Journal of the Mathematical Society of Japan Vol. 12, No. 4. October, 1960

On a smoothing operator for the wave equation.

By J. S. MAYBEE[1)]

(Received Dec. 11, 1959)

1. Introduction. The Cauchy problem for the classical wave equation

$$\Box_n u=\Big(\frac{\partial^2}{\partial x_0{}^2}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i{}^2}\Big)u=f(x)$$

has been the subject of many investigations for more than half a century. During this time several formulas have been deduced for the solution of this problem with the Cauchy data given on the plane $x_0=0$. The difficulties in obtaining explicit formulas all center around the fact that the direct methods of integration lead to singular integrals. These difficulties have been overcome by various methods which either avoid singular integrals or select the appropriate "part" of such integrals. Among the latter methods the best known is that introduced by Hadamard and developed in his lectures on the Cauchy problem [**6**] (numbers in brackets refer to the bibliography at the end of the paper). This work has been extended by Bureau in a number of papers (see, for example, [**7**], [8]). Among the methods which seek to avoid singular integrals the most recent seem to be those of Weinstein [**9**], Diaz and Martin [**10**] and M. Riesz [**1**]. The work of Riesz on the wave equation has been extended by Garding [**2**] to the class of linear hyperbolic equations with constant coefficients. This method depends on the analytic continuation of certain integrals with respect to a complex parameter. Subsequently Leray [3] generalizing the Riesz-Garding method, showed that smoothing operators could be introduced in order to avoid singular integrals and re-derived the Riesz formulas for the wave equation.

In this paper we shall show that a suitable smoothing operator for the wave equation is simply $\partial^m/\partial x_0{}^m$ for m properly chosen and derive new formulas for the solution of the Cauchy problem with data given on the plane $x_0=0$. At the same time we shall show that it is not difficult to construct a solution of the equation

1) Work on this paper was supported, in part, by the Office of Naval Research. The author wishes to thank Professor P. C. Rosenbloom of the University of Minnesota for several stimulating conversations on this subject which resulted in the successful completion of the work.

$$\frac{\partial^m}{\partial x_0{}^m}\Box_n u = f(x) \tag{1}$$

for all values of m and n. It seems likely that a wide class of hyperbolic operators may be treated in a similar way by using the same smoothing operator. The present method appears to be simpler than the other devices known to the author.

2. The Riesz kernel for (1). In the remainder of this paper we shall use the following notation. A fixed point x in space-time, at which we are computing the solution of the Cauchy problem will be written $x=(x_0, x_1, \cdots, x_{n-1})$ while a variable space time point will be written $y=(y_0, y_1, \cdots, y_{n-1})$, y_0 corresponding to the time variable and $y_1, \cdots, y_{n-1}$ to the space variables. The retrograde light cone from the point x will be written D^x. The spacial part of any point will be written Px or Py, i. e. we shall sometimes write $x=(x_0, Px)$ or $y=(y_0, Py)$. The symbol r_x will mean $(\sum_{i=1}^{n-1} x_i^2)^{1/2}$ and r will mean $(\sum_{i=1}^{n-1}(x_i-y_i)^2)^{1/2}$.

Suppose ν is an arbitrary complex number satisfying the inequality $\Re(\nu) > n$, then the Riesz kernel for the wave equation is a function $k(x, \nu)$ with the following properties:

(a) $\Box_n k(x, \nu)=k(x, \nu-1)$,

(b) $k(x, \nu)$ vanishes on the surface of the forward light cone with vertex at the origin and everywhere outside.

To construct a Riesz kernel for equation (1) we seek a function $W(x, \mu, \nu)$ (μ and ν complex numbers with $\Re(\nu) > n$) with the following properties:

(a′) $\dfrac{\partial^m}{\partial x_0{}^m}\Box_n W(x, \mu, \nu) = W(x, \mu-1, \nu-1)$,

(b′) $\dfrac{\partial^m}{\partial x_0{}^m} W(x, \mu, \nu) = W(x, \mu-1, \nu)$,

(c′) $\Box_n W(x, \mu, \nu) = W(x, \mu, \nu-1)$,

(d′) $W(x, \mu, \nu)$ vanishes on the surface and outside of the forward light cone with vertex at the origin. Now, the light cone with vertex at the origin is described by the inequality $x_0^2-r_x^2 \geqq 0$, the equation $x_0^2-r_x^2=0$ giving its surface. We define

$$W(x, \mu, \nu) = \frac{1}{\Gamma(m\mu)}\int_{-\infty}^{x_0}(x_0-\tau)^{m\mu-1}k(\tau, Px, \nu)d\tau\,. \tag{2}$$

This is the classical Riemann-Liouville integral applied to the Riesz kernel for the wave equation. By making use of the properties of the Riemann-Liouville integral, some rather routine computations show that (a′), (b′), and (c′) are satisfied by $W(x, \mu, \nu)$ as defined by (2). We next introduce the Riesz kernel for the wave equation in the following form:

$$k(\tau, Px, \nu) = \begin{cases} \dfrac{(\tau^2 - r_x^2)^{\nu - n/2}\Gamma(\nu+1/2)}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\nu+1-n/2)} & \text{inside the light cone,} \\ 0 & \text{outside.} \end{cases}$$

Let $H_n(\nu) = \pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\nu+1-n/2)/\Gamma(\nu+1/2)$, then we compute (2) for $x_0^2 - x_x^2 \geqq 0, x_0 > 0$ and find

$$\Gamma(m\mu)H_n(\nu)W(x, \mu, \nu) = \int_{r_x}^{x_0} (x_0 - \tau)^{m\mu-1}(\tau^2 - r_x^2)^{\nu - n/2} d\tau\,. \tag{3}$$

In order to evaluate the integral in (3), make the substitution $v = (x_0 - \tau)/(x_0 - r_x)$. Then, after some computation, one finds [4]

$$\begin{aligned} \Gamma(m\mu)H_n(\nu)W(x, \mu, \nu) &= (x_0 - r_x)^\gamma (x_0 + r_x)^{-\alpha} \\ &\quad \times \int_0^1 v^{\beta-1}(1-v)^{-\alpha}\Big(1 - \frac{x_0 - r_x}{x_0 + r_x}\, v\Big)^{-\alpha} dv \\ &= \frac{(x_0 - r_x)^\gamma (x_0 + r_x)^{-\alpha}\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(\gamma+1)}\, F\Big(\alpha, \beta\,; \gamma+1\,; \frac{x_0 - r_x}{x_0 + r_x}\Big), \end{aligned}$$

where $\alpha = n/2 - \nu$, $\beta = m\mu$, $\gamma = m\mu + \nu - n/2$ and F is the hypergeometric function. Substituting for $H_n(\nu)$ its value, we find

$$W(x, \mu, \nu) = \frac{(x_0 - r_x)^\gamma (x + r_x)^{-\alpha}\Gamma(\nu+1/2)}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\gamma+1)}\, F\Big(\alpha, \beta\,; \gamma+1\,; \frac{x_0 - r_x}{x_0 + r_x}\Big). \tag{4}$$

The condition on the parameters insuring convergence of the hypergeometric series is that the inequalities $\Re(\beta) > 0$, $\Re(\gamma+1) > 0$ both hold. The first inequality is satisfied for $\Re(\mu) > 0$, independently of m, and the second if $\Re(m\mu + \nu) > n/2 - 1$. On the other hand, the condition on the argument of the hypergeometric series for convergence is that $|(x_0 - r_x)/(x_0 + r_x)| < 1$. This inequality is satisfied everywhere inside the light cone except when $r_x = 0$, i.e. along the axis of the light cone, at which points $(x_0 - r_x)/(x_0 + r_x) = 1$. This does not necessarily mean that the function has a singularity along the axis, but we shall defer a careful investigation of this point until later. The factor $(x_0 - r_x)^\gamma$ is well behaved as long as $m\mu + \nu > n/2$ and the factor $(x_0 + r_x)^{-\alpha}$ is analytic for all values of ν, since $x_0 > 0 \geqq -r_x$. Thus it follows that the Riesz kernel $W(x, \mu, \nu)$ is an analytic function of μ and ν, except possibly for x on the axis of the light cone, as long as $\Re(\mu) > 0$ and $\Re(m\mu + \nu) > n/2$.

3. The Riesz operators $J^{\mu,\nu}$. We are now ready to introduce the operators

$$(J^{\mu,\nu} f)(x) = \int_{D^x} W(x - y, \mu, \nu) f(y) dy_{(n)} \tag{5}$$

where f is the function occurring on the right side of equation (1). The notation $dy_{(n)}$ means that we are computing the volume integral and has obvious modifications for surface integrals. The integral is taken over the retrograde light cone with vertex at the point x.

Let $S(Px, x_0-y_0)$ be the $n-1$ dimensional sphere with center at Px and radius x_0-y_0, then we can write

$$(J^{\mu,\nu}f)(x)=\int_{-\infty}^{x_0}\Big[\int_{S(Px,x_0-y_0)}W(x-y,\mu,\nu)f(y)dy_{(n-1)}\Big]dy_0\,.$$

One observes that $W(x-y,\mu,\nu)$ actually depends on the variables y_0 and r, hence

$$(J^{\mu,\nu}f)(x)=\int_{-\infty}^{x_0}\Big\{\int_0^{x_0-y_0}\Big[\int_{S(Px,r)}W(x_0-y_0,r,\mu,\nu)f(y)dy_{(n-2)}\Big]dr\Big\}dy_0\,.$$

Let

$$M(r,y_0,x,f)=\frac{1}{\omega_{n-1}r^{n-2}}\int_{S(Px,r)}f(x)dy_{(n-2)}\,, \tag{6}$$

the spherical mean of f, then

$$(J^{\mu,\nu}f)(x)=\int_{-\infty}^{x_0}\int_0^{x_0-y_0}\omega_{n-1}r^{n-2}W(x_0-y_0,r,\mu,\nu)M(r,y_0,x,f)dr\,dy_0\,.$$

Here, ω_{n-1} is the surface area of the $n-1$ dimensional unit sphere, $\omega_{n-1}=2\pi^{(n-2)/2}/\Gamma((n-1)/2))$. Now, let $y_0=x_0-\tau$, $r=\tau(1-\sigma)$, then our last formula becomes

$$(J^{\mu,\nu}f)(x)=\omega_{n-1}\int_0^\infty\int_0^1\tau^{n-1}(1-\sigma)^{n-2}W(\tau,\sigma,\mu,\nu)M(\tau,x,f)d\sigma\,d\tau\,. \tag{7}$$

We also have

$$W(\tau,\sigma,\mu,\nu)=\frac{\sigma^\gamma(2-\sigma)^{-\alpha}\tau^{\beta-2\gamma}\Gamma(\gamma+1/2)}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\gamma+1)}F\Big(\alpha,\beta;\gamma+1;\frac{\sigma}{2-\sigma}\Big)\,.$$

Consider

$$F\Big(\alpha,\beta;\gamma+1;\frac{\sigma}{2-\sigma}\Big)=\frac{\Gamma(\gamma+1)}{\Gamma(\beta)\Gamma(\alpha)}\sum_{j=0}^\infty\frac{\Gamma(\beta+j)\Gamma(\alpha+j)}{j\,!\,\Gamma(\gamma+j+1)}\Big(\frac{\sigma}{2-\sigma}\Big)^j\,.$$

It should be pointed out that in the case where $\nu=n/2+k$, $k=1,2,\cdots$, this hypergeometric series terminates after $k+1$ terms, and in the analysis that follows we need not consider these values. We next write

$$F\Big(\alpha,\beta;\gamma+1;\frac{\sigma}{2-\sigma}\Big)=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha)\Gamma(\beta)}\Big\{\sum_{j=0}^{p-1}\frac{\Gamma(\alpha+j)\Gamma(\beta+j)}{j\,!\,\Gamma(\gamma+j+1)}\Big(\frac{\sigma}{2-\sigma}\Big)^j + \frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{p\,!\,\Gamma(\gamma+p+1)}\Big(\frac{\sigma}{2-\sigma}\Big)^p G_{\mu,\nu,p}\Big\}$$

where

$$G_{\mu,\nu,p}=1+\frac{(m\mu+p)(n/2+p-\nu)}{(p+1)(m\mu+\nu+p+1-n/2)}\frac{\sigma}{2-\sigma}+\cdots .$$

Then we can write

$$W(\tau,\sigma,\mu,\nu)=\frac{\sigma^{\gamma}(2-\sigma)^{-\alpha}\tau^{\beta-2\alpha}\Gamma(\nu+1/2)}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\alpha)\Gamma(\beta)}\Big\{\sum_{j=0}^{p-1}\frac{\Gamma(\alpha+j)\Gamma(\beta+j)}{j\,!\,\Gamma(\gamma+j+1)}\Big(\frac{\sigma}{2-\sigma}\Big)^{j}$$
$$+\frac{\Gamma(\alpha+p)\Gamma(\beta+p)}{p\,!\,\Gamma(\gamma+p+1)}\Big(\frac{\sigma}{2-\sigma}\Big)^{p}G_{\mu,\nu,p}\Big\}.$$

By substitution into (7) one arrives at the result

$$(J^{\mu,\nu}f)(x)=\sum_{j=0}^{p-1}\frac{\Gamma(\alpha+j)}{\Gamma(\theta)\Gamma(\gamma+j+1)}\int_0^{\infty}\int_0^1\tau^{\theta}\sigma^{\gamma+j}(2-\sigma)^{-\alpha-j}M_j d\sigma\,d\tau$$
$$+\frac{\Gamma(\alpha+p)}{\Gamma(\theta)\Gamma(\gamma+p+1)}\int_0^{\infty}\int_0^1\tau^{\theta}\sigma^{\gamma+p}(2-\sigma)^{-\alpha-p}H_{\mu,\nu,p}d\sigma\,d\tau\,,$$

where $\theta=\beta+2\nu-1=m\mu+2\nu-1$ and where

(8) $$M_j(\sigma,\tau,\mu,\nu)=\frac{\omega_{n-1}\Gamma(\theta+1)\Gamma(\nu+1/2)\Gamma(\beta+j)(1-\sigma)^{n-2}}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\alpha)\Gamma(\beta)\,j\,!}M(\tau,\sigma,x,f)\,,$$

and

(9) $$H_{\mu,\nu,p}(\sigma,\tau,\mu,\nu)=\frac{\omega_{n-1}\Gamma(\theta+1)\Gamma(\nu+1/2)\Gamma(\beta+p)(1-\sigma)^{n-2}}{\pi^{(n-1)/2}\Gamma(2\nu)\Gamma(\alpha)\Gamma(\beta)\,p\,!}G_{\mu,\nu,p}M(\tau,\sigma,x,f)\,.$$

Next, for arbitrary a and b, we define the integral

(10) $$R^{a,b}M=\frac{\Gamma(a+n-1-b)}{\Gamma(a)\Gamma(b)}\int_0^{\infty}\int_0^1\sigma^{a-1}\tau^{b-1}(2-\sigma)^{b+1-n-a}M\,d\sigma\,d\tau\,,$$

then we can write

(11) $$(J^{\mu,\nu}f)(x)=\sum_{j=0}^{p-1}R^{m\mu+\nu+j+1-n/2,2\nu+m\mu}M_j$$
$$+R^{m\mu+\nu+p+1-n/2,2\nu+m\mu}H_{\mu,\nu,p}\,.$$

Our aim is to study $J^{\mu,\nu}f$ given by (11) in order to show that $J^{0,0}f=f$. It is not difficult to see that the functions M_j have singularities only for certain negative values of μ and ν and are analytic for all other values of these parameters. The same is true for $H_{\mu,\nu,p}$ if p is sufficiently large. Therefore, the study of $J^{\mu,\nu}f$ reduces to the study of the nature of the integral $R^{a,b}M$ and its dependence on the parameters a and b. To assure the convergence of this integral, we assume M is continuous in σ and τ and $M\equiv 0$ for $\tau>\tau_0$. For such an M, $R^{a,b}M$ converges absolutely for $\Re(a)>0$ and $\Re(b)>0$ and the result is an analytic function of a and b. The only difficulty is at $\sigma=\tau=0$, i.e.

$$\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{1}\sigma^{a-1}\tau^{b-1}(2-\sigma)^{b+1-n-a}M(\sigma,\tau)d\sigma\,d\tau$$

is analytic for all a and b if $\varepsilon > 0$.

Assume now that $M \in C_\delta^{q,r}$ in some neighborhood of $\sigma=\tau=0$, where by $C_\delta^{q,r}$ we mean the class of functions $g(\sigma, \tau)$ which are q times continuously differentiable with respect to σ, r times continuously differentiable with respect to τ and such that

$$|g_{q,r}(\sigma,\tau)-g_{q,r}(0,0)| < K\sigma^\delta\tau^\delta, \qquad 0<\delta\leqq 1,$$

$\left(\text{by } g_{q,r}(\sigma,\tau) \text{ we mean } \dfrac{\partial^{p+r} g(\sigma,\tau)}{\partial\sigma^q\partial\tau^r}\right)$. Adopting some lemmas proved by Riesz [1] to the present integral, we find, for $a, b \geqq 0$,

$$R^{-a,-b}M = \Gamma(b+n-1-a)\, 2^{-(b+n-1-a)} M_{a,b}(0,0)\; (-1)^{a+b},$$

$$R^{-a,b}M = \frac{\Gamma(n-1-b-a)\, 2^{b+1-n+a}(-1)^a}{\Gamma(b)} \int_0^\infty \tau^{b-1} M_a(0,\tau)\, d\tau,$$

$$R^{a,-b}M = \frac{\Gamma(a+n-1+b)\,(-1)^b}{\Gamma(a)} \int_0^1 \sigma^{a-1}(2-\sigma)^{-b+1-n-a} M_b(\sigma,0)\, d\sigma.$$

Thus we see that, for a, b zero or negative integers, the integral $R^{a,b}M$ depends only on the local properties of the function M. From what we have done, it is clear that the functions M_j and $H_{\mu,\nu,p}$ have all the desired properties except for certain negative values of μ and ν and if p is sufficiently large. The requirement that $M\equiv 0$ for $\tau > \tau_0$ can be satisfied by choosing f to vanish for large negative values of y_0.

We can now compute $J^{0,0}f$. One observes that $M_j(\sigma,\tau,0,0)=0$ except for the case where $j=0$. Likewise $H_{0,0,p}=0$ if $p > n/2-1$. Thus, we find

$$(J^{0,0}f)(x) = R^{1-n/2,0}M_0(0,0).$$

But $M_0(0,0) = \dfrac{\omega_{n-1}\Gamma(1/2)(1-\sigma)^{n-2}}{\pi^{(n-1)/2}\Gamma(n/2)} M(\tau(1-\sigma), y_0-\tau, x, f)$. On the other hand, using the above formula for $R^{a,-b}$, one finds

$$R^{a,0}M_0(0,0) = \frac{\Gamma(a+n-1)}{\Gamma(a)} \int_0^1 \sigma^{a-1}(1-\sigma)^{n-2}(2-\sigma)^{1-n-a} \frac{\omega_{n-1}M(0,x_0,x,f)}{\pi^{(n-1)/2}\Gamma(n/2)}\, d\sigma.$$

Now, we make use of the value of ω_{n-1} and the fact that $M(0,x_0,x,f)=f(x)$ to find

$$R^{a,0}M_0(0,0) = \frac{2\pi^{(n-1)/2}\Gamma(a+n-1)f(x)}{\pi^{(n-2)/2}\Gamma(a)\Gamma(n/2)\Gamma((n-1)/2)} \int_0^1 \sigma^{a-1}(1-\sigma)^{n-2}(2-\sigma)^{1-n-a}\, d\sigma.$$

One can show that [4]

$$\int_0^1 \sigma^{a-1}(1-\sigma)^{n-2}(2-\sigma)^{1-n-a}\, d\sigma = 2^{1-n}\, \frac{\Gamma(a)\Gamma(n-1)}{\Gamma(a+n-1)}.$$

Thus,

$$R^{a,0}M_0(0,0) = f(x)\, \frac{2^{2-n}\pi^{1/2}\Gamma(n-1)}{\Gamma((n-1)/2)\Gamma(n/2)}.$$

Finally, an application of the Legendre duplication formula gives

$$(J^{0,0}f)(x)=R^{a,0}M_0(0,0)=f(x)\,. \tag{12}$$

We have already observed that $W(x,\mu,\nu)$ will be an analytic function if $\mu>0$ and $\Re(m\mu+\nu)>n/2$. Suppose $\nu=1$, then the second inequality assures us that $W(x,\mu,1)$ will be analytic at $\mu=1$ if $m>(n-2)/2$. Thus, $W(x,1,1)$ is analytic if $m>(n-2)/2$ where n is the number of dimension and m the number of partial differentiations with respect to x_0 in equation (1). (It will be remembered that the analyticity may break down along the axis of D^x, but we shall continue to ignore this possiblity for the moment.) It is now meaningful to write the operator

$$(J^{1,1}f)(x)=\int_{D^x}W(x-y,1,1)f(y)dy_{(n)}$$

or

$$(J^{1,1}f)(x)=\frac{\pi^{-(n-1)/2}}{\Gamma(m+2-n)/2}\int_{D^x}f(y)\frac{(x_0-y_0-r)^m}{[(x_0-y_0)^2-r^2]^{n/2-1}} \tag{13}$$

$$\times F\Big(n/2-1,m\,;\,m+2-n/2\,;\,\frac{x_0-y_0-r}{x_0-y_0+r}\Big)dy_{(n)}$$

where $m>(n-2)/2$ and f vanishes for sufficiently large negative values of y_0 so that the integral converges.

4. Behavior of the integrand of $J^{1,1}f$ for y on the axis of D_x. We have already pointed out that the hypergeometric function occurring in the Riesz kernel has argument 1 when $r=0$. The value of this function for argument 1 is given by the formula [**4**],

$$F(n/2-1,m\,;\,m+2-n/2\,;1)=\frac{\Gamma(m+2-n/2)\Gamma(3-n)}{\Gamma(2-n/2)\Gamma(m+3-n)}\,,$$

and the formula is valid if $m+2-n/2\neq 0,-1,-2,\cdots$ and if $\Re(3-n)>0$. At first glance it would appear that the integrand of $J^{1,1}f$ will be singular if $n>2$, but we might still hope to choose m (in accord with the inequality above) in such a way that the singularity will be cancelled out. In fact, if we choose $m=n/2$ and use the Legendre duplication formula, we find

$$F(n/2-1,n/2\,;2\,;1)=\frac{2^{2-n}\pi^{1/2}\Gamma\Big(\frac{3-n}{2}\Big)}{\Gamma\Big(\frac{6-n}{2}\Big)}\,. \tag{14}$$

This function has simple poles at $n=3,5,7,9,\cdots$ and zeros for $n=6,8,10,\cdots$. Therefore, for n even the integrand of $J^{1,1}f$ has no singularity and equation (13) is valid with the Riesz kernel given by (14) for $r=0$.

On the other hand, if we choose $m=n/2-1/2$, we also satisfy the inequality for m given in the last section and we find

$$F(n/2-1, n/2-1/2\,;3/2\,;1)=\frac{2^{1-n}\Gamma((3-n)/2)}{\Gamma((5-n)/2)}\,. \tag{15}$$

This function has simple poles in the numerator at $n=3,5,7,9,\cdots$ as before and simple poles in the denominator at $n=5,7,9,\cdots$. Therefore, for $n=5,7,9,\cdots$ the singularity of the integrand on the axis of D^x is removed and equation (13) is again valid with the kernel given as 2^{1-n} for $r=0$.

It remains to determine how the Riesz kernel behaves in the case of 3-dimensions. We choose $m=1$ in conformity with the situation for the odd dimensional case in general. Then one finds [4],

$$F(1/2, 1\,;3/2\,;z)=\frac{1}{2\sqrt{z}}\log\frac{1+\sqrt{z}}{1-\sqrt{z}}$$

where we have set $z=(x_0-y_0-r)/(x_0-y_0+r)$. This means that the Riesz kernel becomes

$$\frac{1}{2}\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\,. \tag{16}$$

This kernel vanishes on the surface of D^x and has a logarithmic singularity along the axis of D^x. However, the kernel (16) is the same as the kernel in the classical solution of Volterra [5] and, by a transformation to polar coordinates, we can show that the singularity on the axis does not cause any trouble in the integration. Therefore, the formal calculation is valid and the Riesz operator for the equation

$$\frac{\partial}{\partial x_0}\Big(\frac{\partial^2}{\partial x_0{}^2}-\frac{\partial^2}{\partial x_1{}^2}-\frac{\partial^2}{\partial x_2{}^2}\Big)u=f$$

can be written explicitly in the form

$$(J^{1,1}f)(x)=\frac{1}{2\pi\Gamma(3/2)}\int_{D^x}f(y)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\,dy_{(3)}\,. \tag{17}$$

Our analysis so far has shown that for certain particular values of m we can generate Riesz operators for equation (1) for which the integrands are analytic at $\mu=\nu=1$ when y is inside the light cone D^x. We must now justify the remark made in the introduction by showing that this can be done for all combinations of m and n. For the case where n is even we can try $m=n/2+k$, then we find

$$F(n/2-1, n/2+k\,;k+2\,;1)=\frac{2^{2-n}\pi^{-1/2}\Gamma(k+2)\Gamma((3-n)/2)}{\Gamma((2k+6-n)/2)}$$

This choice again gives singularities in the odd dimensional cases, but reduces to a constant in the even dimensional cases. It follows that we have an analytic Riesz kernel for equation (1) for every integer value of m greater than or equal to $n/2$, when the number of dimensions is even. Now, if one applies the method we shall use in the next section for obtaining the Riesz operator for $\square_n$, one can find a Riesz operator for every integer value of m if n is even. Further, once the operator has been found for given combination of m and n (even), Hadamard's method of descent [**6**] allows us to deduce the operator for m and $n-1$. In this way we can find the Riesz operator for every conbination of m and n.

5. The Riesz operator for $\square_n$. From our development of formula (13) it is clear that

$$\frac{\partial^m}{\partial x_0{}^m}\square_n(J^{1,1}f)(x)=\square_n\frac{\partial^m}{\partial x_0{}^m}(J^{1,1}f)(x)=(J^{0,0}f)(x)=f(x)\,.$$

Therefore, we set

$$v=\frac{\partial^m}{\partial x_0{}^m}(J^{1,1}f)(x)$$

and we have the Riesz operator for $\square_n$. Moreover, we have shown that if n is even we can chose $m=n/2$ and if n is odd we can chose $m=(n-1)/2$. It follows that the Riesz operator for the wave equation is given by the formula

$$(I^{1,1}f)(x)=\frac{\partial^{[n/2]}}{\partial x_0{}^{[n/2]}}(J^{1,1}f)(x)\,. \tag{18}$$

Let us rewrite the Riesz operator in the following form:

$$(I^{1,1}f)(x)=\frac{\partial^{[n/2]}}{\partial x_0{}^{[n/2]}}\int_{-\infty}^{x_0}\int_{S(Px,x_0-y_0)}f(y)W(x-y,1,1)dy_{(n-1)}dy_0\,.$$

It is not difficult to show that one can move $n-1$ differentiations with respect to x_0 inside the first integral sign. Therefore, we can write

$$(I^{1,1}f)(x)=\int_{-\infty}^{x_0}\frac{\partial^{[n/2]}}{\partial x_0{}^{[n/2]}}\int_{S(Px,x_0-y_0)}f(y)W(x-y,1,1)dy_{(n-1)}dy_0\,.$$

We define $W(x_0)$, a one parameter set of operators on $L^2(E^{n-1})$, such that for $f(Py)\in L^2(E^{n-1})$

$$W(x_0)(f,Px)=\frac{\partial^{[n/2]}}{\partial x_0{}^{[n/2]}}\int_{S(Px,x_0)}f(Py)W(x_0,Py,1,1)dy_{(n-1)}\,. \tag{19}$$

With the aid of this set of operators we shall solve the Cauchy problem for data given on the plane $x_0=0$.

6. Solution of the Cauchy problem for $\square_n u = f$. Let us write the wave equation in the form

$$\frac{\partial^2 u}{\partial x_0^2} + Lu(x_0) = f(x_0) \tag{20}$$

where $L = -\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$. Suppose that we look upon equation (20) as if it were an ordinary differential equation with L a constant and that we seek the solution of this equation for initial data given at $x_0 = 0$. Let the data be $u(0) = u_0$, $\frac{\partial u}{\partial x_0}(0) = u_1$. Then we can write the solution of the homogeneous equation

$$\frac{d^2 u}{dx_0^2} + Lu(x_0) = 0$$

in the form

$$u(x_0) = \cos(\sqrt{L}\, x_0)\, u_0 + \frac{\sin(\sqrt{L}\, x_0)}{\sqrt{L}}\, u_1 .$$

Let $W(x_0) = \frac{\sin(\sqrt{L}\, x_0)}{\sqrt{L}}$, then we have

$$u(x_0) = \frac{dW}{dx_0}(x_0)\, u_0 + W(x_0)\, u_1 . \tag{21}$$

Formally,

$$\frac{\sin(\sqrt{L}\, x_0)}{\sqrt{L}} = x_0 - \frac{Lx_0^3}{3\,!} + \frac{L^2 x_0^5}{5\,!} - \cdots .$$

If $W(x_0)$ has the desired formal properties, then equation (21) gives the solution of the homegeneous equation. To solve the non-homogeneous equation we write

$$u(x_0) = \frac{\partial W}{\partial x_0}(x_0)\, u_0 + W(x_0)\, u_1 + \int_0^{x_0} W(x_0 - y_0) f(y_0)\, dy_0 . \tag{22}$$

Writing $f(y) = f(y_0)(Py)$ we see that the last integral is the operator we have called $(I^{1,1}f)(x)$ so that we can write out the solution of the Cauchy problem in the form

$$\begin{aligned} u(x) = {} & \frac{\partial^{[n/2+1]}}{\partial x_0^{[n/2+1]}} \int_{S(Px, x_0)} u_0(Py) W(x_0, Py, 1, 1)\, dy_{(n-1)} \\ & + \frac{\partial^{[n/2]}}{\partial x_0^{[n/2]}} \int_{S(Px, x_0)} u_1(Py) W(x_0, Py, 1, 1)\, dy_{(n-1)} \\ & + \frac{\partial^{[n/2]}}{\partial x_0^{[n/2]}} \int_0^{x_0} \int_{S(Px, x_0 - y_0)} f(y) W(x - y, 1, 1)\, dy_{(n)} . \end{aligned} \tag{23}$$

We leave aside here the problem of determining under what conditions the one parameter semi-group of operators $W(x_0)$ can be represented by the operator series, but an operational calculus of the kind we have just used can be justified under quite general conditions.

University of Oregon.

Bibliography

[1] M. Riesz, L'integrale de Riemann-Liouville et la probleme de Cauchy, Acta Math., **81** (1949), 1-223.
[2] L. Garding, Linear hyperbolic partial differential equations with constant coefficients, Acta. Math., **85** (1951), 1-62.
[3] J. Leray, Hyperbolic differential equations, mimeographed notes of lectures delivered at the Institute for Advanced Study, Princeton, 1953.
[4] A. Erdelyi, etc., Higher transcendental functions, vol. 1, McGraw-Hill Book Company, Inc., New York.
[5] E. Goursat, Cours D'Analyse Mathematique, Gauthier-Villars, vol. 3, Paris, 1927.
[6] J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Dover Publications, New York, 1952.
[7] F. Bureau, Divergent integrals and partial differential equations, Comm. Pure Appl. Math., **8** (1955), 143-202.
[8] F. Bureau, Sur l'integration des equations lineaires aux derivees partielles du second ordre et du type hyperbolique normal, Mem. Soc. Roy. Sci. Liège, Scr. 4, **3** (1938).
[9] A. Weinstein, On the Cauchy problem for the Euler-Poisson-Darboux equation, Bull. Amer. Math. Soc., **59** (1953), 454.
[10] Diaz and Martin, Riemann's method and the problem of Cauchy. II. The wave equation in n dimensions, Proc. Amer. Math. Soc., **3** (1952), 476-483.