{\min}(x)$ is just a restatement of the envelope theorem.

2°. Let $u_i(x)$ $(i=1,2)$ be in $\mathfrak{U}^*$ such that $u_1(x) \leqq u_2(x)$ and let $c \leqq 0$ be the maximum of the function $u_2(x) - u_1(x)$ in $\bar{G}$. Then

$$\bar{\lambda}(x) = \min(u_1(x) + c/2, u_2(x))$$

and

$$\underline{\lambda}(x) = \max(u_1(x), u_2(x) - c/2)$$

are a quasi-supersolution in G and a quasi-subsolution in G respectively. Hence, according to the main theorem, there exists a solution $u(x)$ in $C^2(\bar{G})$ such that

$$\underline{\lambda}(x) \leqq u(x) \leqq \bar{\lambda}(x)$$

in $\bar{G}$. The solution $u(x)$, therefore, satisfies the condition

$$u_1(x) \leqq u(x) \leqq u_1(x) + c/2, \quad \text{and} \quad u_2(x) - c/2 \leqq u(x) \leqq u_2(x)$$

in $\bar{G}$.

---

12) Suppose $c_0 > 0$ be the maximum of $u(x) - \bar{\omega}(x)$ in $\bar{G}$. Then, at a maximum point x_0, we would get

$$0 \geqq a_{ij}^{(u)}(x)\partial^2(u-\bar{\omega})/\partial x_i \partial x_j > f(x_0, u(x_0), \operatorname{grad} u(x_0)) - f(x_0, \bar{\omega}(x_0) \operatorname{grad}\bar{\omega}(x_0)) > 0$$

since $\operatorname{grad}\bar{\omega}(x_0) = \operatorname{grad} u(x_0)$. This is absurd.

3°. From the preceding discussion we see that for any pair (x^*, u^*) such that $x^* \in \bar{G}$ and $u_1(x^*) \leqq u^* \leqq u_2(x^*)$ there exists a sequence of solutions $\{u_k(x)\}_{k=1}^{\infty}$ in $\mathfrak{U}^*$ such that $\lim_{k\to\infty} u_k(x^*) = u^*$. From (48) in the proof of the envelope theorem there exists a subsequence $\{u_{k'}(x)\}$ such that the subsequence converges uniformly in $\bar{G}$ to a function $u(x)$ with their derivatives up to the second order. The function $u(x)$ is easily seen in $\mathfrak{U}^*$ and satisfies the condition $u(x^*) = u^*$.

4°. The proof of Theorem follows immediately from 2° and 3° if we set $u_1(x) \equiv u_{\min}(x)$ and $u_2(x) \equiv u_{\max}(x)$.

3. Corollary 1. *Let $u_i(x) \in \mathfrak{U}^*$ $(i = 1, 2)$ be such that $u_1(x) \leqq u_2(x)$ in $\bar{G}$. And let (x^*, u^*) be any pair with the properties that $x^* \in \bar{G}$, $u_1(x^*) \leqq u^* \leqq u_2(x^*)$. Then there exists a function $u(x) \in \mathfrak{U}^*$ which satisfies the inequalities*

$$u_1(x) \leqq u(x) \leqq u_2(x)$$

in $\bar{G}$ as well as the additional condition $u(x^*) = u^*$.

The proof follows from 2° and 3° in the proof of the theorem.

Corollary 2. *Suppose the solution of the Dirichlet problem $[D_\varphi]$ is not unique. Then there exists a positive constant δ such that the solution of the Dirichlet problem $[D_\psi]$ is not unique provided that $|\varphi(x) - \psi(x)| < \delta$ on the boundary Γ.*

Proof. Let $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$ be the maximal and the minimal solution of the Dirichlet problem $[D_\varphi]$ and let $\delta = \frac{1}{4}\max(\tilde{u}_1(x) - \tilde{u}_2(x))$ (> 0). Suppose that $\psi(x)$ is in $C^{2,\sigma}(\bar{G})$ with some σ $(0 < \sigma < 1)$ and satisfies the inequality $|\psi(x) - \varphi(x)| < \delta$ on Γ.

Since $\tilde{u}_2(x) + \delta$ and $\tilde{u}_1(x) - \delta$ are a quasi-supersolution and a quasi-subsolution in G, there exist two solutions $u_1(x)$ and $u_2(x)$ of the Dirichlet problem $[D_\psi]$ such that

$$u_1(x) \geqq \tilde{u}_1(x) - \delta \quad \text{and} \quad u_2(x) \leqq \tilde{u}_2(x) + \delta$$

in G. But at the maximum point x_0 of the function $\tilde{u}_1(x) - \tilde{u}_2(x)$ we get

$$u_2(x_0) \leqq \tilde{u}_2(x_0) + \delta < \tilde{u}_1(x_0) - \delta \leqq u_1(x_0),$$

which shows that $u_1(x)$ and $u_2(x)$ are mutually different solutions of the Dirichlet problem $[D_\psi]$.

Corollary 3. *Let $\varphi(x)$ be in $C^{2,\sigma}(\bar{G})$ for some σ, $0 < \sigma < 1$ and assume that the solution of the Dirichlet problem $[D_\varphi]$ is unique. Let $u(x)$ be the unique solution of the problem $[D_\varphi]$ and let $\tilde{u}(x)$ be any solution in $C^2(\bar{G})$ of the differential equation* (1). *If $\varepsilon \geqq 0$ and*

$$\varepsilon \geqq \tilde{u}(x) - \varphi(x) \qquad (\tilde{u}(x) - \varphi(x) \geqq -\varepsilon)$$

on the boundary Γ of G, then the inequality

$$\varepsilon \geqq \tilde{u}(x) - u(x) \qquad (\tilde{u}(x) - u(x) \geqq -\varepsilon)$$

is valid in $\bar{G}$.

REMARK. This corollary indicates that the uniqueness of solutions induces the weak maximum principle. (The converse is evidently true.)

PROOF. Suppose $\tilde{u}(x_0) > u(x_0)+\varepsilon$ at some point x_0 of G. Since $\tilde{u}(x)-\varepsilon$ is a quasi-subsolution in G there would exist a solution $v(x)$ of the Dirichlet problem $[D_\varphi]$ such that $v(x) \geqq \tilde{u}(x)-\varepsilon$. But, by virtue of the uniqueness of the Dirichlet problem $[D_\varphi]$, we would get $v(x) \equiv u(x)$ and hence $u(x_0) \geqq \tilde{u}(x_0)-\varepsilon$, which is absurd.

4. An example: We shall consider the following

Dirichlet problem $[D]$:

Domain G: $1 < |x| < 3$,

Differential equation: $\Delta u = (n-1)x_k \partial u/\partial x_k / |x|^2 + \sqrt{|\operatorname{grad} u|}$ [13],

Boundary condition: $u(x) = 0$ on $|x| = 1$ and $|x| = 3$.

Then we get the following results:

Maximal solution: $u_{\max}(x) \equiv 0$.

Minimal solution: $u_{\min}(x) = \{||x|-2|^3-1\}/12$.

A family of solutions

$$u_a(x) = \begin{cases} \{||x|-1-a|^3-a^3\}/12 & \text{for } 1 \leqq |x| \leqq 1+a, \\ 0 & \text{for } 1+a < |x| < 3-a, \\ \{||x|-3+a|^3-a^3\}/12 & \text{for } 3-a \leqq |x| \leqq 3, \end{cases}$$

with $0 < a < 1$ *fill the gap* between $u_{\max}(x)$ and $u_{\min}(x)$ when a moves in the interval $(0, 1)$.

PROOF. Let $u(x)$ be any solution in $C^2(\bar{G})$ of the problem $[D]$. Then, by virtue of Lemma B, we get $u(x) \leqq 0$. Hence the identically zero function is the maximal solution of $[D]$.

Next let ε be an arbitrary small positive number and consider the function

$$v_\varepsilon(x) \equiv \frac{1}{12}\{||x|-2|^3-1\} + \frac{\varepsilon}{2}\{(|x|-2)^2-1\}.$$

The function $v_\varepsilon(x)$ satisfies in G the differential inequality

$$\Delta v_\varepsilon - (n-1)x_k \partial v_\varepsilon/\partial x_k/|x|^2 = v''_\varepsilon = \frac{1}{2}||x|-2|+\varepsilon$$

$$> \sqrt{\frac{1}{4}(|x|-2)^2+\varepsilon||x|-2|} \geqq \sqrt{|v'_\varepsilon|} = \sqrt{|\operatorname{grad} v_\varepsilon|},$$

where we denote by v'_ε and v''_ε the derivatives of v_ε with respect to $r = |x|$. Again, let $u(x)$ be any solution of $[D]$. Then we get

$$v_\varepsilon(x) \leqq u(x)$$

13) Δ is the Laplacian $\delta_{ij}\partial^2/\partial x_i \partial x_j$.

in $\bar{G}$, for otherwise we would get an absurdity

$$\Delta(u-v_\varepsilon)>0$$

at positive maximum points of the function $u(x)-v_\varepsilon(x)$. Thus we have shown that

$$u_{\min}(x)=\{||x|-2|^3-1\}/12$$

is the minimal solution of the problem [D].

University of Tokyo

Bibliography

As for the earlier references in the same field as in this note the reader may consults with C. Miranda [**9**] and L. Nirenberg [**11**]. After the author had prepared the present note he could get the splendid article [**1**]. The article [**1**] develops an exhaustive study on the estimations of general elliptic differential equations of arbitary order and is very important.

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Part I, Comm. Pure Appl. Math., **12** (1959), 623-727.

[2] L. Bers and L. Nirenberg, On linear and non-linear elliptic boundary value problems in the plane, Convegno Internazionale sulle Equazioni Derivate Parziali, Trieste, (1954), 141-167.

[3] H. O. Cordes, Über die erste Randwertaufgabe bei quasi-linearer Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen, Math. Ann., **131** (1956), 278-312.

[4] N. Dunford and J. T. Schwartz, Linear Analysis, Part I, Interscience, New York, 1958.

[5] I. Hirai and K. Akô, On generalized Peano's theorem concerning the Dirichlet problem of semi-linear elliptic differential equations of the second order, to appear.

[6] E. Hopf, Elementare Bemerkungen zweiter Ordnung vom elliptischen Typus, Sitzungsberichte Preuss. Akad. Wiss. Berlin, (1927), 145-152.

[7] M. Hukuhara, Jô Bibun Hôteishiki (Ordinary Differential Equations), 1950 (Iwanami, Tokyo, in Japanese).

[8] E. Kamke, Differentialgleichungen, Lösungsmethoden und Lösungen, Akademie Verlag, Leibzig, 1943.

[9] C. Miranda, Equazioni alle Derivate Parziali di Tipo Ellitico, Springer, Berlin, 1955.

[10] M. Nagumo, On principally linear elliptic differential equations of the second order, Osaka Math. J., **6**, (1954), 207-229.

[11] L. Nirenberg, On nonlinear elliptic differential equations and Hölder continuity, Comm. Pure Appl. Math., **6** (1953), 103-156.

[12] L. Nirenberg, Estimates and Existence of solutions of elliptic equations, Comm. Pure Appl. Math., **9** (1956), 509-530.

[13] J. Schauder, Über lineare elliptische Differentialgleichungen zweiter Ordnung, Math. Z., **38** (1934), 257-282.

[14] J. Schauder, Numerische Abschätzungen in elliptischen linearen Differentialgleichungen, Studia Math., **5** (1934), 35-42.