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Homogeneous hypersurfaces in euclidean spaces.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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S. Kobayashi [3] proved that a compact connected homogeneous Riemannian manifold M of dimension n is isometric to the sphere if it is isometrically imbedded in the euclidean space E of dimension $n+1$. In this paper we shall prove that a connected homogeneous Riemannian space M (compact or not) of dimension n is isometric to the Riemannian product of a sphere and a euclidean space if M is isometrically imbedded in the euclidean space E of dimension $n+1$ and the rank of the second fundamental form H is of rank $\neq 2$ at some point.

Manifolds and mappings between them will always be of differentiability class C^{∞}.

1. Preliminaries.

Let M be a connected Riemannian manifold. Assume that there exists an isometric map f of M into a euclidean space E, in which we fix a cartesian coordinate system. f is isometric in the sense that the dual map of the differential f' of f carries the Riemannian metric of E to that of M.

Assigning to a point p of M the A-th coordinate component of $f(p)$, $1\leqq A\leqq \dim E$, we obtain a function f^{A} on M. For any vector X tangent to M at x, we denote by Xf the vector tangent to E at $f(x)$ whose A-th component is Xf^{A} and call Xf the covariant differentiation of f in X. We shall write X for ∇_X or $X^{\mu}\nabla_{\mu}$ in coordinates as long as no ambiguity might be feared. In the same way one can define the covariant differentiation Xf' of the differential f' of f and other objects such as a map of M into the tangent bundle of E or into the isometry group of E. It goes without saying that, when X has x as the origin, Xf' is a linear map of the tangent space M_x to M at x into the tangent space $E_{f(x)}$ for any x in M, and that $Xf=f'X$.

It is easy to see that $(Xf')Y$ is normal to $f(M)$ for any vectors X and Y at a point x. Thus $(Xf')Y$ is a linear combination of the normal vectors

$$(Xf')Y=\sum_{1\leqq t\leqq d} H_t \mathfrak{n}_t\,,$$

where $\mathfrak{n}_t$ are linearly independent vectors normal to $f(M)$ at $f(x)$ and d equals $\dim E - \dim M$. Each $H_t = H_t(X, Y)$ is a bilinear form on M_x. The *rank* of f at x is by definition the minimum number of linear forms on M_x in which H_t can be expressed; it is independent of the choice of the normal vectors.

From now on we shall assume that $d=1$; $f(M)$ is a hypersurface of E. Given an orientable neighborhood U in M, we fix a map $\mathfrak{n}$ of U into the tangent bundle of E such that $\mathfrak{n}(x)$ is a unit normal to $f(U)$ at $f(x)$ for each x in U. Then a covariant tensor field H of degree 2 is defined by

$$(1.1) \qquad (Xf')Y = H(X, Y)\mathfrak{n}(x), \qquad X, Y \in M_x.$$

H is the second fundamental form of f, which depends on the choice of $\mathfrak{n}$ and is determined on U up to a constant e with $e^2=1$ if U is connected.

From (1.1) follows

(1.2) *$X\mathfrak{n}$ is tangent to $f(M)$ and the inner product of $X\mathfrak{n}$ with $f'Y$ equals $-H(X, Y)$; $(X\mathfrak{n}, f'Y) = -H(X, Y)$.*

Some of the following propositions in this section are known. (See [1] and [5]).

THEOREM 1.1. *Let f and $\hat{f}$ be isometric maps of M into E. Assume that for any connected orientable neighborhood U in M there exists a constant e with $e^2=1$ such that we have $H = e\hat{H}$ on U, H and $\hat{H}$ being the second fundamental forms of f and $\hat{f}$ respectively. Then there exists an isometry α of E onto itself satisfying $\alpha f = \hat{f}$.*

Note that M is not necessarily orientable.

PROOF. For a point x of M, take a connected orientable neighborhood U of x and consider the isometry α_x of E onto itself defined by

$$(1.3) \qquad \alpha_x(f(x)) = \hat{f}(x),$$

$$(1.4) \qquad \alpha_x' f' = \hat{f}' \quad \text{on } M_x,$$

$$(1.5) \qquad \alpha_x' \mathfrak{n} = e\hat{\mathfrak{n}}.$$

α_x is independent of the choice of U, as is easily seen. Thus we obtain a map α of M into the isometry group of E such that $\alpha(x) = \alpha_x$. By (1.1), (1.4) and (1.5) together with $H = e\hat{H}$, we have

$$\begin{aligned}(X\alpha')(f'Y) &= X(\alpha' f' Y) - \alpha'(Xf')Y - \alpha' f' XY = X\hat{f}'Y - \alpha' H(X, Y)\mathfrak{n} - \hat{f}'XY \\ &= X\hat{f}'Y - \hat{H}(X, Y)\hat{\mathfrak{n}} - \hat{f}'XY \\ &= X\hat{f}'Y - (X\hat{f}')Y - \hat{f}'XY = 0\end{aligned}$$

for any vector X tangent to U and a vector field Y on U. To prove $X\alpha' = 0$ we have to show $(X\alpha')\mathfrak{n} = 0$. By (1.2), (1.4) and (1.5) we get

$$(X\alpha')\mathfrak{n} = X(\alpha'\mathfrak{n}) - \alpha' X\mathfrak{n} = eX\hat{\mathfrak{n}} - \hat{f}' f'^{-1} X\mathfrak{n} = 0;$$

in fact by (1.2) the inner product

$$(f'f'^{-1}X\mathfrak{n}, \hat{f}'Y) = (f'^{-1}X\mathfrak{n}, Y) = (X\mathfrak{n}, f'Y) = -H(X, Y) = -e\hat{H}(X, Y)$$
$$= (eX\hat{\mathfrak{n}}, \hat{f}'Y)$$

for any tangent vector Y with the same origin as X.

Therefore we have $X\alpha' = 0$; i.e. the rotation part α' of α is constant. Finally (1.3) and (1.4) imply that

$$(X\alpha)f = X\alpha f - \alpha' Xf = X\hat{f} - \alpha' f' X = \hat{f}'X - \hat{f}'X = 0 .$$

Hence α is constant on M, and we have $\alpha f = \hat{f}$.

LEMMA 1.2. *Let f and $\hat{f}$ be as in Theorem 1.1. Denote by $r = r(x)$ and $\hat{r} = \hat{r}(x)$ the ranks at x of f and $\hat{f}$ respectively. Then r equals either $\hat{r}$ or $1-\hat{r}$. In particular the inequality $1 < r$ gives $\hat{r} = r$.*

For the proof we recall the Gauss formula:

(1.6) *K denoting the curvature tensor of M, the vector $K(X, Y)Z$, with the components $K_{\alpha\beta\gamma}{}^{\lambda}X^{\alpha}Y^{\beta}Z^{\gamma}$, is the dual of the one-form θ*

$$\theta:\ W \to H(X, W)H(Y, Z) - H(Y, W)H(X, Z) = (K(X, Y)Z, W) .$$

Fix a basis of M_x, and denote by ϕ_ν the form (on M_x): $Y \to H(X, Y)$ where X is the ν-th vector of the basis. $\hat{\phi}_\nu$ is defined analogously by means of $\hat{H}$. Then r equals the number of linearly independent forms in the system $\{\phi_\nu\}$. Hence the number of linearly independent forms in the system $\{\phi_\mu \wedge \phi_\nu\}$ is $r(r-1)/2$. On the other hand $\phi_\mu \wedge \phi_\nu$ equals $\hat{\phi}_\mu \wedge \hat{\phi}_\nu$ by (1.6). Hence we have $r(r-1)/2 = \hat{r}(\hat{r}-1)/2$, and so we have $(r-\hat{r})(r+\hat{r}-1) = 0$.

COROLLARY 1.3. *Let f be an isometric map of M into E, and ρ an isometry of M onto itself. Then the rank $r(x)$ of f at x is equal to either $r(\rho(x))$ or $1-r(\rho(x))$. In particular $1 < r(x)$ implies $r(x) = r(\rho(x))$.*

Put $\hat{f} = f\rho$. Since ρ is an isometry, ρ commutes with the covariant differentiation; in particular we have $(X(f'\rho'))Y = ((\rho'X)f')\rho'Y$. Hence we have $\hat{H}(X, Y)\mathfrak{n}(x) = H(\rho'X, \rho'Y)\mathfrak{n}(\rho(x))$. From Lemma 1.2 thus follows Corollary 1.3.

THEOREM 1.4. *Let f and $\hat{f}$ be as in Theorem 1.1. If $r \geqq 3$ at every point, then there exists an isometry α of E onto itself such that $\alpha f = \hat{f}$.*

PROOF. From $\phi_\mu \wedge \phi_\nu = \hat{\phi}_\mu \wedge \hat{\phi}_\nu$ (see the proof of 1.2) follows

$$\hat{\phi}_\mu \wedge \phi_\mu \wedge \phi_\nu = \hat{\phi}_\mu \wedge \hat{\phi}_\mu \wedge \hat{\phi}_\nu = 0 .$$

If $\hat{\phi}_\mu$ and ϕ_μ are linearly independent, any ϕ_ν is a linear combination of ϕ_μ and $\hat{\phi}_\mu$, contrary to the assumption. Thus we have $\hat{\phi}_\mu = c_\mu \phi_\mu$ for each μ, c_μ being some real number. Hence $\hat{\phi}_\mu \wedge \hat{\phi}_\nu = c_\mu c_\nu \phi_\mu \wedge \phi_\nu$. It follows that c_μ's are all equal to a number e with $e^2 = 1$. From this and the definition of ϕ_ν we conclude $H = e\hat{H}$. Now Theorem 1.4 follows from Theorem 1.1.

COROLLARY 1.5. *Let f be an isometric map of M into E. If an isometry group G of M is transitive and the rank r of f satisfies $3 \leqq r$ at some point,* *then for any ρ in G there exists a unique isometry α of E on itself such that $f\rho=\alpha f$.*

By Corollary 1.4, we have $3\leqq r$ at every point. Thus there exists an isometry α with $f\rho=\alpha f$ by Theorem 1.4. α is unique, for otherwise $f(M)$ would be symmetric with respect to a hyperplane with which $f(M)$ would coincide locally, contrary to $r\geqq 3$ everywhere.

2. The case $3\leqq r$.

This section is devoted to the proof of

LEMMA 2.1. *Assume that there exists an isometric map f of a connected homogeneous Riemannian manifold M onto a hypersurface of a euclidean space E. If the rank r of f satisfies $3\leqq r$ at some point, then M is isometric to the Riemannian product of a sphere and a euclidean space. In particular f is unique up to the composition αf with an isometry α of E.*

For brevity we identify M with $f(M)$. By Corollary 1.5, the connected isometry group G of M can be identified with a subgroup of the isometry group of E. Take an arbitrary line γ normal to M. If there exists a G-orbit $G(p)$ of dimension $<n$, $n=\dim M$, $p\in\gamma$, then o shall be one of such points. Otherwise o shall be an arbitrary point on $\gamma\cap M$. Denote by N the G-orbit $G(o)$ and by F the plane (= a linear subspace) which is the union of the lines normal to N at o.

Now we shall prove the following lemma.

LEMMA 2.2. *If a one-parameter subgroup L of G leaves fixed a point q on F, then L leaves fixed o.*

Let $H=H_o$ be the isotropy subgroup of G at o. ν denoting the dimension of N, there exist ν linearly independent Killing vectors $u_1,\cdots,u_\nu$ which, together with the Lie algebra of H, span the Lie algebra of G. The dual one-forms of u_i will be denoted by the same letters. We have to prove

$$\text{(2.1)} \qquad \text{the form } \rho=u_1\wedge u_2\wedge\cdots\wedge u_\nu\neq 0 \qquad \text{on } F.$$

Let U be the subset of γ consisting of the points p at which we have $\dim G(p)=n$. V shall be the complement of U in γ. The inequality $\rho\neq 0$ holds at each point p of U, for we have $\dim H(p)=\dim F-1=n+1-\nu-1$. Let x be a boundary point of U, if any. Since U is open, x belongs to V. Let H_x be the isotropy subgroup of G at x. H_x leaves invariant the plane $H_x(\gamma)$ and is transitive on the unit sphere in that plane. It follows that every point $y\neq x$ of γ sufficiently near x belongs to U. Hence V is discrete in γ. Further it follows that the point $o'\neq o$ of γ at the same arc length from x as o belongs to V, where we have assumed that V is not empty and contains points x other than o. Thus V is an infinite set. On the other hand every Killing field in E has the components expressed as polynomials in the cartesian coordinates. Hence the form ρ, restricted on γ, has the components expressed as a polynomial, say in the arc length s from o. ρ vanishes on V. We thus infer that V contains at most o only. Hence we have $\rho \neq 0$ on γ, therefore on $F = H(\gamma)$.

We identify F with the tangent space to F at o and E with the tangent space to E at o. The tangent space to N at o is denoted by N_o.

(2.2) H acts naturally on the tangent space E and leaves invariant the subspaces F and N_o.

Every point x of E is identified with the vector $\mathfrak{x} = ox$. Then any element of the Lie algebra G' of G is expressed by the pair $(A, \mathfrak{a})$ of a skew-symmetric matrix A and a vector $\mathfrak{a}$ in E such that $(A, \mathfrak{a})$ maps $\mathfrak{x}$ to $A\mathfrak{x}+\mathfrak{a}$. Let P and Q be the orthogonal projections of E onto F and N_o respectively. Then we have

(2.3) $$P\mathfrak{a} = 0, \text{ i. e. } Q\mathfrak{a} = \mathfrak{a} \text{ for any } (A, \mathfrak{a}) \text{ in } G'.$$

Given a vector $\mathfrak{r}$ in F we define a bilinear form R on G' by $R: ((A, \mathfrak{a}), (B, \mathfrak{b})) \to$ the inner product $(\mathfrak{r}, PA\mathfrak{b})$.

Since the linear map $(A, \mathfrak{a}) \in G' \to \mathfrak{a} = Q\mathfrak{a} \in N_o$ is onto, and $\mathfrak{a} = 0$ implies $PAQ = 0$ by (2.2) and therefore $PA\mathfrak{b} = PAQ\mathfrak{b} = 0$ by (2.3), R can be regarded as a well-defined bilinear form on N_o.

(2.4) The bilinear form R on N_o is symmetric.

Proof. The bracket product $[(A, \mathfrak{a}), (B, \mathfrak{b})]$ in G' equals $([A, B], A\mathfrak{b} - B\mathfrak{a})$. By (2.3) we thus have $P(A\mathfrak{b} - B\mathfrak{a}) = 0$. Hence $R(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{r}, PA\mathfrak{b}) = (\mathfrak{r}, PB\mathfrak{a}) = R(\mathfrak{b}, \mathfrak{a})$.

(2.5) $PAQ = 0$ for any $(A, \mathfrak{a})$ in G'.

Proof. Otherwise we have $R \neq 0$ for some $\mathfrak{r}$ in F. By (2.4) R has an eigenvalue c different from 0. Let $\mathfrak{b} \neq 0$ be the corresponding eigenvector; $R(\mathfrak{b}, \mathfrak{a}) = c(\mathfrak{b}, \mathfrak{a})$ ($= c$ multiplied by the inner product of $\mathfrak{b}$ and $\mathfrak{a}$) for any $\mathfrak{a}$ in N_o. We have $c(\mathfrak{b}, \mathfrak{a}) = R(\mathfrak{b}, \mathfrak{a}) = (\mathfrak{r}, PBQ\mathfrak{a}) = ({}^t(PBQ)\mathfrak{r}, \mathfrak{a})$, where ${}^tK$ denotes the transposed matrix of K. Hence we obtain ${}^t(PBQ)\mathfrak{r} = c\mathfrak{b}$. Let μ be the linear map of G' into N_o (or, more precisely, into the subspace of the tangent space to E at the point $\mathfrak{r}/c$ of F which is parallel to N_o) defined by $\mu((A, \mathfrak{a})) = Q(A\mathfrak{r}/c + \mathfrak{a})$. It follows then that $\mu((B, \mathfrak{b})) = Q(B\mathfrak{r}/c + \mathfrak{b}) = QBP\mathfrak{r}/c + \mathfrak{b} = -{}^t(PBQ)\mathfrak{r}/c + \mathfrak{b} = -\mathfrak{b} + \mathfrak{b} = 0$. This means that the one-parameter group generated by $(B, \mathfrak{b})$ leaves fixed the point $\mathfrak{r}/c$ in F, though it does not leave fixed the point o, contrary to (2.1). Thus (2.5) is proved.

From (2.5) we infer that G which is transitive on N carries N_o to linear subspaces which are parallel to N_o in E. Therefore we have proved that

(2.6) N is a plane.

Hence for any point p in N there exists exactly one perpendicular to N

starting at p. It follows as in [4] that E admits a fibre bundle structure over N with fibre F associated with the principal bundle $(G, G/H, H)$, for the map $(\alpha, x)\in G\times F\rightarrow\alpha(x)\in E$ is onto and we have $\alpha(x)=\beta(y)$ if and only if $\alpha\beta^{-1}$ belongs to H and $x=\alpha^{-1}\beta(y)$. Assume $M\neq N$. Any G-orbit $\neq N$ (and in particular M) is a subbundle with a sphere S of dimension $=n-\dim N$ as the fibre. Since N is a plane, the bundle is trivial. Thus M is homeomorphic to $S\times N$. By (2.5), M is clearly isometric to the Riemannian product $S\times N$, which proves the lemma 2.1; in case $M=N$ the lemma follows directly from (2 6), though this case cannot occur because of the hypothesis $3\leqq r$.

3. The case $r\leqq 1$.

In case $r\leqq 1$, M is locally flat by the Gauss formula (1.6).

THEOREM 3.1. *A connected homogeneous Riemannian manifold M which is locally flat is the Riemannian product of a euclidean space and a torus. A torus is the Riemannian product of a finite number of circles.*

The universal covering Riemannian manifold of M is the euclidean space, which we denote by E here. In E we fix a cartesian coordinate system. Let G be a connected transitive isometry group of M. G induces an isometry group $\hat{G}$ of E so that $\hat{G}$ is an extension of G by the Poincaré group P (= the 1-dimensional homotopy group) of M. Since P is a discrete normal subgroup of G, P is contained in the center of G. Any element of P can be expressed by a pair $(C, \mathfrak{c})$ of an orthogonal matrix C and a vector $\mathfrak{c}$ in E such that $(C, \mathfrak{c})$ carries a point $\mathfrak{x}$ of E to $C\mathfrak{x}+\mathfrak{c}$.

(3.1) $C\mathfrak{c}=\mathfrak{c}$ for any $(C, \mathfrak{c})$ in P.

Since $(C, \mathfrak{c})$ commutes with any element $(A, \mathfrak{a})$ of the Lie algebra G' of $\hat{G}$, we have

$$A\mathfrak{c}+\mathfrak{a}=C\mathfrak{a}\,.$$

A being skew-symmetric, $A\mathfrak{c}$ is orthogonal to $\mathfrak{c}$. For an arbitrary vector $\mathfrak{x}$, $\|\mathfrak{x}\|$ shall denote its length. Since $\hat{G}$ is transitive on E, $\mathfrak{a}$ can be any vector. Putting $\mathfrak{a}=\mathfrak{c}$, we get $\|A\mathfrak{c}\|^2+\|\mathfrak{a}\|^2=\|C\mathfrak{a}\|^2=\|\mathfrak{a}\|^2$. It follows $A\mathfrak{c}=0$, and so (3.1).

(3.2) The n-time composition $(C, \mathfrak{c})^n$ of $(C, \mathfrak{c})$ is $(C^n, n\mathfrak{c})$ for any $(C, \mathfrak{c})$ in P.

This follows from (3.1) easily.

(3.3) $A\mathfrak{c}=\mathfrak{c}$ for any $(A, \mathfrak{a})$ in $\hat{G}$ and any $(C, \mathfrak{c})$ in P.

PROOF. Since $(C, \mathfrak{c})^n$ commutes with $(A, \mathfrak{a})$ for any positive integer n, we obtain from (3.2)

$$nA\mathfrak{c}+\mathfrak{a}=C^n\mathfrak{a}+n\mathfrak{c}\,, \tag{3.4}$$

that is, $n(A\mathfrak{c}-\mathfrak{c})=C^n\mathfrak{a}-\mathfrak{a}$.

Assume that $A\mathfrak{c}\neq\mathfrak{c}$. Then the length $\|n(A\mathfrak{c}-\mathfrak{c})\|$ is not bounded as a function of n, while $\|C^n\mathfrak{a}-\mathfrak{a}\|\leqq\|C^n\mathfrak{a}\|+\|\mathfrak{a}\|=2\|\mathfrak{a}\|$ is obviously bounded. Thus

(3.3) is true.

(3.5) C is the identity matrix for any $(C, \mathfrak{c})$ in P.

From (3.3) and (3.4) follows $C^n\mathfrak{a}=\mathfrak{a}$; in particular $C\mathfrak{a}=\mathfrak{a}$. $\hat{G}$ being transitive, $\mathfrak{a}$ can be any vector and we have (3.5).

By (3.5), P is a free abelian group contained in the translation group of E. Hence M is the Riemannian product of a euclidean space and a subspace N whose undelying manifold is that of a toral group T. T is a transitive isometry group of N. Hence N is a torus and Theorem 3.1 is proved.

LEMMA 3.2. *Lemma* 2.1 *holds good with the condition* $3\leqq r$ *replaced by* $r\leqq 1$. *The sphere is of dimension one or zero.*

M is then locally flat as was remarked before. By Theorem 3.1, M is the Riemannian product of a euclidean space and a torus T. By Corollary 1.3, we have $r\leqq 1$ throughout on the homogeneous space M. Restricted to T, f gives an imbedding of T into E whose rank does not exceed $r\leqq 1$ as is easily seen. Now the following theorem of Chern [2, p. 23] applies: *Let g be an isometric map of a compact Riemannian manifold N into a euclidean space. Let $s(p)$ denote the rank of g at a point p of N. Then we have*

$$\dim N \leqq \operatorname*{Max}_{p\in M} s(p).$$

And we conclude $\dim T\leqq 1$ and the Lemma 3.2 is proved.

The main theorem mentioned in the introduction follows from Lemma 2.1 and Lemma 3.2.

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