Journal of the Mathematical Society of Japan Vol. 12, No. 4, October, 1960

Ordinal Diagrams II.

By Gaisi TAKEUTI

(Received Nov. 12, 1959)

In a former paper [1] the author developed the theory of ordinal diagrams, which represent the ordinal numbers in a certain "Abschnitt" of the second number class and are useful for the consistency proof of some logical systems. In this paper we shall generalize the notion of ordinal diagrams and their ordering relations, and we shall prove that the generalized system $\mathrm{Od}(I, A, S)$ of ordinal diagrams is well-ordered.

The well-ordering property of the system of ordinal diagrams will be lost if we generalize the notion of ordinal diagrams in the following directions:

a) Making use of an ordinal diagram in place of $i \in I$.

b) Making use of an ordinal diagram in place of $s \in S$.

In fact, we shall have, in case of a), a strictly descending sequence:

$1 > (1, 0, 0) > ((1, 0, 0), 0, 0) > \cdots$

and in case of b),

$1 > (0, 1) > (0, (0, 1)) > \cdots$

Most of expressions in this paper should be read and understood according to the context.

§ 1. Ordinal diagrams constructed from I, A and S.

1. Let I, A and S be well-ordered sets. The system $\mathrm{Od}(I, A, S)$, called the *system of ordinal diagrams constructed from* I, A and S, is defined recursively by means of the operations of $(\cdot, \cdot), (\cdot, \cdot, \cdot)$ and $\#$ as follows. (The word "ordinal diagram" or "o.d." which was used in [1] is now applied to "element of $\mathrm{Od}(I, A, S)$." If no confusion is feared, the same symbol $<$ is used to denote the order of I or A or S; the symbol $<$, as well as the equality $=$, should be understood according to the context.)

1.1. If $a \in A$, then a is an o.d.

1.2. If α is an o.d. and $s \in S$, then (α, s) is an o.d.

1.3. If α and β are o.d.'s and $i \in I$, then (i, α, β) is an o.d.

This work was done under Appointment supported by the International Cooperation Administration under the Visting Research Scientists Program administered by the National Academy of Science of the United States of America.

1.4. If α and β are o.d.'s, then $\alpha \# \beta$ is an o.d.

2. $\mathrm{O}(I, A, S)$, which is a subsystem of $\mathrm{Od}(I, A, S)$ and is called the *system of ordinal diagrams in the narrow sense*, is defined recursively as follows.

2.1. If $a \in A$, then $a \in \mathrm{O}(I, A, S)$.

2.2. If $a \in A$ and $s \in S$, then $(a, s) \in \mathrm{O}(I, A, S)$.

2.3. If $a \in A$, $\alpha \in \mathrm{O}(I, A, S)$ and $i \in I$, then $(i, a, \alpha) \in \mathrm{O}(I, A, S)$.

2.4. If $\alpha \in \mathrm{O}(I, A, S)$ and $\beta \in \mathrm{O}(I, A, S)$, then $\alpha \# \beta \in \mathrm{O}(I, A, S)$.

3. Let α and β be o.d.'s and $i \in I$. We define recursively the relation $\beta \subset_i \alpha$ (to be read: β is an *i-section* of α) as follows:

3.1. If $\alpha \in A$ or α is of the form (α_0, s), then $\beta \subset_i \alpha$ never holds (α has no i-section).

3.2. Let α be of the form (j, α_1, α_2).

a) If $i < j$, then $\beta \subset_i \alpha$ if and only if $\beta \subset_i \alpha_2$.

b) If $i = j$, then $\beta \subset_i \alpha$ if and only if β is α_2.

c) If $j < i$, then $\beta \subset_i \alpha$ never holds.

3.3. Let α be of the form $\alpha_1 \# \alpha_2$. Then $\beta \subset_i \alpha$ if and only if either $\beta \subset_i \alpha_1$ or $\beta \subset_i \alpha_2$ holds.

4. An o.d. α is called a *connected ordinal diagram* (abbreviated by c.o.d.), if and only if the operation used in the final step of construction of α is not $\#$.

5. Let α be an o.d. We define *components* of α recursively as follows:

5.1. If α is a c.o.d., then α has only one component which is α itself.

5.2. If α is an o.d. of the form $\alpha_1 \# \alpha_2$ and components of α_1 and α_2 are $\beta_1, \cdots, \beta_k$ and $\gamma_1, \cdots, \gamma_l$ respectively, then components of $\alpha_1 \# \alpha_2$ are $\beta_1, \cdots, \beta_k$, $\gamma_1, \cdots, \gamma_l$.

6. Let α and β be o.d.'s. We define $\alpha = \beta$ recursively as follows:

6.1. Let $\alpha \in A$. Then $\alpha = \beta$ is equivalent to the equality in A.

6.2. Let α be of the form (α_0, s). Then $\alpha = \beta$, if and only if β is of the form (β_0, t) and $\alpha_0 = \beta_0, s = t$.

6.3. Let α be an o.d. of the form (i, α_1, α_2). Then $\alpha = \beta$ if and only if β is of the form (j, β_1, β_2) and $i = j$, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.

6.4. Let α be a non-connected o.d. with k components $\alpha_1, \cdots, \alpha_k$. Then $\alpha = \beta$, if and only if β has the same number of components, and $\beta_1, \cdots, \beta_k$ being these components, there exists a permutation $(l_1, \cdots, l_k)$ of $(1, \cdots, k)$ such that $\alpha_m = \beta_{l_m}$, $m = 1, \cdots, k$.

6.5. $\beta = \alpha$ holds, if and only if $\alpha = \beta$.

7. Let α be an o.d. and $i \in I$. i is called an *index* of α, if and only if α has an i-section.

8. Let α and β be two o. d.'s and $i \in I$. We define the reations $\alpha <_i \beta$ and $\alpha <_\infty \beta$ recursively as follows:

8.1. If $\alpha, \beta \in A$, then both $\alpha <_i \beta$ and $\alpha <_\infty \beta$ mean $\alpha < \beta$ in A.

8.2. Let the components of α and β be $\alpha_1, \cdots, \alpha_k$ and $\beta_1, \cdots, \beta_h$ respectively. $\alpha <_i \beta$ ($i \in I$ or i is ∞) holds, if and only if one of the following conditions is fulfilled.

a) There exists β_m $(1 \leqq m \leqq h)$ such that for every l $(1 \leqq l \leqq k)$ $\alpha_l <_i \beta_m$ holds.

b) $k = 1$, $h > 1$ and $\alpha_1 = \beta_m$ for suitable m $(1 \leqq m \leqq h)$.

c) $k > 1$, $h > 1$ and there exist α_l $(1 \leqq l \leqq k)$ and β_m $(1 \leqq m \leqq h)$ such that $\alpha_l = \beta_m$ and

$\alpha_1 \# \cdots \# \alpha_{l-1} \# \alpha_{l+1} \# \cdots \# \alpha_k <_i \beta_1 \# \cdots \# \beta_{m-1} \# \beta_{m+1} \# \cdots \# \beta_h$.

8.3. Let α and β be c. o. d.'s and $i \in I$. If there exists no index of α or β larger than i we define j to be ∞, and if there exist any such indices we define j to be their minimum. Then $\alpha <_i \beta$, if and only if one of the following conditions is fulfilled:

a) There exists an i-section β_0 of β such that $\alpha \leqq_i \beta_0$.

b) $\alpha_0 <_i \beta$ for every i-section α_0 of α and $\alpha <_j \beta$.

8.4. Let α and β be c. o. d.'s of the form (i, α_1, α_2) and (j, β_1, β_2) respectively. $\alpha <_\infty \beta$, if and only if one of the following conditions is fulfilled:

a) $\alpha_1 <_\iota \beta_1$ (We assume that I begins with ι).

b) $\alpha_1 = \beta_1$ and $i < j$.

c) $\alpha_1 = \beta_1$, $i = j$ and $\alpha_2 <_i \beta_2$.

8.5. Let α and β be c. o. d.'s of the form (α_0, s) and (j, β_1, β_2) respectively. Then $\alpha <_\infty \beta$, if and only if $\alpha_0 \leqq_\iota \beta_1$. $\beta <_\infty \alpha$, if and only if $\beta_1 <_\iota \alpha_0$.

8.6. Let α and β be c. o. d.'s of the form (α_0, s_1) and (β_0, s_2) respectively. $\alpha <_\infty \beta$, if and only if one of the following conditions is fulfilled:

a) $\alpha_0 <_\iota \beta_0$.

b) $\alpha_0 = \beta_0$ and $s_1 < s_2$.

8.7. $(i, \alpha_1, \alpha_2) <_\infty a$ if and only if $\alpha_1 <_\iota a$. $a <_\infty (i, \alpha_1, \alpha_2)$, if and only if $a \leqq_\iota \alpha_1$. $a <_\infty (\alpha, s)$, if and only if $a \leqq_\iota \alpha$. $(\alpha, s) <_\infty a$ if and only if $\alpha <_\iota a$.

Under these definitions the following propositions are easily proved.

Proposition 1. *$\alpha = \alpha$ and $\alpha = \beta$, $\beta = \gamma$ imply $\alpha = \gamma$.*

Proposition 2. *$\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ imply $\alpha_1 \# \beta_1 = \alpha_2 \# \beta_2$, $(i, \alpha_1, \beta_1) = (i, \alpha_2, \beta_2)$ and $(\alpha_1, s) = (\alpha_2, s)$.*

Proposition 3. *$\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\alpha_1 <_i \beta_1$ imply $\alpha_2 <_i \beta_2$ ($i \in I$ or i is ∞).*

Proposition 4. *Everyone of the relations $<_i$ ($i \in I$ or i is ∞) is a linear order between o. d.'s.*

PROPOSITION 5. *If α and β are o.d.'s and $i \in I$, then $(i, \alpha, \beta)_\iota > \beta$.*

§ 2. Accessibility of $O(I, A, S)$.

Let $\mathfrak{S}$ be a system with a linear order $\prec$. A sequence of elements of $s_1, s_2, s_3, \cdots$ is called a *strictly decreasing sequence* (abbreviated by s.d.s.) for $\prec$, if and only if $s_n \succ s_{n+1}$ for every n $(n = 1, 2, \cdots)$.

An element s_1 of $\mathfrak{S}$ is called '*inaccessible* in this system (or inaccessible for this order)', if there exists an s.d.s. $s_1, s_2, s_3, \cdots$ in this system (or for this order).

An elements s of $\mathfrak{S}$ is called '*accessible* in this system (or accessible for this order)', if and only if s is not inaccessible in this system (or for this order).

PROPOSITION 1. *Let α be an o.d. (in the narrow sense). If every o.d. (in the narrow sense) less than α in the sense of $<_i$ is accessible for $<_i$ (in $O(I, A, S)$), then α is accessible for $<_i$ (in the narrow sense).*

PROPOSITION 2. *Let α be an o.d. (in the narrow sense). If α is accessible for $<_i$ (in $O(I, A, S)$), then every o.d. (in the narrow sense) less than α in the sense of $<_i$ is accessible for $<_i$ (in $O(I, A, S)$).*

PROPOSITION 3. *Let $\alpha_1, \cdots, \alpha_k$ be o.d.'s (in the narrow sense). If $\alpha_1, \cdots, \alpha_k$ are accessible for $<_i$ (in $O(I, A, S)$), then $\alpha_1 \# \cdots \# \alpha_k$ is accessible for $<_i$ (in $O(I, A, S)$).*

Let α be an o.d. The *rank* of α means the sum of the number of $\#$ and ( ) in α.

Let $\alpha \in O(I, A, S)$. α is called *ι-fan*, if and only if every ι-section of α is accessible for $<_\iota$ in $O(I, A, S)$.

Let $\alpha \in O(I, A, S)$ and $i \in I$. i is called an *index* of α, if and only if α has an i-section.

PROPOSITION 4. *Let $\alpha_1, \alpha_2, \cdots$ be an s.d.s. for $<_\iota$ in $O(I, A, S)$. Then there exists an s.d.s. $\beta_1, \beta_2, \cdots$ for $<_\iota$ in $O(I, A, S)$ such that the following conditions are fulfilled:*

a) *$\beta_1, \beta_2, \cdots$ are ι-fans.*

b) *If $\alpha_1, \cdots, \alpha_n$ are ι-fans, then β_i is α_i for every i $(1 \leqq i \leqq n)$.*

c) *If $\alpha_1 = \beta_1, \cdots, \alpha_n = \beta_n$ and $\alpha_{n+1} \neq \beta_{n+1}$, then the rank of β_{n+1} is less than the rank of α_{n+1}.*

PROOF. If $\alpha_1, \cdots, \alpha_n$ are ι-fans and α_{n+1} is not an ι-fan, then we may and shall make an s.d.s. $\alpha_1, \cdots, \alpha_n, \beta'_{n+1}, \beta'_{n+2}, \cdots$ for $<_\iota$ in $O(I, A, S)$ such that β_{n+1}'

is an inaccessible ι-section of α_{n+1}. The proposition is verified by repeating this method.

$W(I, A, S)$ is defined to be the well-ordered system of the accessible o. d.'s for $<_\iota$ in $O(I, A, S)$.

$\widetilde{W}(I, A, S)$ is the well-ordered system, which is $S \cup W(I, A, S)$ as a set, and, whose order $\prec$ is defined as follows:

a) If $s_1, s_2 \in S$, then $s_1 \prec s_2$ means $s_1 < s_2$.

b) If $s \in S$ and $\alpha \in W(I, A, S)$, then $s \prec \alpha$.

c) If $\alpha, \beta \in W(I, A, S)$ then $\alpha \prec \beta$ means $\alpha <_\iota \beta$.

Let α be an ι-fan in $O(I, A, S)$. Then α^* is defined to be an element of $O(I, A, \widetilde{W}(I, A, S))$ as follows:

a) If α has no ι-section, then α^* is α itself.

b) If α is of the form $\alpha_1 \# \cdots \# \alpha_n$, then α^* is $\alpha_1{}^* \# \cdots \# \alpha_n{}^*$.

c) If α is of the form (i, a, α_0) and $i > \iota$, then α^* is $(i, a, \alpha_0{}^*)$.

d) If α is of the form (ι, a, α_0), then α^* is (a, α_0).

The following proposition is easily proved.

Proposition 5. *Let α and β be ι-fans in* $O(I, A, S)$ *and $\alpha <_\iota \beta$. Let every ι-section of β be less than α in the sense of $<_\iota$. Then $\alpha^* <_\iota \beta^*$ holds in* $O(I, A, \widetilde{W}(I, A, S))$ *and also $\alpha^* <_{\iota+1} \beta^*$ holds, because α^* and β^* have no ι-section.*

Let $i \in I$. Then I^i is defined to be the subsystem of I satisfying the condition: $j \in I^i$ if and only if $i \leqq j$.

Let us consider I and A as fixed, assume I begins with 0 and $i \in I$. We define S^i as follows:

a) S^0 is S itself.

b) S^{i+1} is $\widetilde{W}(I^i, A, S^i)$.

c) If i is a limit number, then $S^i = \bigcup_{j<i} S^j$, where S^j is considered as a subsystem of $\widetilde{W}(I^j, A, S^j)$.

Proposition 6. *Let $\alpha_1, \alpha_2, \cdots$ be an s. d. s. for $<_0$ in* $O(I, A, S)$. *Then there exists an s. d. s. $\alpha_1{}^i, \alpha_2{}^i, \cdots$ for $<_0$ in* $O(I^i, A, S^i)$ *for every $i \in I$ such that the following conditions are fulfilled:*

a) *$\alpha_k{}^0 = \alpha_k$ for every k $(k = 1, 2, \cdots)$.*

b) *If $\alpha_1{}^i, \cdots, \alpha_n{}^i$ has no index i, then $\alpha_1{}^{i+1}, \cdots, \alpha_n{}^{i+1}$ are $\alpha_1{}^i, \cdots, \alpha_n{}^i$ themselves respectively.*

c) *If $\alpha_1{}^i = \alpha_1{}^{i+1}, \cdots, \alpha_k{}^i = \alpha_k{}^{i+1}$ and $\alpha_{k+1}^i \neq \alpha_{k+1}^{i+1}$, then the rank of α_{k+1}^{i+1} is less than the rank of α_{k+1}^i.*

d) *If i is a limit number, then for every k $(k = 1, 2, \cdots)$ there exists j_k such that $j_k < i$ and for every i_0 $(j_k \leqq i_0 \leqq i)$ $\alpha_k{}^i = \alpha_k{}^{i_0}$ holds.*

Proof. If we can construct $\alpha_k{}^j$ $(k = 1, 2, \cdots)$ satisfying the proposition for all $j \leqq i$, then we have α_k^{i+1} $(k=1,2,\cdots)$ by Propositions 5 and 4.

Therefore, we have only to prove for every limit number i that for every k $(k=1,2,\cdots)$ there exists j $(<i)$ such that for every i_0 $(j \leqq i_0 \leqq i)$ $\alpha_k^{i_0}=\alpha_k^j$ holds under the hypothesis that α_k^j $(k=1,2,\cdots, j<i)$ satisfy the proposition. However this is clear by the condition c) in the proposition.

THEOREM 1. $\mathrm{O}(I, A, S)$ *is well-ordered for* $<_0$.

PROOF. If $\mathrm{O}(I, A, S)$ is not well-ordered, then there exists an s. d. s. $\bar{\alpha}_1, \bar{\alpha}_2, \cdots$ for $<_0$ in $\mathrm{O}(I, A, S^*)$, where $S^*=\bigcup_{i \in I} S^i$, such that $\bar{\alpha}_1, \bar{\alpha}_2, \cdots$ has no index. It is a contradiction.

THEOREM 2. $\mathrm{O}(I, A, S)$ *is well-ordered for* $<_i$ *for every* $i \in I$ *(or for* $<_\infty$*).*

PROOF. Theorem follows from Theorem 1 by considering $\mathrm{O}(I^i, A, S^i)$ for $<_i$ (or $\mathrm{O}(I, A, S^*)$ for $<_\infty$ respectively).

§ 3. Accessibility of Od(I, A, S).

Let α and β be o. d.'s. We define '*β is a value of α*' as follows:

a) If $\alpha \in A$, then α has no value.

b) Let α be not a c. o. d. and have components $\alpha_1, \cdots, \alpha_n$. Then β is a value of α, if and only if β is a value of α_i for a suitable i $(1 \leqq i \leqq n)$.

c) Let α be of the form (α_0, s). Then β is a value of α, if and only if β is α_0 or β is a value of α_0.

d) Let α be of the form (i, α_1, α_2). Then β is a value of α, if and only if β is α_1 or β is a value of α_1 or α_2.

Let α and β be o. d.'s. β is called an $(i_1, \cdots, i_n)$-*section* of α, if and only if the following conditions are fulfilled:

a) $i_1 \leqq i_2 \leqq \cdots \leqq i_n \in I$.

b) There exist $\alpha=\alpha_0, \alpha_1, \cdots, \alpha_n=\beta$ such that α_k is the maximal component of an i_k-section of α_{k-1} in the sense of $<_{i_k}$ for every k $(k=1, \cdots, n)$.

The following proposition is easily proved.

PROPOSITION 1. *Let α and β be c. o. d.'s. If $\alpha <_i \beta$ for an $i \in I$, then $\alpha <_\infty \beta$ or there exists an $(i_1, \cdots, i_n)$-section β_0 of β such that $i \leqq i_1$ and $\alpha \leqq_\infty \beta_0$.*

PROPOSITION 2. *Let α be an o. d. and α_1 be a value of α. Then $\alpha_1 <_0 \alpha$.*

PROOF. We prove this by induction on the rank of α. Since the other cases can be treated similarly, we treat only the case, where α is of the form (i, α_1, α_2) and α_1 is of the form (j, β_1, β_2).

We assume that $\alpha <_0 \alpha_1$. If $\alpha <_\infty \alpha_1$, then we have $\alpha_1 <_0 \beta_1$, which contradicts the hypothesis of induction. Therefore there exists an $(i_1, \cdots, i_n)$-section

γ of α_1 such that $(i, \alpha_1, \alpha_2) \leqq_\infty \gamma$. Without loss of generality we can assume that γ is of the form (l, γ_1, γ_2). From this and Proposition 4 of §1 and the hypothesis of induction follows $\alpha_1 \leqq_0 \gamma_1 <_0 \gamma <_0 \alpha_1$, which is a contradiction.

PROPOSITION 3. *If there exists an s.d.s. $\alpha_1, \alpha_2, \alpha_3, \cdots$ for $<_0$ in o.d.'s, then there exists an s.d.s. $\beta_1, \beta_2, \beta_3, \cdots$ for $<_0$ such that every value of each β_k $(k=1, 2, \cdots)$ is accessible for $<_0$ in o.d.'s.*

PROOF. We consider the following condition C for an o.d. α.

C: Every value of α is accessible for $<_0$ in o.d.'s. If $\alpha_1, \cdots, \alpha_n$ satisfy C and α_{n+1} does not satisfy C, then we may and shall make an s.d.s. $\alpha_1, \cdots, \alpha_n, \beta'_{n+1}, \beta'_{n+2}, \cdots$ for $<_0$ in o.d.'s such that β'_{n+1} is a value of α_{n+1}. By repeating this method, the proposition is verified.

THEOREM 3. $\mathrm{Od}(I, A, S)$ *is well-ordered by* $<_0$.

PROOF. Theorem follows from Proposition 3 and the fact that $O(I, W^*(I, A, S), S)$ is well-ordered by $<_0$, where $W^*(I, A, S)$ is the system of accessible o.d.'s for $<_0$ and A and ordered by $<_0$.

THEOREM 4. $\mathrm{Od}(I, A, S)$ *is well-ordered by $<_i$ for every $i \in I$ (or by $<_\infty$).*

PROOF. Theorem follows from Theorems 2 and 3 and the accessibility of $O(I, \mathrm{Od}(I, A, S), S)$ for $<_i$.

Institute for Advanced Study,
Tokyo University of Education

Reference

[1] G. Takeuti, Ordinal diagrams, J. Math. Soc. Japan, **9** (1957), 386-394.

Added in proof. A correction to 'On the recursive functions of ordinal numbers' in this Journal **12** (1960), 119-128. In Note of p. 125, l. 12, for "ordinal number", read "cardinal number".