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A diffeomorphy invariant of quotient manifolds.

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Introduction.

Let $\tilde{M}$ be a compact connected orientable odd-dimensional C^{∞}-manifold which is the boundary manifold of a compact connected orientable C^{∞}-manifold (with boundary) and let $\boldsymbol{G}=\{G\}$ be a family of finite groups whose elements are orientation-preserving C^{∞}-transformations of $\tilde{M}$ having no fixed point. In this note we shall define an invariant Π of quotient manifolds $M=\tilde{M}/G$ $(G\in\boldsymbol{G})$, making use of P-classes of C^{∞}-M-spaces introduced in the previous paper (Tamura [**7**]). Π becomes a diffeomorphy invariant, if $\tilde{M}$ satisfies some conditions (Section 2, (a)-(d)).

In particular we shall consider the diffeomophy invariant Π of lens spaces. As we shall see in the case of 3-dimensional lens spaces, Π is not homotopy type invariant in general.

The lens spaces have a combinatorial invariant, the Reidemeister's torsion (Reidemeister [**4**], Franz [**1**]), which turned out to be homeomorphy invariant, in virtue of the result of Moise [**2**], for 3-dimensional cases. On the other hand, the homotopy classification of lens spaces was given by J. H. C. Whitehead [**8**], P. Olum [**3**]. In comparison with this, one sees that the Reidemeister's torsion is not homotopy type invariant.

Now we have shown in our former papers (Tamura [**5**], [**6**]) that the Pontrjagin classes have the like property. It seemed desirable to us to have an analogous approach in the case of lens spaces, i. e. to have a homeomorphy invariant connected with characteristic classes of differentiable structures naturally defined on lens spaces, which is not homotopy type invariant.

But since the Pontrjagin classes vanish for 3-dimensional manifolds, they cannot be used for such purpose. So we shall use P-classes (a generalization of Pontrjagin classes) of C^{∞}-M-spaces (a generalization of C^{∞}-manifolds) associated to lens spaces.

A finer classification of these spaces than the homotopy classification will be given by the invariant Π (Theorem 4).

1. Definition of $\Pi(M)$.

Let $\tilde{M}$ be a compact connected oriented $(2n+1)$-dimensional C^{∞}-manifold

($n \geqq 1$) and let G be a finite group of orientation-preserving C^∞-transformations of $\tilde{M}$ having no fixed point. If we identify those points of $\tilde{M}$ which correspond under a transformation of G, we obtain a compact connected oriented $(2n+1)$-dimensional C^∞-manifold $M=\tilde{M}/G$. We shall call M the *quotient manifold* of $\tilde{M}$.

$\tilde{M}$ is a C^∞-differentiable covering manifold of M. Denote the covering map by $p:\tilde{M}\to M$. Then we have a principal fibre bundle $\{\tilde{M}, p, M, G, G\}$.

We assume that $\tilde{M}$ is the boundary manifold of a compact connected oriented $(2n+2)$-dimensional C^∞-manifold (with boundary) N.

Now we construct the mapping cylinder $M_0(p)$ of $p:\tilde{M}\to M$ as follows. Let $I=[0,1]$ be the closed unit interval of real numbers. $\tilde{M}\times I$ has the natural differentiable structure. Suppose that G operates on $\tilde{M}\times 0$ which is $(2n+1)$-dimensional submanifold of $\tilde{M}\times I$, in an obvious manner. If we identify the points of $\tilde{M}\times I$ which are transformed by an element of G, we obtain a topological space $M_0(p)$.

$\boldsymbol{M}_0(\boldsymbol{p})=\{M_0(p), \tilde{M}\times I, \tilde{M}\times 0, G, \varphi\}$ is a $(2n+2)$-dimensional C^∞-M-space (Tamura [**7**, Definition 1.1]).

Under the identification of the boundary of N and $\tilde{M}\times 1$, we obtain a topological space $M(p)=M_0(p)\cup N$. $M(p)$ becomes a $(2n+2)$-dimensional C^∞-M-space. In fact, $M(p)$ is nothing but a topological space obtained from N identifying the points transformed by an element of G which operates on the boundary $\tilde{M}$ of N. That is, $\boldsymbol{M}(\boldsymbol{p})=\{M(p), N, \tilde{M}, G, \varphi\}$.

Obviously we have

$$H_{2n+2}(M(p);Z_m)\approx Z_m, \quad H^{2n+2}(M(p);Z_m)\approx Z_m,$$

where m denotes the order of G and Z_m the group of integers mod m. Denote by $[M(p)]$ and $\{M(p)\}$ their generators determined by the orientation of N concordant with the orientation of $\tilde{M}$.

Clearly the normal vector bundle of $\tilde{M}$ in N is a product bundle. Let F be the cross section over $\tilde{M}$ of this vector bundle defined by the vector field of $\tilde{M}$ consisting of normal vectors with outward direction at each point of $\tilde{M}$.

Let $\mathfrak{T}(\boldsymbol{M}(\boldsymbol{p}))=\{\boldsymbol{M}(\boldsymbol{p}), \mathfrak{T}(\tilde{M}), E^{2n+2}, SO(2n+2)\}$ be the tangent D-bundle of C^∞-M-space $\boldsymbol{M}(\boldsymbol{p})$ (Tamura [**7**, Definition 3.3]) namely a collection as follows:

(i) The tangent vector bundle $\mathfrak{T}(N)=\{T(N), p, N, E^{2n+2}, SO(2n+2)\}$ of N in the usual sense.

(ii) The tangent vector bundle $\mathfrak{T}(\tilde{M})=\{T(\tilde{M}), p, \tilde{M}, E^{2n+1}, SO(2n+1)\}$ of $\tilde{M}$ in the usual sense.

(iii) An isomorphism α of G into the group of C^∞-bundle maps of $\mathfrak{T}(\tilde{M})$ onto itself defined by

$$\alpha: g\to dg \qquad (g\in G).$$

(iv) The natural injection $\lambda : \mathfrak{T}(\tilde{M}) \to \mathfrak{T}(N) | \tilde{M}$.

Moreover let $\mathfrak{T}^{[n+2]}(\boldsymbol{M}(\boldsymbol{p}))(F) = \{\boldsymbol{M}(\boldsymbol{p}), \mathfrak{T}^{[n+2]}(N), U(2n+2)/U(n), SO(2n+2)\}$ be the generalized associated D-bundle of $\mathfrak{T}(\boldsymbol{M}(\boldsymbol{p}))$ defined by a collection as follows:

(i) The associated bundle $\mathfrak{T}^{[n+2]}(N) = \{T^{[n+2]}(N), p, N, U(2n+2)/U(n), SO(2n+2)\}$ of $\mathfrak{T}(N)$ in the usual sense.

(ii) The associated bundle $\mathfrak{T}^{[n+1]}(\tilde{M}) = \{T^{[n+1]}(\tilde{M}), p, \tilde{M}, U(2n+1)/U(n), SO(2n+1)\}$ of $\mathfrak{T}(\tilde{M})$ in the usual sense.

(iii) An isomorphism $\alpha^{[n+2]}$ of G into the group of C^∞-bundle maps of $\mathfrak{T}^{[n+1]}(\tilde{M})$ onto itself determined by α in an obvious manner.

(iv) An injection $\lambda^{[n+2]} : \mathfrak{T}^{[n+1]}(\tilde{M}) \to \mathfrak{T}^{[n+2]}(N) | \tilde{M}$ defined by

$$\lambda^{[n+2]}(V_x(n+1)) = \lambda(V_x(n+1)) \vee F(x) \quad (x \in \tilde{M}),$$

where $V_x(n+1)$ and $\lambda(V_x(n+1))$ denote the fibre of $\mathfrak{T}^{[n+1]}(\tilde{M})$ over x and its image by λ respectively, and the right hand side means the $(n+2)$-frame obtained as the union of $\lambda(V_x(n+1))$ and $F(x)$.

Since the first non-zero homotopy groups of $U(2n+1)/U(n)$, $U(2n+2)/U(n)$ are $\pi_{2n+1}(U(2n+1)/U(n)) \approx \pi_{2n+1}(U(2n+2)/U(n)) \approx Z$ and each element of G operates on $\tilde{M}$ without fixed points, the primary obstruction class, i. e. $((n+1)/2)$-th P-class $P_{(n+1)/2}(\boldsymbol{M}(\boldsymbol{p}))$ of $\boldsymbol{M}(\boldsymbol{p})$ can be defined (Tamura [7, Section 7]).

In 3-dimensional case, we can take the associated D-bundle $\mathfrak{T}^{[3]}(\boldsymbol{M}^4(\boldsymbol{p}))$, in place of the geneneralized associated D-bundle $\mathfrak{T}^{[3]}(\boldsymbol{M}^4(\boldsymbol{p}))(F)$, to define P-class of $\boldsymbol{M}^4(\boldsymbol{p})$, by virtue of $\pi_2(SU(3)) \approx 0$.

Definition 1. For $M = \tilde{M}/G$ described as above, we define $\Pi(M)$ by

$$\Pi(M) = \langle P_{(n+1)/2}(\boldsymbol{M}(\boldsymbol{p})), [M(p)] \rangle .$$

$\Pi(M)$ is an integer mod m (m is the order of G).

Obviously $\Pi(M)$ depends on the choice of $\tilde{M}$ and N.

2. Diffeomorphy invariance of Π.

In order to prove diffeomorphy invariance of Π, let us introduce assumptions about $\tilde{M}$ and N as follows:

(a) $\tilde{M}$ is the universal covering manifold of M (i. e. $\pi_1(\tilde{M}) = 0$).

(b) Let $\mathfrak{T}^{[n+2]}(N) = \{T^{[n+2]}(N), p, N, U(2n+2)/U(n), SO(2n+2)\}$ be the associated bundle of the tangent bundle $\mathfrak{T}(N)$ of N. Then $\mathfrak{T}^{[n+2]}(N)$ has a cross section over N.

(c) Let $\tilde{h}_0$ be an arbitrary diffeomorphic map of $\tilde{M}$ onto itself. Then there exist two differentiable cellular decompositions K and K' of N such that $\tilde{h}_0$ is extendable to a homeomorphic cellular map $\tilde{h} : K \to K'$ which is diffeomorphic on $(2n+1)$-section K^{2n+1}.

(d) Let $\tilde{h}_0$ be an arbitrary diffeomorphic map of $\tilde{M}$ onto itself and let $N_1 \cup N_2$ be a closed $(2n+2)$-dimensional C^∞-manifold obtained from two copies N_1, N_2 of N by the identification of points corresponding under $\tilde{h}_0$. Then the $((n+1)/2)$-th Pontrjagin class of $N_1 \cup N_2$ vanishes.

THEOREM 1. *Under the above assumptions* (a)-(d), $\Pi(M)$ *defined by* $\tilde{M}$ *and* N *is a diffeomorphy invariant.*

PROOF. Let M and M' be two diffeomorphic compact connected orientable $(2n+1)$-dimensional C^∞-manifolds. Denote the diffeomorphic map of M onto M' by h_0. M and M' have the same universal covering manifold $\tilde{M}$. Obviously there exists a diffeomorphic map $\tilde{h}_0$ of $\tilde{M}$ onto itself, for which the following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{h}_0} & \tilde{M} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{h_0} & M' , \end{array} \tag{2.1}$$

where p and p' are covering maps.

Now we have two C^∞-M-spaces $\boldsymbol{M}(\boldsymbol{p}) = \{M(p), N, \tilde{M}, G, \varphi\}$, $\boldsymbol{M}(\boldsymbol{p}') = \{M(p'), N, \tilde{M}, G', \varphi'\}$.

Let $\mathfrak{T}^{[n+2]}(\boldsymbol{M}(\boldsymbol{p}))(F)$, $\mathfrak{T}^{[n+2]}(\boldsymbol{M}(\boldsymbol{p}'))(F)$ be two C^∞-D-bundles as in Section 1. For a G-cross section f_0 of $\mathfrak{T}^{[n+1]}(\tilde{M})$ over $\tilde{M}$, $f_0' = (d\tilde{h}_0)(f_0)$ is a G'-cross section of $\mathfrak{T}^{[n+1]}(\tilde{M}')$ over $\tilde{M}'$ by (2.1). Denote cross sections $\lambda^{[n+2]}(f_0)$ and $\lambda'^{[n+2]}(f_0')$ of $\mathfrak{T}^{[n+2]}(N)$ over $\tilde{M}$ by f and f' respectively.

By the assumption (c) we can find a homeomorphic map $\tilde{h}: N \to N$ which is an extension of $\tilde{h}_0$. Suppose that U_i (resp. $\tilde{h}(U_i)$) $(i=1,\cdots,r)$ are coordinate neighbourhoods and $\phi_i(x,y)$ (resp. $\phi_i'(x,y)$) $(x \in N)$ $(i=1,\cdots,r)$ are coordinate functions of $\mathfrak{T}(N)$ with respect to U_i (resp. $\tilde{h}(U_i)$). Then, by the assumption (c), $\tilde{h}$ defines the maps

$$\tilde{h}_i : \phi_i(x,y) \to \phi_i'(\tilde{h}(x), v_i(\tilde{h}(x))y) \qquad (i=1,\cdots,r)\ (x \in K^{2n+1}), \tag{2.2}$$

where $\phi_i'(x, v_i(x))$ $(i=1,\cdots,r)$ $(x \in K'^{2n+1})$ is a cross section v of the associated principal bundle of $\mathfrak{T}(N)$ over K'^{2n+1}.

Let $\phi_i^{[n+2]}(x,y)$ (resp. $\phi_i'^{[n+2]}(x,y)$) $(i=1,\cdots,r)$ be coordinate functions of $\mathfrak{T}^{[n+2]}(N)$ associated to ϕ_i (resp. ϕ_i'). Then f and f' are expressed by $\phi_i^{[n+2]}(x, f_i(x))$ and $\phi_i'^{[n+2]}(x, f_i'(x))$ $(x \in \tilde{M})$ respectively. Obviously we have

$$f_i'(x) = (\iota(v_i(x)))(f_i(\tilde{h}_0^{-1}(x))) \qquad (x \in \tilde{M}),$$

where ι is the natural injection $SO(2n+2) \to U(2n+2)$.

Let $\bar{f}$ be a cross section $\phi_i^{[n+2]}(x, \bar{f}_i(x))$ $(i=1,\cdots,r)$ $(x \in K^{2n+1})$ of $\mathfrak{T}^{[n+2]}(N)$ which is an extension of f over K^{2n+1}. Then the set of

$$\phi_i'^{[n+2]}(\tilde{h}(x), (\iota(v_i(\tilde{h}(x))))(\bar{f}_i(x)) \qquad (i=1,\cdots,r)\ (x \in K^{2n+1}) \tag{2.3}$$

defines a cross section $\bar{f}'$ of $\mathfrak{T}^{[n+2]}(N)$ over K'^{2n+1} which is an extension of f'. We have by (2.3)

$$\tilde{c}(\bar{f})(\sigma^{2n+2})=\tilde{c}(\bar{f}')(\tilde{h}(\sigma^{2n+2}))-\tilde{c}(p_1(\iota(v)))(\tilde{h}(\sigma^{2n+2})) \qquad (\sigma^{2n+2}\in K)\,, \tag{2.4}$$

where p_1 denotes the natural projection $U(2n+2)\rightarrow U(2n+2)/U(n+2)$.

On the other hand, by the assumption (b), there exists a cross section $\mathring{f}$ of $\mathfrak{T}^{[n+2]}(N)$ over N defined by $\phi_i^{[n+2]}(x, \mathring{f}_i(x))$ $(i=1,\cdots,r)$. Then the set of

$$\phi_i'^{[n+2]}(\tilde{h}(x), (\iota(v_i(\tilde{h}(x))))(\mathring{f}_i(x))) \qquad (i=1,\cdots,r) \ (x\in K^{2n+1})$$

defines a cross section $\mathring{f}'$ of $\mathfrak{T}^{[n+2]}(N)$ over K'^{2n+1}. K and K' define a cellular decomposition $K\cup K'$ of $N_1\cup N_2$. Moreover $\mathring{f}$ and $\mathring{f}'$ define a cross section $\mathring{f}\cup\mathring{f}'$ of $\mathfrak{T}^{[n+2]}(N_1\cup N_2)$ over $(K\cup K')^{2n+1}$. Then, by the assumption (d), we have

$$\sum_{\sigma\in K}\tilde{c}(p_1(\iota(v)))(\tilde{h}(\sigma^{2n+2}))=0\,, \tag{2.5}$$

because

$$\tilde{c}(\mathring{f})(\sigma^{2n+2})=0 \qquad (\sigma^{2n+2}\in K)\,.$$

Combining (2.4) and (2.5), we have

$$\sum_{\sigma\in K}\tilde{c}(\bar{f})(\sigma^{2n+2})=\sum_{\sigma'\in K'}\tilde{c}(\bar{f}')(\sigma'^{\,2n+2})\,.$$

Thus we have proved

$$P_{(n+1)/2}(\boldsymbol{M}(\boldsymbol{p}))=\pm P_{(n+1)/2}(\boldsymbol{M}(\boldsymbol{p}'))\,,$$

$$\Pi(M)=\pm\Pi(M')\,. \qquad \text{q. e. d.}$$

Now suppose that $\tilde{M}$ is the $(2n+1)$-sphere $S^{2n+1}=\{(x_0, x_1,\cdots, x_{2n+1})\,|\,x_0^2+x_1^2+\cdots+x_{2n+1}^2=1\}$ with the natural differentiable structure and that N is the $(2n+2)$-dimensional closed cell $\Sigma^{2n+2}=\{(x_0, x_1,\cdots, x_{2n+1})\,|\,x_0^2+x_1^2+\cdots+x_{2n+1}^2\leqq 1\}$ with the natural differentiable structure. Clearly they satisfy the conditions (a), (b) and (c). The condition (d) is an immediate consequence of the index theorem of Thom-Hirzebruch, because $\Sigma_1^{2n+2}\cup\Sigma_2^{2n+2}$ is a $(2n+2)$-dimensional sphere (with an arbitrary differentiable structure). Hence we obtain the following theorem:

THEOREM 2. *Let S^{2n+1} be the $(2n+1)$-sphere and G a finite group of orientation-preserving C^∞-transformations of S^{2n+1}. Then $\Pi(S^{2n+1}/G)$ with respect to S^{2n+1} and Σ^{2n+2} is a diffeomorphy invariant.*

As we shall see in Section 4, $\Pi(S^{2n+1}/G)$ is not homotopy type invariant in general.

3. Lens spaces.

In order to consider the invariant Π of lens spaces, we recall here the definition and some properties of lens spaces.

Let S^{2n+1} be the unit $(2n+1)$-sphere in Euclidean $(2n+2)$-space E^{2n+2} given in terms of $n+1$ complex coordinates $(z_0, z_1,\cdots, z_n)$, $z_j = x_{2j}+ix_{2j+1}$ $(j=0,1,\cdots,n)$, with $\sum_{j=0}^{n} z_j\bar{z}_j = \sum_{j=0}^{2n+1} x_j^2 = 1$. Let $m \geqq 2$ be a fixed integer and let $l_0, l_1,\cdots, l_n$ be $n+1$ integers relatively prime to m. We define a rotation γ of S^{2n+1} onto itself by

$$\gamma : z_j \to (\exp(2\pi i l_j/m))z_j \qquad 0 \leqq j \leqq n .$$

γ generates a cyclic group of order m consisting of rotations of S^{2n+1}, none of which (except the identity) has a fixed point; we call this group G.

If we now identify those points of S^{2n+1} transformed by an element of G, we get an orientable $(2n+1)$-dimensional manifold $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ called the *lens space*. Obviously $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ is determined by $l_0, l_1,\cdots, l_n \bmod m$. The universal covering space of $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ is S^{2n+1} and G is the group of covering transformations.

As is easily verified we obtain the same (i. e. diffeomorphic) lens space in replacing $(l_0, l_1,\cdots, l_n)$ as follows:

(i) $(l_0, l_1,\cdots, l_n) \to (ll_0, ll_1,\cdots, ll_n)$, where l is an integer relatively prime to m.

(ii) $(l_0, l_1,\cdots, l_n) \to$ (a permutation of $l_0, l_1,\cdots, l_n$).

(iii) $(l_0, l_1,\cdots, l_j,\cdots, l_n) \to (l_0, l_1,\cdots, -l_j,\cdots, l_n)$.

Therefore we can write $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ briefly as $L^{2n+1}(m; r_1, r_2,\cdots, r_n)$ by $r_j = l_0^{-1}l_j \bmod m$ $(j=1,2,\cdots,n)$, where l_0^{-1} denotes an integer $\bmod m$ such that $l_0^{-1}l_0 = 1$. The orientation is preserved by (i), (ii), and is reversed by (iii).

The combinatorial classification of lens spaces is given by the following theorem (Franz [**1**]):

THEOREM A. *Two lens spaces* $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ *and* $L^{2n+1}(m'; l_0', l_1',\cdots, l_n')$ *are combinatorially equivalent if and only if* $m = \pm m'$ *and* $(l_0', l_1',\cdots, l_n')$ *can be obtained by the composition of* (i), (ii), (iii) *from* $(l_0, l_1,\cdots, l_n)$.

The homotopy classification of lens spaces is given by the following theorem (Olum [**3**]):

THEOREM B. *Two lens spaces* $L^{2n+1}(m; l_0, l_1,\cdots, l_n)$ *and* $L^{2n+1}(m'; l_0', l'_1,\cdots, l_n')$ *have the same homotopy type if and only if*

$$m = \pm m'; \quad l_0 l_1 \cdots l_n = \pm k^{n+1} l_0 l_1 \cdots l_n \pmod m$$

for some integer k relatively prime to m.

4. The invariant Π of 3-dimensional lens spaces.

In this section, let us compute $\Pi(L^3(m; r))$.

Let $z_0 = x_0 + ix_1$, $z_1 = x_2 + ix_3$ be complex numbers and $q = z_0 + z_1 j$ be the quaternion having multiplicative formulae as follows:

$$zj=j\bar{z}\,,\quad j^2=-1\,.$$

We can regard q as a point (x_0, x_1, x_2, x_3) of E^4 and q with norm $\|q\|=|z_0|^2+|z_1|^2=|x_0|^2+|x_1|^2+|x_2|^2+|x_3|^2=1$ as a point of S^3.

Let $L^3(m; l_0, l_1)=L^3(m; r)$ be a 3-dimensional lens space. Then the operation of the generator γ of the group G defined in Section 3 is expressed by quaternions as follows:

$$\gamma(q)=z_0e^{\frac{2\pi i}{m}l_0}+z_1e^{\frac{2\pi i}{m}l_1}j=e^{\frac{2\pi i}{m}((l_0+l_1)/2)}qe^{\frac{2\pi i}{m}((l_0-l_1)/2)}\,. \tag{4.1}$$

Now let $\boldsymbol{M}_L{}^4=\{M_L{}^4, \Sigma^4, S^3, G, \varphi\}$ be the C^∞-M-space associated to $L^3(m; l_0, l_1)$ and let $\mathfrak{T}^{[3]}(\boldsymbol{M}_L{}^4)(F)$ be the generalized associated D-bundle of the tangent D-bundle $\mathfrak{T}(\boldsymbol{M}_L{}^4)$ of $\boldsymbol{M}_L{}^4$ (Section 1).

Let us first construct a G-cross section of $\mathfrak{T}^{[2]}(S^3)=\{T^{[2]}(S^3), p, S^3, U(3)/U(1), SO(3)\}$ over S^3. Let $q=z_0+z_1j$ be a point of S^3. Tangent vectors $X_q{}^{(k)}=\left(\frac{\partial}{\partial x_k}\right)_q$ $(k=0, 1, 2, 3)$ at q form an orthogonal basis of the tangent space $T(q, \Sigma^4)$ of Σ^4 at q. A point of $T(q, \Sigma^4)$ can be expressed by a quaternion, say q'. Since the transformation $L(q)$ of $T(q, \Sigma^4)$ given by

$$L(q): q' \to qq' \qquad q' \in T(q, \Sigma^4)$$

is orthogonal, tangent vectors $Y_q{}^{(k)}$ $(k=0, 1, 2, 3)$ defined by

$$\begin{aligned}(Y^{(0)}, Y^{(1)}, Y^{(2)}, Y^{(3)})_q&=L(q)((X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)})_q)\\ &=(X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)})_q\begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3\\ x_1 & x_0 & -x_3 & x_2\\ x_2 & x_3 & x_0 & -x_1\\ x_3 & -x_2 & x_1 & x_0\end{pmatrix}\end{aligned}$$

form an orthogonal basis of $T(q, \Sigma^4)$. Obviously $Y_q{}^{(k)}$ $(k=1, 2, 3)$ form an orthogonal basis of the tangent space $T(q, S^3)$ of S^3 at q; and for each k, $Y_q{}^{(k)}$ $(q\in S^3)$ is a left invariant vector field of the group manifold of quaternions with norm 1.

On the other hand, γ induces the map $d\gamma: T(q, S^3)\to T(\gamma q, S^3)$ $(q\in S^3)$. By the direct computation we have by (4.1)

$$\begin{aligned}&d\gamma((Y^{(1)}, Y^{(2)}, Y^{(3)})_q)\\ &\quad=(Y^{(1)}, Y^{(2)}, Y^{(3)})_{\gamma q}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(2\pi(l_0-l_1)/m) & \sin(2\pi(l_0-l_1)/m)\\ 0 & -\sin(2\pi(l_0-l_1)/m) & \cos(2\pi(l_0-l_1)/m)\end{pmatrix}.\end{aligned} \tag{4.2}$$

Clearly $d\gamma$ generates a group $\alpha(G)$ of C^∞-D-bundle maps.

Let us put $\Theta_0=l_0^{-1}(l_0-l_1)\theta_0$, $\Theta_1=l_1^{-1}(l_0-l_1)\theta_1$, where $q=z_0+z_1j$, $z_0=\rho_0e^{i\theta_0}$, $z_1=\rho_1e^{i\theta_1}$. We consider now the following two continuous fields of three real vectors on S^3:

$$q \to (Y^{(1)}, Y^{(2)}, Y^{(3)})_q \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_0 \cos \Theta_0 & \rho_0 \sin \Theta_0 \\ 0 & -\rho_0 \sin \Theta_0 & \rho_0 \cos \Theta_0 \end{pmatrix}, \tag{4.3}$$

$$q \to (Y^{(1)}, Y^{(2)}, Y^{(3)})_q \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_1 \cos \Theta_1 & \rho_1 \sin \Theta_1 \\ 0 & -\rho_1 \sin \Theta_1 & \rho_1 \cos \Theta_1 \end{pmatrix}. \tag{4.4}$$

More precisely, (4.3) means that at each point q of S^3 we attach three vectors $Y_q^{(1)}$, $(\rho_0 \cos\Theta_0) Y_q^{(2)} - (\rho_0 \sin\Theta_0) Y_q^{(3)}$, $(\rho_0 \sin\Theta_0) Y_q^{(2)} + (\rho_0 \cos\Theta_0) Y_q^{(3)}$. These three vectors are mutually orthogonal. It is to be noticed however that there are points where the second or the third of these vectors vanishes. It is easily verified by (4.2) that these vector fields (4.3), (4.4) are invariant under the transformation $d\gamma$.

Let us denote by $q_0 + Jq_1$ $(q_0, q_1 \in E^4)$ the complexification of E^4, where J is a symbol satisfying $J^2 = -1$. Then we have a continuous field of two mutually orthogonal *unit* complex vectors on S^3 which is invariant under the transformation $d\gamma$, as follows:

$$q \to (Y^{(1)}, Y^{(2)}, Y^{(3)})_q \begin{pmatrix} 1 & 0 \\ 0 & \rho_0 \cos \Theta_0 + J\rho_1 \cos \Theta_1 \\ 0 & -\rho_0 \sin \Theta_0 - J\rho_1 \sin \Theta_1 \end{pmatrix}. \tag{4.5}$$

(4.5) defines the map $f: S^3 \to T^{[2]}(S^3)$.

Since the cross section F of $\mathfrak{T}(\Sigma^4)$ over S^3 defined in Section 1 is the map $F: q \to Y_q^{(0)}$ $(q \in S^3)$, the image $\lambda^{[3]}(f)$ of f is a G-cross section ([7, Definition 2.5]) of $\mathfrak{T}^{[3]}(\Sigma^4)$ over S^3 such that

$$q \to (Y^{(0)}, Y^{(1)}, Y^{(2)}, Y^{(3)})_q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho_0 \cos \Theta_0 + J\rho_1 \cos \Theta_1 \\ 0 & 0 & -\rho_0 \sin \Theta_0 - J\rho_1 \sin \Theta_1 \end{pmatrix}$$

$$= (X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)})_q \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho_0 \cos \Theta_0 + J\rho_1 \cos \Theta_1 \\ 0 & 0 & -\rho_0 \sin \Theta_0 - J\rho_1 \sin \Theta_1 \end{pmatrix},$$

which will be denoted by the same notation f. f determines the map $f_1: S^3 \to U(4)/U(1)$ as follows:

$$f_1(q) = (X^{(0)}, X^{(1)}, X^{(2)}, X^{(3)})_0 \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho_0 \cos \Theta_0 + J\rho_1 \cos \Theta_1 \\ 0 & 0 & -\rho_0 \sin \Theta_0 - J\rho_1 \sin \Theta_1 \end{pmatrix}.$$

In order to compute the homotopy class $\{f_1\}$, let us introduce two maps $f_2, f_3 : S^3 \to U(4)$ defind by

$$f_2(q)=\begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix},$$

$$f_3(q)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_0 \cos\Theta_0 + J\rho_1 \cos\Theta_1 & \rho_0 \sin\Theta_0 - J\rho_1 \sin\Theta_1 \\ 0 & 0 & -\rho_0 \sin\Theta_0 - J\rho_1 \sin\Theta_1 & \rho_0 \cos\Theta_0 - J\rho_1 \cos\Theta_1 \end{pmatrix}$$

respectively. The natural projection $\bar{p} : U(4) \to U(4)/U(1)$ gives an isomorphism $\bar{p}_* : \pi_3(U(4)) \approx \pi_3(U(4)/U(1))$. Clearly we have

$$\bar{p}_*\{f_2 \cdot f_3\} = \{f_1\}\,. \tag{4.6}$$

As is well-known $\pi_3(U(n)) \approx Z$ $(n \geqq 2)$. Let μ_2 be a generator of $\pi_3(U(2))$ which has the map

$$f_4 : q = z_0 + z_1 j \to \begin{pmatrix} z_0 & -z_1 \\ \bar{z}_1 & \bar{z}_0 \end{pmatrix}$$

as representative and let μ_n $(n \geqq 3)$ be a generator of $\pi_3(U(n))$ defined by $\mu_n = i_*''\mu_2$, where i'' denotes the inclusion map $i'' : U(2) \to U(n)$. The maps $(x_0, x_1, \cdots, x_{n-1}) \to (z_0, z_1, \cdots, z_{n-1})$ and $(z_0, z_1, \cdots, z_{n-1}) \to (x_0, x_1, \cdots, x_{2n-1})$ $(z_k = x_{2k} + i x_{2k+1})$, induce maps $j_n : SO(n) \to U(n)$ and $k_n : U(n) \to SO(2n)$ respectively. Moreover let $i' : SO(n) \to SO(n')$ $(n \leqq n')$ be the inclusion map. Then the following diagram is commutative:

$$\begin{tikzcd} & \pi_3(U(2)) \arrow[r, "(k_2)_*"] \arrow[dl, "(k_2)_*"'] \arrow[d, "i_*''"] & \pi_3(SO(4)) \arrow[d, "i_*'"] \\ \pi_3(SO(4)) \arrow[r, "(j_4)_*"] & \pi_3(U(4)) \arrow[r, "(k_4)_*"] & \pi_3(SO(8)) \end{tikzcd}$$

Since

$$j_4(k_2(f_4)) = f_2\,, \qquad (k_4)_* \circ (j_4)_* = 2i_*' \text{ (Tamura [5])},$$

we have

$$\{f_2\} = 2\mu_4\,. \tag{4.7}$$

Furthermore let $f_5 : S^3 \to U(2)$ be the map defined by

$$f_5(q) = \begin{pmatrix} \rho_0 \cos\Theta_0 + J\rho_1 \cos\Theta_1 & \rho_0 \sin\Theta_0 - J\rho_1 \sin\Theta_1 \\ -\rho_0 \sin\Theta_0 - J\rho_1 \sin\Theta_1 & \rho_0 \cos\Theta_0 - J\rho_1 \cos\Theta_1 \end{pmatrix}.$$

Obviously we have

$$\{f_3\} = i_*''\{f_5\}\,. \tag{4.8}$$

Since the projection $p: U(2) \to S^3$ defined by $p(U)=(1,0)U$ $(U \in U(2))$ gives an isomorphism $p_*: \pi_3(U(2)) \approx \pi_3(S^3)$ such that

$$p_*(\mu_2) = \iota_3 \tag{4.9}$$

(ι_3 denotes the canonical generator of $\pi_3(S^3)$), we can compute $\{f_5\}$ from $\{p \circ f_5\} \in \pi_3(S^3)$.

Clearly we have

$$p(f_5(q)) = (\rho_0 \cos \Theta_0,\ \rho_1 \cos \Theta_1,\ \rho_0 \sin \Theta_0,\ -\rho_1 \sin \Theta_1)\,.$$

Let $f_6: S^3 \to S^3$ be the map defined by

$$f_6(q) = (\rho_0 \cos \Theta_0,\ \rho_0 \sin \Theta_0,\ \rho_1 \cos \Theta_1,\ \rho_1 \sin \Theta_1)\,.$$

Clearly we have

$$\{p \circ f_5\} = \{f_6\}\,. \tag{4.10}$$

Making use of the local degree of f_6, we have

$$\{f_6\} = l_0^{-1} l_1^{-1} (l_0 - l_1)^2 \iota_3\,. \tag{4.11}$$

(4.8), (4.9), (4.10), (4.11) enable us to compute $\{f_3\}$:

$$\{f_3\} = l_0^{-1} l_1^{-1} (l_1 - l_0)^2 \mu_4\,. \tag{4.12}$$

Moreover by (4.6), (4.7), (4.12) we have

$$\{f_1\} = \bar{p}_*(\{f_2\} + \{f_3\}) = \bar{p}_*((l_0^{-1} l_1 + l_0 l_1^{-1}) \mu_4)\,.$$

Consequently the first P-class $P_1(M_L{}^4)$ of $\boldsymbol{M}_L{}^4$ is given by

$$P_1(M_L{}^4) = (l_0^{-1} l_1 + l_0 l_1^{-1}) \{M_L{}^4\}\,.$$

Hence we obtain the following theorem:

THEOREM 3. $\Pi(L^3(m;r)) = (r + r^{-1}) \mod m.$

Π is a diffeomorphy invariant (Theorem 2) but not homotopy type invariant. In fact, $L^3(7;1)$ and $L^3(7;2)$ have the same homotopy type (Theorem B), but $\Pi(L^3(7;1)) \neq \pm \Pi(L^3(7;2))$.

Therefore P-class is not homotopy type invariant for 4-dimensional C^∞-M-spaces. On the contrary, since the obstruction of the vector field $q \to Y_q{}^{(0)}$ gives the 4-th SW-class of $M_L{}^4$ (Tamura [**7**, Definition 7.4]), we have

$$SW_4(M_L{}^4) = \{M_L{}^4\}\,,$$

which is homotopy type invariant.

Suppose that $L^3(m;r)$ and $L^3(m;r')$ have the same invariant Π, i.e.

$$\Pi(L^3(m;r)) = \pm \Pi(L^3(m;r'))\,.$$

Then we have

$$\begin{aligned} (r + r^{-1}) &= \pm (r' + r'^{-1}) && (\mathrm{mod}\ m)\,, \\ (r \mp r')(1 \mp r^{-1} r'^{-1}) &= 0 && (\mathrm{mod}\ m)\,. \end{aligned}$$

Therefore if m is a prime, we have

$$r = \pm r' \quad \text{or} \quad r = \pm r'^{-1} \qquad (\mathrm{mod}\, m).$$

In either case $L^3(m; r)$ and $L^3(m; r')$ are naturally diffeomorphic by (i), (ii), (iii) of Section 3. Hence we obtain the following theorem:

THEOREM 4. *Under the assumption that m is a prime, 3-dimensional lens spaces $L^3(m; r)$ and $L^3(m; r')$ are diffeomorphic if and only if they have the same invariant Π.*

Now the result of Moise [**2**] implies, together with Theorem A, that the diffeomorphic and the homeomorphic classifications of 3-dimensional lens spaces amount to the same. Therefore we can replace the word "diffeomorphic" by "homeomorphic" in Theorem 4. That this replacement is allowed, would follow, independently of the result of Moise and of Reidemeister, if one of the following (1), (2) could be proved as valid:

(1) Each 3-dimensional lens space (or more generally each 3-dimensional manifold) admits only one differentiable strucure (up to diffeomorphism).[1]

(2) The P_1-class of 4-dimensional C^∞-M-spaces is topologically invariant.

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Bibliography

[1] W. Franz, Über die Torsion einer Überdeckung, J. Reine Angew. Math., **173** (1935), 245-254.

[2] E. Moise, Affine structures in 3-manifolds, V. The triangulation theorem and Hauptvermutung, Ann. of Math., **56** (1952), 96-114.

[3] P. Olum, Mappings of manifolds and the notion of degree, Ann. of Math., **58** (1953), 458-480.

[4] K. Reidemeister, Homotopieringe und Linsenräume, Abh. Math. Sem. Hansischen Univ., **11** (1935), 102-109.

[5] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250-262.

[6] I. Tamura, Homeomorphy classification of total spaces of sphere bundles over spheres, J. Math. Soc. Japan, **10** (1958), 29-43.

[7] I. Tamura, Characteristic classes of M-spaces I, J. Math. Soc. Japan, **11** (1959), 312-342.

[8] J. H. C. Whitehead, On incidence matrices, nuclei and homotopy types, Ann. of Math., **42** (1941), 1197-1239.

1) It is reported that J. Munkres, S. Smale and J. H. C. Whitehead have proved recently the uniqueness of differentiable structure of a 3-dimensional topological manifold.