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Some theorems on almost Kählerian spaces.

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An almost Hermitian structure is called an almost Kählerian structure, if the associated differential form $\omega = F_{ji}\, dx^j \wedge dx^i$ is closed, and the space with an almost Kählerian structure is called an almost Kählerian space. (See K. Yano [**10**][1].)

In the present paper, we remark that an almost Kählerian space is not necessarily only the space which might be called by this name, and assert that there exists a more general space which may be called by this name. Furthermore, we define a certain almost Hermitian space (called an almost semi-Kählerian space) having some interesting characters. Some results on almost Kählerian spaces, for example, those on almost analytic vectors (S. Tachibana [7] and [8]), may be generalized to these new spaces.

In § 1 we shall prove some identities valid in an almost Hermitian space, and obtain a necessary and sufficient condition that an almost Hermitian space is Hermitian. In § 2 we shall define various almost Kählerian spaces, and deduce some identities and theorems in these spaces. In § 3 we shall discuss the curvatures in almost (semi-) Kählerian spaces. In § 4 we shall define an almost analytic tensor which is a generalization of an analytic tensor in a Kählerian space and consider contravariant almost analytic vectors in almost (semi-) Kählerian spaces. In the last section, we shall give necessary and sufficient conditions that a contravariant vector is almost analytic in almost (semi-) Kählerian spaces.

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§ 1. Almost Hermitian spaces.

Let X_{2n} be a $2n$-dimensional real differentiable space of class C^∞ with local coordinates $\{x^i\}$ admitting an almost complex structure defined by the tensor

1) The numbers between brakets refer to the Biblography at the end of this paper.

field $F_i^{\ h}$ satisfying

(1.1) $$F_i^{\ l}F_l^{\ h} = -A_i^h\,, \qquad h, i, \cdots = 1, 2, \cdots, 2n\,,$$

where A_i^h denotes the unit tensor.[2)]

It is a well known fact that a space X_{2n} with an almost complex structure $F_i^{\ h}$ always admits a positive definite Riemannian metric such that

(1.2) $$F_i^{\ m}F_h^{\ l}g_{ml} = g_{ih}\,.$$

By an almost Hermitian space we shall mean a space which admits an almost complex structure $F_i^{\ h}$ and an almost Hermitian metric g_{ih}, that is, a space which satisfies (1.1) and (1.2). It is easily seen that $F_{ih} = F_i^{\ l}g_{lh}$ is anti-symmetric in its lower indices.

In an almost Hermitian space, we define the following linear operators operating on the tensors

(1.3) $$O_{ih}^{ml} = \frac{1}{2}(A_i^m A_h^l - F_i^{\ m}F_h^{\ l})\,,$$

(1.4) $$*O_{ih}^{ml} = \frac{1}{2}(A_i^m A_h^l + F_i^{\ m}F_h^{\ l})\,.$$

A tensor is called pure (hybrid) in two indices if it is annihilated by transvection of $*O$ (O) on these indices.

Since (1.2) can be written in the form

$$O_{ih}^{ml}g_{ml} = 0\,,$$

the metric tensor g_{ih} is hybrid in i and h.

From (1.1), we have

(1.5) $$F_i^{\ l}\nabla_j F_l^{\ h} + F_l^{\ h}\nabla_j F_i^{\ l} = 0\,,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Riemannian connection. Thus from (1.2) and the last equation, we find that F_{ih} is hybrid in i and h, but $\nabla_j F_{ih}$ is pure in i and h, that is,

(1.6) $$O_{ih}^{ml}F_{ml} = 0\,, \qquad \text{or} \qquad *O_{ih}^{ml}F_{ml} = F_{ih}\,,$$

(1.7) $$*O_{ih}^{ml}\nabla_j F_{ml} = 0\,, \qquad \text{or} \qquad O_{ih}^{ml}\nabla_j F_{ml} = \nabla_j F_{ih}\,.$$

For the two operators with the same indices, we have

(1.8) $$O_{ih}^{ml}O_{ml}^{ts} = O_{ih}^{ts}\,, \qquad *O_{ih}^{ml}{*O}_{ml}^{ts} = *O_{ih}^{ts}\,,$$

$$O_{ih}^{ml}{*O}_{ml}^{ts} = *O_{ih}^{ml}O_{ml}^{ts} = 0\,, \qquad *O_{ih}^{ml} + O_{ih}^{ml} = E\,,$$

where E denotes an identity operator.

The Nijenhuis tensor of an almost complex structure is defined by

(1.9) $$N_{ji}^{\ \ h} = F_j^{\ l}(\nabla_l F_i^{\ h} - \nabla_i F_l^{\ h}) - F_i^{\ l}(\nabla_l F_j^{\ h} - \nabla_j F_l^{\ h})\,.$$

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2) Notations are those of K. Yano [10].

From the definition we easily find that $N_{ji}{}^{h}$ is antisymmetric in j and i.

Now, in an almost Hermitian space, if we put

$$(1.10)\qquad 2P_{jih}=N_{jih}-N_{jhi}-N_{ihj}\,,\qquad N_{jih}=N_{ji}{}^{l}g_{lh}\,,$$

then from the above definitions and (1.5), we find

$$(1.11)\qquad P_{jih}=F_j{}^l\nabla_l F_{ih}+F_i{}^l\nabla_j F_{lh}\,,$$

$$(1.12)\qquad N_{jih}=P_{jih}-P_{ijh}\,.$$

If, in an almost Hermitian space, the Nijenhuis tensor vanishes identically, then the space is called a Hermitian space.

Now we shall obtain a necessary and sufficient condition that an almost Hermitian space is a Hermitian space.

In a Hermitian space, from the definition, on taking account of (1.10), we find $P_{jih}=0$, i. e.,

$$(1.13)\qquad F_j{}^l\nabla_l F_{ih}+F_i{}^l\nabla_j F_{lh}=0\,,\qquad \text{or}\qquad O_{ji}^{ml}\nabla_m F_l{}^h=0\,.$$

Conversely, if, in an almost Hermitian space, (1.13) holds good, then from (1.12), we get $N_{ji}{}^{h}=0$. Thus we have

THEOREM 1.1. *A necessary and sufficient condition that an almost Hermitian space is a Hermitian space is that $\nabla_j F_{ih}$ is hybrid in j and i, that is,*

$$O_{ji}^{ml}\nabla_m F_l{}^h=0\,.$$

§ 2. Almost Kählerian spaces.

I). Definitions.

Theorem 1.1 suggests us that there exists a new almost Kählerian space, so we state;

In an almost Hermitian space, if its structure tensor $F_i{}^h$ satisfies

$$(2.1)\qquad {}^*O_{ji}^{ml}\nabla_m F_l{}^h=0\,,\qquad \text{or}\qquad O_{ji}^{ml}\nabla_m F_l{}^h=\nabla_j F_i{}^h\,,$$

then we shall call the space an **O*-almost Kählerian space (or briefly an **O*-space) and if it satisfies

$$(2.2)\qquad F_i\equiv\nabla_l F_i{}^l=0\,,$$

or

$$(2.3)\qquad F_{jih}\equiv\nabla_j F_{ih}+\nabla_i F_{hj}+\nabla_h F_{ji}=0\,,$$

or

$$(2.4)\qquad \nabla_j F_{ih}+\nabla_i F_{jh}=0\,,$$

then we shall call the space an almost semi-Kählerian space, or an H-almost Kählerian space (an H-space[3]), or a K-almost Kählerian space (a K-space[4]),

3) Usually an H-space is called an almost Kählerian space. K. Yano [10, p. 231].
4) S. Tachibana [8].

respectively.

Now, in these spaces we shall deduce some identities and theorems which are useful in the later sections.

Let $K_{kji}{}^{h}$ be the Riemannian curvature tensor;

$$K_{kji}{}^{h}=\partial_k\{{}_j{}^h{}_i\}-\partial_j\{{}_k{}^h{}_i\}+\{{}_k{}^h{}_l\}\{{}_j{}^l{}_i\}-\{{}_j{}^h{}_l\}\{{}_k{}^l{}_i\}\,. \tag{2.5}$$

where $\partial_k=\partial/\partial x^k$, and

$$K_{ji}=K_{lji}{}^{l}\,,\qquad K=g^{ml}K_{ml}\,,\qquad K_{kjih}=K_{kji}{}^{l}g_{lh}\,, \tag{2.6}$$

$$H_{ji}=\frac{1}{2}F^{ml}K_{mlji}{}^{5)}\,,\qquad H=-F^{ml}H_{ml}\,. \tag{2.7}$$

Applying the Ricci formulae to $F_i{}^h$, we have the following identities which are valid in an almost Hermitian space;

$$\nabla_k\nabla_jF_i{}^h-\nabla_j\nabla_kF_i{}^h=K_{kjl}{}^{h}F_i{}^l-K_{kji}{}^{l}F_l{}^h\,, \tag{2.8}$$

$$F^{ml}\nabla_m\nabla_lF_{ih}=F_i{}^lH_{lh}+F_h{}^lH_{il}\,. \tag{2.9}$$

And the following identities are well known;

$$\underset{v}{\pounds}\,\nabla_jF_i{}^h-\nabla_j\underset{v}{\pounds}\,F_i{}^h=F_i{}^l\underset{v}{\pounds}\,\{{}_j{}^h{}_l\}-F_l{}^h\underset{v}{\pounds}\,\{{}_j{}^l{}_i\}\,, \tag{2.10}$$

$$\underset{v}{\pounds}\,\{{}_j{}^h{}_i\}=\frac{1}{2}g^{hl}[\nabla_j\underset{v}{\pounds}\,g_{li}+\nabla_i\underset{v}{\pounds}\,g_{jl}-\nabla_l\underset{v}{\pounds}\,g_{ji}]\,, \tag{2.11}$$

where $\underset{v}{\pounds}$ denotes the operator of Lie differentiation with respect to v^i.

II) Almost semi-Kählerian spaces.

Now, we shall assume we are in an almost semi-Kählerian space.

From (2.8), by virtue of the definition (2.2), we find

$$\nabla_l\nabla_jF_i{}^l=F_i{}^lK_{jl}-H_{ji}\,. \tag{2.12}$$

From (1.1) and (2.2), we find

$$F^{ji}\nabla_jF_{ih}=\nabla_j(F^{ji}F_{ih})=0\,. \tag{2.13}$$

Operating $\nabla^h=g^{lh}\nabla_l$ to the last equation and taking account of (2.12), we have

$$H-K=(\nabla_hF^{ml})(\nabla_lF_m{}^h)\,, \tag{2.14}$$

which is useful in § 3.

On the other hand, from (2.10), by virtue of (2.2), we find

$$-\nabla_l\underset{v}{\pounds}\,F_i{}^l=F_i{}^m\underset{v}{\pounds}\,\{{}_m{}^l{}_l\}-F_m{}^l\underset{v}{\pounds}\,\{{}_l{}^m{}_i\}\,, \tag{2.15}$$

and substituting (2.11) in the last equation, we get

$$-\nabla_l\underset{v}{\pounds}\,F_i{}^l=\frac{1}{2}F_i{}^mg^{ls}(\nabla_m\underset{v}{\pounds}\,g_{ls})+F^{ml}(\nabla_m\underset{v}{\pounds}\,g_{li})\,, \tag{2.16}$$

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5) This notation differs from K. Yano's only in a factor. K. Yano [10, p. 235].

which are useful in § 4.

III) *O-spaces.

In an *O-space, by definition (2.1), we have

$$\nabla_j F_i^{\ h} + F_j^{\ m} F_i^{\ l} \nabla_m F_l^{\ h} = 0 . \tag{2.17}$$

Transvecting the last equation with g^{ji} and on taking account of (1.2), we find $F_i = 0$. Thus we have

THEOREM 2.1. *An *O-space is an almost semi-Kählerian space.*

Next, since in an *O-space the tensor $\nabla_j F_{ih}$ is pure in its lower indices, we find that F_{jih} is also pure in its lower indices, i. e.,

$$F_j^{\ l} F_{lih} = F_i^{\ l} F_{jlh} = F_h^{\ l} F_{jil} . \tag{2.18}$$

From the definition (2.1), we find

$$F_j^{\ l} \nabla_l F_{ih} = F_h^{\ l} \nabla_j F_{il} .$$

Operating ∇^h to the last equation and taking account of (2.12), we obtain

$$(\nabla_h F_j^{\ l})(\nabla_l F_i^{\ h}) = F^{hl} \nabla_h \nabla_j F_{il} - F_j^{\ m} F_i^{\ l} K_{ml} + F_j^{\ l} H_{li} . \tag{2.19}$$

Now, in an *O-space, if the Nijenhuis tensor vanishes, then from (1.10), we have $P_{jih} = 0$, i. e.,

$$F_j^{\ l} \nabla_l F_{ih} + F_i^{\ l} \nabla_j F_{lh} = 0 ,$$

from which taking account of (2.1), we get $\nabla_j F_i^{\ h} = 0$. Conversely, it is easily seen that the Nijenhuis tensor vanishes if $\nabla_j F_i^{\ h} = 0$.

Thus we find that in an *O-space, the conditions $\nabla_j F_i^{\ h} = 0$ and $N_{ji}^{\ \ h} = 0$ are equivalent to each other.

IV) H-spaces.

Now, we shall assume we are in an H-space. If we put

$$T_{jih} = 2{}^*O_{ji}^{ml} \nabla_m F_{lh} ,$$

then from the definition (2.3), we have

$$\begin{aligned} T_{jih} &= \nabla_j F_{ih} + F_j^{\ m} F_i^{\ l} \nabla_m F_{lh} \\ &= -(\nabla_i F_{hj} + \nabla_h F_{ji}) - F_j^{\ m} F_i^{\ l} (\nabla_l F_{hm} + \nabla_h F_{ml}) \\ &= \nabla_i F_{jh} + F_i^{\ l} F_j^{\ m} \nabla_l F_{mh} \\ &= T_{ijh} , \end{aligned}$$

by virtue of (1.7). On the other hand, taking account of (1.5), we have

$$\begin{aligned} T_{jih} &= -\nabla_j F_{hi} - F_j^{\ m} F_h^{\ l} \nabla_m F_{li} \\ &= -T_{jhi} . \end{aligned}$$

It is a well-known fact that if a tensor T_{jih} is symmetric in j and i, and antisymmetric in i and h, then T_{jih} is the zero tensor.

Thus we obtain ${}^*O_{ji}^{ml}\nabla_m F_{lh}=0$. Hence we have

THEOREM 2.2. *An* H-*space is an *O-space.*

V) K-spaces.

In a K-space, from the definition (2.4), we find

$$T_{jih}=\nabla_j F_{ih}+F_j{}^m F_i{}^l \nabla_m F_{lh}$$
$$=-(\nabla_h F_{ij}+F_j{}^m F_i{}^l \nabla_h F_{lm}),$$

thus on taking account of (1.7), we find $T_{jih}=0$. Hence we have

THEOREM 2.3. *A* K-*space is an *O-space.*

Next, operating ∇^h to (2.4), we obtain

$$F_j{}^l K_{li}+F_i{}^l K_{jl}=0, \quad \text{or} \quad O_{ji}^{ml}K_{ml}=0, \tag{2.20}$$

which shows that in a K-space the Ricci tensor K_{ji} is hybrid in j and i.

From (2.19), taking account of (2.4), (2.9) and (2.20), we obtain

$$(\nabla_j F^{ml})(\nabla_i F_{ml})=K_{ji}-2F_j{}^l H_{li}-F_i{}^l H_{jl}.$$

As the left hand side of the last equation is symmetric in j and i, we find

$$F_j{}^l H_{li}+F_i{}^l H_{jl}=0, \quad \text{or} \quad O_{ji}^{ml}H_{ml}=0, \tag{2.21}$$

which shows that in a K-space H_{ji} is hybrid in j and i.

Consequently, we have

$$(\nabla_j F^{ml})(\nabla_i F_{ml})=K_{ji}-F_j{}^l H_{li}. \tag{2.22}$$

Now, we state;

THEOREM 2.4. *A necessary and sufficient condition that an *O-space is a* K-*space is that the Nijenhuis tensor N_{jih} is antisymmetric in its all lower indices*[6].

In fact, if in an *O-space the Nijenhuis tensor N_{jih} is antisymmetric in its all indices, then from (1.9) and (2.1), we have

$$0=N_{jih}+N_{jhi}=2F_j{}^l(\nabla_i F_{hl}+\nabla_h F_{il}),$$

from which we have (2.4). Conversely, in a K-space, since the Nijenhuis tensor has the form

$$N_{jih}=4F_j{}^l \nabla_l F_{ih}, \tag{2.23}$$

which shows that N_{jih} is antisymmetric in i and h, thus it is antisymmetric in its all indices. q.e.d.

Furthermore we can obtain the following;

THEOREM 2.5. *A necessary and sufficient condition that a* K-*space is Kählerian*

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6) If, in an almost Hermitian space, the Nijenhuis tensor N_{jih} is antisymmetric in its all lower indices, then the space is called a half-Hermitian space. Thus, we see that as an almost Hermitian space corresponds to an *O-space so a half-Hermitian space corresponds to a K-space. S. Sawaki and S. Kotō [5].

is that $F_i^{\ h}$ *satisfies*

$$K_{ji}=F_j^{\ l}H_{li}\,. \tag{2.24}$$

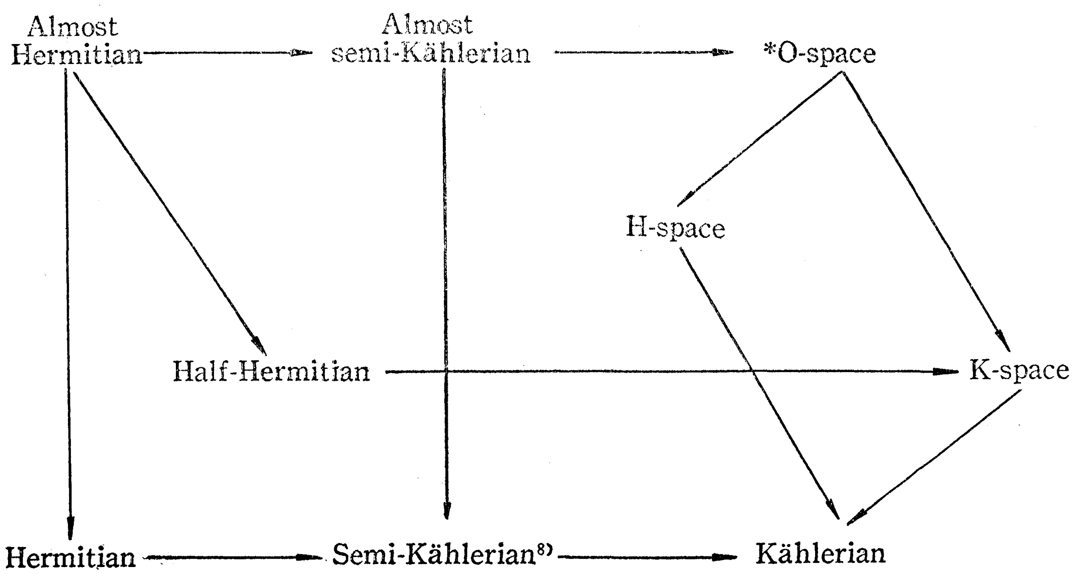
In fact, if a K-space satisfies (2.24), then from (2.22), we find

$$(\nabla_j F^{ml})(\nabla_i F_{ml})=0\,,$$

hence we have $\nabla_j F_i^{\ h}=0$. Converely, in a Kählerian space, since $\nabla_j F_i^{\ h}=0$ holds good, thus from (2.12), we have (2.24). q. e. d.

VI) Diagram.

The relation between these spaces may be seen in the following diagram[7]:



§ 3. Curvatures.

In this section, we shall assume we are in an almost semi-Kählerian space. From (2.14), we have

Theorem 3.1. *In an almost semi-Kählerian space, if it satisfies*

$$(\nabla_h F_m^{\ l})(\nabla_l F^{mh})\geqq 0\,, \qquad (\leqq 0)\,,$$

then the inequality $K\leqq H$ $(K\geqq H)$ *is valid.*

Especially, in an H-space, we have

$$\begin{aligned}(\nabla^h F^{ml})(\nabla_l F_{mh}) &= \frac{1}{2}(\nabla^h F^{ml})(\nabla_l F_{mh}-\nabla_m F_{lh}) \\ &= \frac{1}{2}(\nabla^h F^{ml})(\nabla_h F_{ml})\geqq 0\,,\end{aligned} \tag{3.1}$$

7) Cf., K. Yano [**10**, p. 231].

8) A semi-Kählerian space has remarkable characters and M. Apte [**1**] has considered such a space. We shall discuss this space in another place.

by virtue of (2.3).

In a K-space, by definition (2.4), we have

$$(3.2)\qquad (\nabla_h F_m{}^l)(\nabla_l F^{mh}) = -(\nabla^h F^{ml})(\nabla_h F_{ml}) \leqq 0\,.$$

Thus we have

COROLLARY[9]. *In an* H-*space* (*a* K-*space*), *the relation* $K \leqq H$ ($K \geqq H$) *holds. The equality holds if and only if the space is Kählerian.*

Now, we assume that an almost semi-Kählerian space is conformally flat, so that the curvature tensor has the form[10];

$$(3.3)\qquad 2(n-1)K_{kjih} = (K_{ji}g_{kh} - K_{jh}g_{ki} + K_{kh}g_{ji} - K_{ik}g_{jh}) - \frac{K}{(2n-1)}(g_{ji}g_{kh} - g_{ki}g_{jh})\,.$$

Transvecting this equation with $F^{kj}F^{ih}$, we find

$$(2n-1)(K-H) = 2(n-1)K\,.$$

Thus, using the Theorem 3.1, we have

THEOREM 3.2. *In an almost semi-Kählerian space, if the relation*

$$(\nabla_h F^{ml})(\nabla_l F_m{}^h) \geqq 0\,, \qquad (\leqq 0)\,,$$

holds, then there does not exist a conformally flat almost semi-Kählerian space with $K>0$ ($K<0$).

COROLLARY[11]. *There does not exist a conformally flat* H-*space* (K-*space*) *with* $K>0$ ($K<0$).

COROLLARY. *In an almost semi-Kählerian space, if the relation*

$$(\nabla_h F^{ml})(\nabla_l F_m{}^h) \geqq 0\,, \qquad (\leqq 0)\,,$$

holds, then there does not exist an almost semi-Kählerian space of positive (*negative*) *constant curvature.*

COROLLARY[11]. *There does not exist an* H-*space* (K-*space*) *of positive* (*negative*) *constant curvature.*

§ 4. Almost analytic vectors.

In a $2n$-dimensional Riemannian space, if a vector field v^i satisfies each of the following conditions;

$$(4.1)\qquad \underset{v}{\pounds}\, g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 0\,,$$

or

$$(4.2)\qquad \underset{v}{\pounds}\, g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 2\phi g_{ji}\,,$$

9) S. Tachibana [8] and [9].

10) J. A. Schouten [6].

11) S. Tachibana [8] and [9].

or

$$\underset{v}{\pounds}\left\{ {}_j{}^h{}_i \right\} \equiv \nabla_j \nabla_i v^h + K_{lji}{}^h v^l = A_j{}^h \psi_i + A_i{}^h \psi_j , \tag{4.3}$$

then it is called a Killing vector, a conformal Killing vector, a projective Killing vector, respectively, where

$$\phi = \frac{1}{2n} \nabla_l v^l , \qquad \psi_i = \frac{1}{2n+1} \nabla_i \nabla_l v^l .$$

It is a well known fact that, in a compact orientable Riemannian space, a necessary and sufficient condition that a vector field v^i is a Killing vector is that it satisfies

$$g^{ml} \nabla_m \nabla_l v^i + K_l{}^i v^l = 0 \tag{4.4}$$

and

$$\nabla_l v^l = 0 . \tag{4.5}$$

Now, in an almost semi-Kählerian space, we shall say that a tensor field $T_{j_p \cdots j_1}{}^{i_q \cdots i_1}$, which is pure in all its indices, is an almost analytic tensor if it satisfies[12];

$$\begin{aligned} F_h{}^l \nabla_l T_{j_p \cdots j_1}{}^{i_q \cdots i_1} &- \nabla_h (F_{j_1}{}^l T_{j_p \cdots j_2 l}{}^{i_q \cdots i_1}) \\ &+ \sum_{r=1}^{p} (\nabla_{j_r} F_h{}^l) T_{j_p \cdots \underset{(r)}{l} \cdots j_1}{}^{i_q \cdots i_1} \\ &- \sum_{s=1}^{q} (\nabla_l F_h{}^{i_s} - \nabla_h F_l{}^{i_s}) T_{j_p \cdots j_1}{}^{i_q \cdots \underset{(s)}{l} \cdots i_1} = 0 . \end{aligned} \tag{4.6}$$

We notice that this formula is independent of the connection, that is, it is a differential concomitant[13].

In the next, we shall consider a contravariant almost analytic vector in a $2n$-dimensional compact almost semi-Kählerian space.

From (4.6), we have for a contravariant almost analytic vector (or briefly an analytic vector) v^i,

$$\underset{v}{\pounds} F_j{}^i \equiv v^l \nabla_l F_j{}^i - F_j{}^l \nabla_l v^i + F_l{}^i \nabla_j v^l = 0 . \tag{4.7}$$

If a conformal Killing vector v^i is at the same time analytic, then substituting (4.2) and (4.7) in (2.16), we find

$$(n-1) F_i{}^m \nabla_m \nabla_l v^l = 0 .$$

From which we get $\nabla_i \nabla_l v^l = 0$, $(n > 1)$.

As the space is compact, using the Green's theorem, we deduce $\nabla_l v^l = 0$. Thus we have

12) This definition is valid in an almost complex space.

13) A. Nijenhuis [4]. J. A. Schouten [6].

THEOREM 4.1[14]. *In a $2n$-dimensional compact almost semi-Kählerian space $(n>1)$, if a conformal Killing vector is at the same time analytic, then it is a Killing vector.*

For a projective Killing vector v^i which is at the same time analytic, from (2.15), we find

$$F_i^m \underset{v}{\pounds} \{{}_m{}^l{}_l\} - F_m{}^l \underset{v}{\pounds} \{{}_l{}^m{}_i\} = 0 .$$

Substituting (4.3) in the last equation, we get $\nabla_i\nabla_l v^l = 0$, therefore, as the space is compact, we have (4.4) and (4.5), that is, the vector becomes a Killing one. Thus we have

THEOREM 4.2[14]. *In a compact almost semi-Kählerian space, if a projective Killing vector is at the same time analytic, then it is a Killing vector.*

§ 5. Integral formulae.

Now, in a compact almost semi-Kählerian space X_{2n}, we shall obtain a necessary and sufficient condition that a contravariant vector field v^i is analytic. For analytic vector v^i, from (4.7), we find

$$v^l\nabla_l F^{ji} - F^{jl}\nabla_l v^i + F_l{}^i\nabla^j v^l = 0 . \tag{5.1}$$

Operating ∇_j to the last equation and taking account of (2.9) and (2.12), we have

$$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l + F_i{}^l(\nabla^s v^m)(\nabla_m F_{sl} + \nabla_s F_{ml}) = 0 . \tag{5.2}$$

On the other hand, multiplying (5.1) by $\frac{1}{2}(\nabla_h F_{ji}) + \nabla_j F_{ih}$, and contracting, we obtain

$$\frac{1}{2} v^s F_{iml}(\nabla_s F^{ml}) + (\nabla^s v^m)(F_s{}^l\nabla_i F_{lm} + F_s{}^l\nabla_l F_{mi} + F_i{}^l\nabla_s F_{ml}) = 0 .$$

Subtracting the last equation from (5.2), we have

$$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l - \frac{1}{2} F_{iml}(\underset{v}{\pounds} F^{ml}) = 0 . \tag{5.3}$$

This equation is a necessary condition for a vector v^i to be analytic.

Next, we shall get a sufficient condition. For a vector field v^i, if we put

$$a_{ji} \equiv (\underset{v}{\pounds} F_j{}^l)F_{li} = v^m(\nabla_m F_j{}^l)F_{li} + F_j{}^m F_i{}^l\nabla_m v_l - \nabla_j v_i$$

then we have

$$\frac{1}{2} a_{ji}a^{ji} = (\nabla_j v_i)(\nabla^j v^i) + \frac{1}{2}(\nabla_s F_{jt})(\nabla_m F^{jt})v^s v^m \tag{5.4}$$
$$+ v^i F^{jt}(\nabla_i F_j{}^l)(\nabla_t v_l) + F_i{}^l(\nabla_s F_{jl})(\nabla^j v^i)v^s - F_j{}^m F_i{}^l(\nabla_m v_l)(\nabla^j v^i) ,$$

and

14) For an H-space or a K-space, see S. Tachibana [7] and [8].

(5.5) $$\nabla^j(a_{ji}v^i) = -[g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l - (\nabla^j v^m)(\nabla_m F_i^{\ l})F_{jl} + (\nabla_m F_j^{\ l})(\nabla^j F_{il})v^m - F_j^{\ m}(\nabla^j F_i^{\ l})(\nabla_m v_l)]v^i - F_i^{\ l}(\nabla_m F_{jl})(\nabla^j v^i)v^m + F_j^{\ m}F_i^{\ l}(\nabla_m v_l)(\nabla^j v^i) - (\nabla^j v^i)(\nabla_j v_i).$$

From (5.4) and (5.5), we get

$$\frac{1}{2}a_{ji}a^{ji} + \nabla^j(a_{ji}v^i) = -\Big[g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l - \frac{1}{2}F_{iml}(\underset{v}{\pounds}F^{ml})\Big]v^i.$$

Hence, applying the Green's theorem, we have

LEMMA[15]. *In a compact almost semi-Kählerian space X_{2n}, the integral formula*

(5.6) $$\int_{X_{2n}}\Big[\Big\{g_m\nabla_m\nabla_l v_i + K_{li}v^l - \frac{1}{2}F_{iml}(\underset{v}{\pounds}F^{ml})\Big\}v^i + \frac{1}{2}a_{ji}a^{ji}\Big]d\sigma = 0,$$

is valid for any vector field v^i, where $d\sigma$ means the volume element of the X_{2n}, and $a_{ji} = (\underset{v}{\pounds}F_j^{\ l})F_{li}$.

From this lemma, we have

THEOREM 5.1. *In a compact almost semi-Kählerian space, a necessary and sufficient condition that a contravariant vector v^i is analytic is that it satisfies* (5.3), *i. e.,*

$$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l - \frac{1}{2}F_{iml}(\underset{v}{\pounds}F^{ml}) = 0.$$

In an *O-space, for an analytic vector, the third term of (5.3) may be written in the form;

$$-\frac{1}{2}F_{iml}(\underset{v}{\pounds}F^{ml}) = -\frac{1}{2}F_{iml}F_s^{\ l}\underset{v}{\pounds}g^{sm}$$
$$= -\frac{1}{2}F_i^{\ l}F_{lms}\underset{v}{\pounds}g^{sm}$$
$$= 0,$$

by virtue of (2.18) and (4.7). Thus we have

THEOREM 5.2. *In a compact *O-space, a necessary and sufficient condition that a contravariant vector v^i is analytic is that it satisfies*

(5.7) $$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l = 0$$

and

(5.8) $$F_{iml}\underset{v}{\pounds}F^{ml} = 0.$$

Next, in an *O-space, for an analytic vector using (5.7), we find

$$\nabla^j(v^i\nabla_j v_i) = (\nabla^j v^i)(\nabla_j v_i) - K_{ji}v^j v^i,$$

Hence, by Green's theorem, we have

15) For an H-space or a K-space, see S. Tachibana [7] and [8], and for Kählerian case, see A. Lichnerowicz [3], and also K. Yano [10, p. 238].

THEOREM[16] 5.3. *If a compact *O-space has a negative definite Ricci tensor, there does not exist an analytic vector field other than the zero vector.*

From (5.7) and taking account of (4.4) and (4.5), we have

COROLLARY. *In a compact *O-space, if an analytic vector v^i satisfies $\nabla_l v^l = 0$, then it is a Killing vector.*

In an H-space, F_{jih} is identically zero by the definition, hence we have

COROLLARY[17]. *In a compact* H*-space, a necessary and sufficient condition that a contravariant vector v^i is analytic is that it satisfies*

$$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l = 0 .$$

Next, in a K-space, from (2.22) and (2.23), the equation (5.8) becomes

$$\frac{1}{2} N_{iml}(\nabla^m v^l) + (K_{li} - F_l^{\ m} H_{mi})v^l = 0 ,$$

by virtue of $F_{jih} = 3\nabla_j F_{ih}$. Thus we have

COROLLARY[18]. *In a compact* K*-space, a necessary and sufficient condition that a contravariant vector v^i is analytic is that it satisfies*

$$g^{ml}\nabla_m\nabla_l v_i + K_{li}v^l = 0$$

and

$$\frac{1}{2} N_{iml}(\nabla^m v^l) + (K_{li} - F_l^{\ m} H_{mi})v^l = 0 .$$

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Bibliography

[1] Apte, M., Sur certains variétés hermitiques, C. R. Acad. Sci. Paris, **241** (1954)' 1091-1093.

[2] Bochner, S., Vector fields and Ricci curvature, Bull. Amer. Math. Soc., **2** (1946), 776-797.

[3] Lichnerowicz, A., Sur les transformations analytiques des variétés kähleriennes compactes, C. R. Acad. Sci. Paris, **244** (1957), 3011-3013.

[4] Nijenhuis, A., Jacobi-type identities for bilinear differential concomitants of certain tensor fields, I; II, Indag. Math., **17** (1955), 390-397; 398-403.

[5] Sawaki, S. and Kotō, S., On some F-connections in almost Hermitian manifolds, J. fac. sci. Niigata Univ., **1** (1958), 85-96.

[6] Schouten, J. A., Ricci-Calculus, second edition, Springer, 1954.

[7] Tachibana, S., On almost-analytic vectors in almost-Kählerian manifolds, Tôhoku Math. J., **11** (1959), 247-265.

[8] Tachibana, S., On almost-analytic vectors in certain almost Hermitian manifolds, Tôhoku Math. J., **11** (1959), 351-363.

[9] Tachibana, S., Note on conformally flat almost-Kählerian space, Ochanomizu Univ. sci. rep., **10** (1959), 41-43.

[10] Yano, K., The theory of Lie derivatives and its applications, Amsterdam, 1955.

16) For an H-space or a K-space, see S. Tachibana [7] and [8], and for Kählerian case, see S. Bochner [2], and also K. Yano [**10**, p. 237].

17) S. Tachibana [7].

18) S. Tachibana [8].