Journal of the Mathematical Society of Japan Vol. 12, No. 1, January, 1960

Some remarks on Einstein spaces and spaces of constant curvature.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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(Received July 31, 1959)

§ 1. Preliminaries.

The object of the present paper is to generalise some of recent results of André Avez [1]* to the case of non-compact Einstein spaces and to the case of spaces of constant curvature.

We shall here give notations and the formulas which will be used in the sequel.

Let M be an n dimensional Riemannian space of class C^4 with the fundamental metric tensor $g_{\mu\lambda}$ whose signature is not necessarily positive definite. We denote by ∇_μ the covariant differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$, by $K_{\nu\mu\lambda\kappa}$ the curvature tensor, by $K_{\mu\lambda}$ the Ricci tensor and by K the curvature scalar.

For an arbitrary skew-symmetric tensor field $w: w_{\lambda_1\lambda_2\cdots\lambda_p}$ of order p, we write

(1.1) $$(dw)_{\mu\lambda_1\lambda_2\cdots\lambda_p}=(p+1)\nabla_{[\mu}w_{\lambda_1\lambda_2\cdots\lambda_p]}$$

and

(1.2) $$(\delta w)_{\lambda_2\lambda_3\cdots\lambda_p}=\nabla_\mu w^\mu{}_{\lambda_2\lambda_3\cdots\lambda_p}\,.$$

Then the de Rham operator $\Delta=d\delta+\delta d$ applied to w gives [2]

$$(\Delta w)_{\lambda_1\lambda_2\cdots\lambda_p}=g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda_1\lambda_2\cdots\lambda_p}$$
$$-pK_{[\lambda_1}{}^\mu w_{|\mu|\lambda_2\cdots\lambda_p]}-\frac{p(p-1)}{2}K_{[\lambda_1\lambda_2}{}^{\nu\mu}w_{|\nu\mu|\lambda_3\cdots\lambda_p]}\,.$$

Especially, if w is a vector field,

(1.3) $$(\Delta w)_\lambda=g^{\nu\mu}\nabla_\nu\nabla_\mu w_\lambda-K_\lambda^\kappa w_\kappa$$

and if w is a skew-symmetric tensor field of order two,

$$(\Delta w)_{\lambda\kappa}=g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa}-2K_{[\lambda}{}^\mu w_{|\mu|\kappa]}-K_{\lambda\kappa}{}^{\nu\mu}w_{\nu\mu}$$

or

(1.4) $$(\Delta w)_{\lambda\kappa}=g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa}-(2K_{[\lambda}{}^{[\nu}A^{\mu]}_{\kappa]}+K_{\lambda\kappa}{}^{\nu\mu})w_{\nu\mu}\,.$$

* See the Bibliography at the end of the paper.

A skew-symmetric tensor field w is said to be closed (or coclosed) if $dw=0$ (or $\delta w=0$). By the well known properties of the operators d and δ, we have $d^2=0$, $\delta^2=0$, $d\Delta=\Delta d$ and $\delta\Delta=\Delta\delta$. Thus if w is closed (or coclosed), then Δw is also closed (or coclosed).

§ 2. Two theorems.

Let $T_{\mu\lambda}$ be a tensor field of class C^1 and put $T_\lambda^{\kappa}=T_{\lambda\alpha}g^{\alpha\kappa}$. The following theorem is essentially due to A. Avez [**1**], but we shall omit the condition of symmetry for $T_{\mu\lambda}$ and give a simple proof.

THEOREM 1. *If the dimension n of M is greater than* 1, *the following three properties of $T_{\mu\lambda}$ are equivalent:*

(a) $T_{\mu\lambda}=cg_{\mu\lambda}$ *where c is a constant.*

(b) $T_\lambda^{\alpha}w_\alpha$ *is closed for all closed vector field w of class* C^3.

(c) $T_\lambda^{\alpha}w_\alpha$ *is coclosed for all coclosed vector field w of class* C^3.

PROOF. If we assume (a), then $T_\lambda^{\alpha}w_\alpha=cw_\lambda$ and as c is a constant, it follows easily (b) and (c). So it is sufficient to prove (a) under the condition (b) or (c).

First assume that $T_{\mu\lambda}$ has the property (b). Then for an arbitrary closed vector field w_λ, we have

$$\nabla_{[\mu}(T_{\lambda]}{}^{\alpha}w_\alpha)=(\nabla_{[\mu}T_{\lambda]}{}^{\alpha})w_\alpha+T_{[\lambda}{}^{\alpha}\nabla_{\mu]}w_\alpha=0$$

or

$$(\nabla_{[\mu}T_{\lambda]}{}^{\alpha})w_\alpha+T_{[\lambda}{}^{\alpha}A_{\mu]}^{\beta}\nabla_\beta w_\alpha=0\,. \tag{2.1}$$

If we fix a point of M and consider the above equations at this point, then w_α can take any values and $\nabla_\beta w_\alpha$ can also take any values except the condition $\nabla_{[\beta}w_{\alpha]}=0$ at this point. So we have

$$\nabla_{[\mu}T_{\lambda]}{}^{\alpha}=0 \tag{2.2}$$

and

$$T_{[\lambda}{}^{(\alpha}A_{\mu]}^{\beta)}=0\,. \tag{2.3}$$

From the last equation we obtain, by contraction with respect to β and μ,

$$T_\lambda^{\alpha}=\frac{1}{n}T_\beta{}^{\beta}A_\lambda^{\alpha}$$

or

$$T_{\mu\lambda}=cg_{\mu\lambda}\,.$$

Substituting this into (2.2) we easily find that c is a constant.

Next assume that $T_{\mu\lambda}$ has the property (c). Then for an arbitrary coclosed vector field w_λ, we have

$$\nabla^{\mu}(T_{\mu}{}^{\lambda}w_{\lambda})=(\nabla^{\mu}T_{\mu}{}^{\lambda})w_{\lambda}+T_{\mu\lambda}(\nabla^{\mu}w^{\lambda})=0\,.$$

Considering this equation at a point of M, we get

$$\nabla^{\mu}T_{\mu}{}^{\lambda}=0 \tag{2.4}$$

and

$$T_{\mu\lambda}=cg_{\mu\lambda}\,, \tag{2.5}$$

for w_{λ} can take any values and $\nabla^{\mu}w^{\lambda}$ can take any values except the condition $g_{\mu\lambda}\nabla^{\mu}w^{\lambda}=0$ at the point. Substituting (2.5) into (2.4), we easily find that c is a constant.

Before going to theorem 2, we shall prove some lemmas.

Consider a tensor $T_{\nu\mu\lambda\kappa}$ at a point P of M which is skew-symmetric with respect to the first two indices and also with respect to the last two indices and put $T_{\nu\mu}{}^{\lambda\kappa}=T_{\nu\mu\beta\alpha}g^{\beta\lambda}g^{\alpha\kappa}$.

LEMMA 1. *Under the assumption $n>1$, the equation*

$$T_{[\mu\lambda}{}^{\beta\alpha}S_{\nu]\beta\alpha}=0 \tag{2.6}$$

holds good for an arbitrary tensor $S_{\nu\mu\lambda}$ at P satisfying

$$S_{\nu(\mu\lambda)}=0 \tag{2.7}$$

and

$$S_{[\nu\mu\lambda]}=0\,, \tag{2.8}$$

if and only if

$$T_{\nu\mu\lambda\kappa}=c(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})\,.$$

PROOF. Assume that $T_{\nu\mu\lambda\kappa}=c(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})$, then $T_{[\mu\lambda}{}^{\beta\alpha}S_{\nu]\beta\alpha}=2cS_{[\nu\mu\lambda]}$. Thus if $S_{\nu\mu\lambda}$ satisfies (2.7) and (2.8) we can conclude $T_{[\mu\lambda}{}^{\beta\alpha}S_{\nu]\beta\alpha}=0$.

Conversely suppose that when $S_{\nu\mu\lambda}$ satisfies (2.7) and (2.8) we have $T_{[\mu\lambda}{}^{\beta\alpha}S_{\nu]\beta\alpha}=0$. Let $P_{\nu\mu\lambda}$ be an arbitrary tensor of order three and put

$$S_{\nu\mu\lambda}=P_{\nu\mu\lambda}+P_{\mu\nu\lambda}-P_{\nu\lambda\mu}-P_{\lambda\nu\mu}\,,$$

then $S_{\nu\mu\lambda}$ satisfies (2.7) and (2.8). Thus we have

$$\begin{aligned}T_{[\mu\lambda}{}^{\beta\alpha}S_{\nu]\beta\alpha}&=2T_{[\mu\lambda}{}^{\beta\alpha}P_{\nu]\beta\alpha}+2T_{[\mu\lambda}{}^{\beta\alpha}P_{|\beta|\nu]\alpha}\\&=2(T_{[\mu\lambda}{}^{\beta\alpha}A_{\nu]}^{\gamma}+T_{[\mu\lambda}{}^{\gamma\alpha}A_{\nu]}^{\beta})P_{\gamma\beta\alpha}\\&=0\,.\end{aligned}$$

As $P_{\gamma\beta\alpha}$ is arbitrary, we get from this

$$T_{[\mu\lambda}{}^{\beta\alpha}A_{\nu]}^{\gamma}+T_{[\mu\lambda}{}^{\gamma\alpha}A_{\nu]}^{\beta}=0\,.$$

Contracting with respect to γ and ν, we have

$$T_{\mu\lambda}{}^{\beta\alpha}=\frac{1}{n-1}\,(T_{\mu}{}^{\alpha}A_{\lambda}^{\beta}-T_{\lambda}{}^{\alpha}A_{\mu}^{\beta})\,, \tag{2.9}$$

where $T_{\mu}{}^{\alpha}=T_{\mu\beta}{}^{\beta\alpha}=T_{\beta\mu}{}^{\alpha\beta}$. From this we get by contraction with respect to α and μ

$$T_\lambda{}^\beta = \frac{1}{n} T_\alpha{}^\alpha A_\lambda^\beta ,$$

and substituting this into (2.9) we have

$$T_{\nu\mu\lambda\kappa} = c(g_{\nu\lambda} g_{\mu\kappa} - g_{\nu\kappa} g_{\mu\lambda}) .$$

LEMMA 2. *$T_{\nu\mu\lambda\kappa}$ is as in Lemma 1 and assume $n>1$. The equation*

$$T_{\gamma\lambda}{}^{\beta\alpha} S^\gamma{}_{\beta\alpha} = 0 \tag{2.10}$$

holds good for an arbitrary tensor $S^\nu{}_{\mu\lambda}$ at P satisfying

$$S^\nu{}_{\mu\lambda} + S^\nu{}_{\lambda\mu} = 0 \tag{2.11}$$

and

$$S^\alpha{}_{\alpha\lambda} = 0 , \tag{2.12}$$

if and only if

$$T_{\nu\mu\lambda\kappa} = c(g_{\nu\lambda} g_{\mu\kappa} - g_{\nu\kappa} g_{\mu\lambda}) .$$

PROOF. If $T_{\nu\mu\lambda\kappa} = c(g_{\nu\lambda} g_{\mu\kappa} - g_{\nu\kappa} g_{\mu\lambda})$, then $T_{\gamma\lambda}{}^{\beta\alpha} S^\gamma{}_{\beta\alpha} = c(S^\alpha{}_{\alpha\lambda} - S^\alpha{}_{\lambda\alpha})$. Thus if $S^\nu{}_{\mu\lambda}$ satisfies (2.11) and (2.12) we easily find that $T_{\gamma\lambda}{}^{\beta\alpha} S^\gamma{}_{\beta\alpha} = 0$.

Conversely suppose that when $S^\nu{}_{\mu\lambda}$ satisfies (2.11) and (2.12), we have $T_{\gamma\lambda}{}^{\beta\alpha} S^\gamma{}_{\beta\alpha} = 0$. Let $P^\nu{}_{\mu\lambda}$ be an arbitrary tensor at a point P and put

$$S^\nu{}_{\mu\lambda} = P^\nu{}_{[\mu\lambda]} - \frac{1}{n-1} \{ P^\alpha{}_{\alpha[\lambda} A^\nu_{\mu]} - P^\alpha{}_{[\lambda|\alpha|} A^\nu_{\mu]} \} ,$$

then $S^\nu{}_{\mu\lambda}$ satisfies (2.11) and (2.12). Thus we have

$$\begin{aligned} T_{\gamma\lambda}{}^{\beta\alpha} S^\gamma{}_{\beta\alpha} &= T_{\gamma\lambda}{}^{\beta\alpha} P^\gamma{}_{\beta\alpha} - \frac{1}{n-1} \{ T_{\gamma\lambda}{}^{\gamma\alpha} P^\sigma{}_{\sigma\alpha} - T_{\gamma\lambda}{}^{\gamma\alpha} P^\sigma{}_{\alpha\sigma} \} \\ &= \Big[T_{\gamma\lambda}{}^{\beta\alpha} + \frac{1}{n-1} (T_\lambda{}^\alpha A_\gamma^\beta - T_\lambda{}^\beta A_\gamma^\alpha) \Big] P^\gamma{}_{\beta\alpha} \\ &= 0 . \end{aligned}$$

As $P^\nu{}_{\mu\lambda}$ is arbitrary we get

$$T_{\nu\mu}{}^{\lambda\kappa} = \frac{1}{n-1} (T_\mu{}^\lambda A_\nu^\kappa - T_\mu{}^\kappa A_\nu^\lambda) . \tag{2.13}$$

From this we have easily

$$T_\lambda{}^\kappa = \frac{1}{n} T_\alpha{}^\alpha A_\lambda^\kappa .$$

Substituting this into (2.13) we find

$$T_{\nu\mu\lambda\kappa} = c(g_{\nu\lambda} g_{\mu\kappa} - g_{\nu\kappa} g_{\mu\lambda}) .$$

Using these lemmas we shall prove the following theorem.

THEOREM 2. *Let $T_{\nu\mu\lambda\kappa}$ be a tensor field of class C^1 and skew-symmetric with respect to the first two indices and also with respect to the last two indices and assume that the dimension n of M is greater than 2, then the following three* *conditions for $T_{\nu\mu\lambda\kappa}$ are equivalent.*

(a) $T_{\nu\mu\lambda\kappa}=c(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})$, *where c is a constant.*

(b) $T_{\mu\lambda}{}^{\beta\alpha}w_{\beta\alpha}$ *is closed for any closed tensor field w of class* C^3.

(c) $T_{\mu\lambda}{}^{\beta\alpha}w_{\beta\alpha}$ *is coclosed for any coclosed tensor field w of class* C^3.

PROOF. Suppose (a), then $T_{\mu\lambda}{}^{\beta\alpha}w_{\beta\alpha}=2cw_{\mu\lambda}$ and as c is a constant, it follows easily (b) and (c). So it is sufficient to prove (a) under the condition (b) or (c).

Suppose first (b), then we have for an arbitrary closed tensor field $w_{\mu\lambda}$

$$\nabla_{[\nu}(T_{\mu\lambda]}{}^{\beta\alpha}w_{\beta\alpha})=(\nabla_{[\nu}T_{\mu\lambda]}{}^{\beta\alpha})w_{\beta\alpha}+T_{[\mu\lambda}{}^{\beta\alpha}\nabla_{\nu]}w_{\beta\alpha}=0\,. \tag{2.14}$$

Considering the above equation at a point of M we get

$$\nabla_{[\nu}T_{\mu\lambda]}{}^{\beta\alpha}=0 \tag{2.15}$$

and

$$T_{[\mu\lambda}{}^{\beta\alpha}\nabla_{\nu]}w_{\beta\alpha}=0\,, \tag{2.16}$$

for $w_{\beta\alpha}$ can take any values except the condition $w_{\beta\alpha}+w_{\alpha\beta}=0$ at the point. As $\nabla_\nu w_{\beta\alpha}$ can take any values except the condition $\nabla_\nu w_{\beta\alpha}+\nabla_\nu w_{\alpha\beta}=0$ and $\nabla_{[\nu}w_{\beta\alpha]}=0$, we have, from (2.16) and Lemma 1,

$$T_{\nu\mu\lambda\kappa}=c(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})\,.$$

Substituting this into (2.15) we find by the assumption $n>2$ that c is a constant.

Next suppose (c) then for an arbitrary coclosed tensor field $w_{\mu\lambda}$ we have

$$\nabla^\mu(T_{\mu\lambda}{}^{\beta\alpha}w_{\beta\alpha})=(\nabla^\mu T_{\mu\lambda}{}^{\beta\alpha})w_{\beta\alpha}+T_{\mu\lambda}{}^{\beta\alpha}(\nabla^\mu w_{\beta\alpha})=0\,.$$

If we fix a point in the space, $w_{\beta\alpha}$ can take any values except the condition $w_{\beta\alpha}+w_{\alpha\beta}=0$ and $\nabla^\mu w_{\beta\alpha}$ can also take any values except the conditions $\nabla^\mu w_{\beta\alpha}+\nabla^\mu w_{\alpha\beta}=0$ and $\nabla^\mu w_{\mu\alpha}=0$. Thus from Lemma 2, we get

$$\nabla^\mu T_{\mu\lambda}{}^{\beta\alpha}=0 \quad\text{and}\quad T_{\nu\mu\lambda\kappa}=c(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})\,.$$

Substituting the second into the first we find that c is a constant. Thus the proof of the theorem is complete.

§ 3. Applications to Einstein spaces and spaces of constant curvature.

We shall prove in this section the necessary and sufficient conditions for a space to be an Einstein space and these for a space to be a space of constant curvature using the results of the above section.

THEOREM 3. *If the dimension n of M is greater than* 2, *the following three conditions are equivalent:*

(a) *M is an Einstein space.*

(b) $g^{\nu\mu}\nabla_\nu\nabla_\mu w_\lambda$ *is closed for any closed vector field* w_λ *of class* C^3.

(c) $g^{\nu\mu}\nabla_\nu\nabla_\mu w_\lambda$ *is coclosed for any coclosed vector field* w_λ *of class* C^3.

A. Avez has proved in [**1**] the equivalence of (a) and (b) and also proved the equivalence of (a) and (c) in the case M is compact.

PROOF OF THEOREM 3. From the formula (1.3) we find that when w_λ is closed (or coclosed), $g^{\nu\mu}\nabla_\nu\nabla_\mu w_\lambda$ is closed (or coclosed) if and only if $K_\lambda{}^\alpha w_\alpha$ is closed (or coclosed). Thus (b) is equivalent to the condition:

(b′) $K_\lambda{}^\alpha w_\alpha$ is closed for any closed vector field w_λ of class C^3,

and (c) is equivalent to the condition:

(c′) $K_\lambda{}^\alpha w_\alpha$ is coclosed for any coclosed vector field w_λ of class C^3.

From Theorem 1 these (b′) and (c′) are equivalent to

$$K_{\mu\lambda}=cg_{\mu\lambda}$$

where c is a constant and this is just the condition (a).

REMARK. If the dimension of M is equal to 2, it is well known that M is an Einstein space and consequently $K_{\mu\lambda}=cg_{\mu\lambda}$ but as in this case c is not always a constant Theorem 1 cannot be applied.

The following theorem 4 is an extension of Theorem 3 to the case of a space of constant curvature.

THEOREM 4. *If the dimension n of M is greater than* 2 *and not equal to* 4, *the following three conditions are equivalent:*

(a) *M is a space of constant curvature.*

(b) $g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa}$ *is closed for any closed tensor field* $w_{\lambda\kappa}$ *of class* C^3.

(c) $g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa}$ *is coclosed for any coclosed tensor field* $w_{\lambda\kappa}$ *of class* C^3.

PROOF. Put $T_{\nu\mu}{}^{\lambda\kappa}=2K_{[\nu}{}^{[\lambda}A_{\mu]}^{\kappa]}+K_{\nu\mu}{}^{\lambda\kappa}$, then from the formula (1.4) we have

$$(\Delta w)_{\lambda\kappa}=g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\nu\kappa}-T_{\lambda\kappa}{}^{\beta\alpha}w_{\beta\alpha}\,,$$

and we find that when $w_{\lambda\kappa}$ is closed (or coclosed) $g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa}$ is closed (or cocloses) if and only if $T_{\lambda\kappa}{}^{\beta\alpha}w_{\beta\alpha}$ is closed (or coclosed). Thus the condition (b) is equivalent to the condition:

(b′) $T_{\lambda\kappa}{}^{\beta\alpha}w_{\beta\alpha}$ is closed for any closed tensor field $w_{\lambda\kappa}$ of class C^3,

and the condition (c) is equivalent to the condition:

(c′) $T_{\lambda\kappa}{}^{\beta\alpha}w_{\beta\alpha}$ is coclosed for any coclosed tensor field $w_{\lambda\kappa}$ of class C^3.

Applying Theorem 2 to the tensor field $T_{\nu\mu\lambda\kappa}$ these conditions (b′) and (c′) are equivalent to

$$T_{\nu\mu}{}^{\lambda\kappa}=c(A_\nu^\lambda A_\mu^\kappa-A_\nu^\kappa A_\mu^\lambda)$$

or

$$2K_{[\nu}{}^{[\lambda}A_{\mu]}^{\kappa]}+K_{\nu\mu}{}^{\lambda\kappa}=c(A_\nu^\lambda A_\mu^\kappa-A_\nu^\kappa A_\mu^\lambda)\,, \tag{3.1}$$

where c is a constant.

Thus it is sufficient to prove the equivalence of (a) and (3.1).

Assume (a), that is,

$$K_{\nu\mu\lambda\kappa}=k(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})\,, \tag{3.2}$$

where k is a constant, then we have

$$K_{\mu\lambda}=\frac{K}{n}g_{\mu\lambda}\,.$$

Thus (3.1) is easily verified.

Conversely suppose (3.1), then by contraction with respect to κ and ν, we have

$$(n-4)K_{\mu\lambda}=\{2(n-1)c-K\}g_{\mu\lambda}\,. \tag{3.3}$$

Transvecting this by $g^{\mu\lambda}$ we get

$$K=\frac{n(n-1)}{n-2}c\,. \tag{3.4}$$

So, K is a constant and from (3.3) and the assumption $n\neq 4$ we obtain

$$K_{\mu\lambda}=c'g_{\mu\lambda}\,,$$

where c' is a constant. Substituting this into (3.1) we find

$$K_{\nu\mu\lambda\kappa}=k(g_{\nu\lambda}g_{\mu\kappa}-g_{\nu\kappa}g_{\mu\lambda})\,,$$

where k is a constant.

If we assume that the dimension of M is equal to 4 and suppose that (b) or (c) in Theorem 4 holds, we find in the similar way in which we obtained (3.4) in the proof of Theorem 4, that K is a constant. Thus we have

THEOREM 5. *If the dimension n of M is equal to* 4 *and the condition* (b) *or* (c) *in Theorem* 4 *is satisfied, then M has a constant curvature scalar.*

REMARK. The operator $\nabla^{\alpha}\nabla_{\alpha}$ appeared in Theorems 3, 4 and 5, gives an endomorphism of the space of skew-symmetric tensor fields of order p or of the space of p-forms. From Theorem 3, we can easily verify that if $\nabla^{\alpha}\nabla_{\alpha}$ induces an endomorphism of the space of closed 1-forms, then it induces an endomorphism of the space of exact 1-forms, and consequently, it induces an endomorphism of one dimensional homology group of the manifold M. Moreover, if a 1-form w is harmonic then $\nabla^{\alpha}\nabla_{\alpha}w$ is also harmonic and equal to $\frac{K}{n}w$. Thus if M is compact and K does not vanish, then the induced endomorphism of the homology group is an isomorphism onto, and if K vanishes, then the induced endomorphism is trivial.

In a quite similar way, if $\nabla^{\alpha}\nabla_{\alpha}$ induces an endomorphism of the space of closed two-forms, then it induces an endomorphism of 2-dimensional homology group of the manifold M. Moreover if M is compact, and the sectional curvature does not vanish, the induced endomorphism of the homology group is an isomorphism onto and if the sectional curvature vanishes, then the induced isomorphism is trivial.

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