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On Hilbert's modular group.

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In the following lines, we will determine a fundamental domain of Hilbert's modular group. This problem was first treated by O. Blumenthal [**1**], however his argument was not correct, as was indicated by H. Maass, in the case where the class number of the basic field is >1. Correct answers were given by H. Maass, then by O. Herrmann in more explicit form, by using the invariant Riemannian metric in the hyperabelian space. In this note we would like to give another answer to this problem. Our method is arithmetical, and the author expects that similar methods might be applied to investigate related groups, e.g. unimodular groups and modular groups of higher degree in algebraic number fields. For the convenience of readers, we will not omit the proofs of some lemmata appearing in [**1**], [**2**] and [**3**].

Let C^d be the product of d copies of the complex number field C whose elements will be denoted in the form $\tau=(\tau^{(1)},\cdots,\tau^{(d)})$. Let k be a totally real algebraic number field of degree d and $\alpha\rightarrow\alpha^{(1)},\cdots,\alpha\rightarrow\alpha^{(d)}$ distinct isomorphisms of k into C. By the mapping $\alpha\rightarrow(\alpha^{(1)},\cdots,\alpha^{(d)})$ we identify k with a subfield of C^d. We define the following notations for $\tau=(\tau^{(1)},\cdots,\tau^{(d)})$;

$$\mathrm{N}\tau=\tau^{(1)}\cdots\tau^{(d)},$$

$$\mathrm{S}\tau=\tau^{(1)}+\cdots+\tau^{(d)},$$

$$|\tau|=(|\tau^{(1)}|,\cdots,|\tau^{(d)}|),$$

$$\mathrm{Re}\tau=(\mathrm{Re}\tau^{(1)},\cdots,\mathrm{Re}\tau^{(d)}),$$

$$\mathrm{Im}\tau=(\mathrm{Im}\tau^{(1)},\cdots,\mathrm{Im}\tau^{(d)}),$$

where $\mathrm{Re}\tau^{(\nu)}$ and $\mathrm{Im}\tau^{(\nu)}$ denote the real and imaginary parts of $\tau^{(\nu)}$ respectively. If every component of $y\in C^d$ is real and positive we call y totally positive and write $y>0$. P denotes the domain of all $\tau\in C^d$ with $\mathrm{Im}\tau>0$ and R^d denotes the subring of all $x\in C^d$ with $\mathrm{Im}x=0$. If $\bar{M}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a non-singular matrix in k, we denote the corresponding linear fractional transformation

$$\tau\rightarrow\frac{\alpha\tau+\beta}{\gamma\tau+\delta}=\left(\frac{\alpha^{(1)}\tau^{(1)}+\beta^{(1)}}{\gamma^{(1)}\tau^{(1)}+\delta^{(1)}},\cdots,\frac{\alpha^{(d)}\tau^{(d)}+\beta^{(d)}}{\gamma^{(d)}\tau^{(d)}+\delta^{(d)}}\right)$$

by M. If $\bar{M}'$ is another matrix, then the linear transformation corresponding to $\bar{M}'\cdot\bar{M}$ is equal to $M'M$. If the determinant $|\bar{M}|$ of $\bar{M}$ is totally positive, then M is defined at every point on P and maps P onto itself. Let $\bar{\Gamma}$ be the group of all matrices $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ with $\alpha, \beta, \gamma, \delta \in \mathfrak{o}$ and $\alpha\delta-\beta\gamma=1$ where $\mathfrak{o}$ is the integral domain of all algebraic integers in k, and Γ the corresponding group of transformations of P. This group Γ is called Hilbert's modular group in k. It is wellknown that Γ is a properly discontinuous transformation group of P, and our aim is to determine a fundamental domain of Γ.

First we study general lattice groups. Let V be a euclidian vector space of dimension d, (x, y) the inner product defined on V and $\|x\|=(x, x)^{\frac{1}{2}}$ the length of a vector $x \in V$. A lattice Λ in V is a subgroup of all linear combinations of linearly independent vectors $\alpha_1, \cdots, \alpha_{d'}$ with integral coefficients. We call d' the rank of Λ_1 and the square root of the determinant $|(\alpha_i, \alpha_j)|$ the determinant of Λ. The determinant of Λ is equal to the volume of a fundamental parallelotope of Λ if the rank of Λ is equal to d. Let W be the d'-dimensional subspace spanned by vectors in Λ. A vector $x \in V$ is called Λ-reduced if $\|x+\alpha\| \geqq \|x\|$ for all $\alpha \in \Lambda$. The set of all Λ-reduced vectors will be denoted by Π_Λ. The following lemma is wellknown.

LEMMA 1. *Π_Λ is a convex set defined by a finite number of linear inequalities*

$$\pm 2(x, \gamma_\nu) \leqq (\gamma_\nu, \gamma_\nu) \qquad (\nu=1, \cdots, s)$$

where $\gamma_1, \cdots, \gamma_s$ are suitable vectors in Λ, and is a fundamental domain of the translation group $\{x \to x+\alpha\,;\, \alpha \in \Lambda\}$. Furthermore the set $\Pi_\Lambda \frown W$ is compact and $x \in V$ is contained in Π_Λ if and only if x is the sum x_1+x_2 with $x_1 \in \Pi_\Lambda \frown W$ and a vector x_2 orthogonal to W.

Let E be the group of all units in k. Let Y be the set of all $y>0$ and Y_1 the subset of Y of all y with $\mathrm{N}y=1$. For $y=(y^{(1)}, \cdots, y^{(d)})$ we put $\log y = (\log y^{(1)}, \cdots, \log y^{(d)})$. The mapping $y \to \log y$ maps Y onto R^d. Let Λ be the set of all $\log|\varepsilon|$ with $\varepsilon \in E$. It is wellknown that Λ is a lattice of rank $d-1$ in R^d, and spans a hyperplane $H=\{x\,;\, \mathrm{S}x=0\}$. Now we introduce a positive definite inner product $\mathrm{S}(xy)$ defined on R^d, and consider R^d a euclidian vector space. A vector x is orthogonal to H if and only if x is a scalar multiple of the vector $1=(1, \cdots, 1)$. A vector $y \in Y$ will be called strongly reduced if $\log y$ is Λ-reduced. Let K be the set of all strongly reduced y, and put $K_1 = K \frown Y_1$. From Lemma 1 we have

LEMMA 2. *K is bounded by a finite number of analytic hypersurfaces in Y, and K_1 is compact. For every $y \in Y$ there exists $|\varepsilon|$, $\varepsilon \in E$, uniquely determined in general, such that $|\varepsilon|\, y \in K$.*

REMARK. The words "uniquely determined in general" mean that if $|\varepsilon|\, y$ and $|\varepsilon'|\, y$ are in K with $|\varepsilon| \neq |\varepsilon'|$, then both lie on the boundary of K. In the sequel, we use this expression in the similar meaning.

COROLLARY. *If* $y=(y^{(1)},\cdots,y^{(d)})$ *is strongly reduced, then there exists a constant* $c_1>1$ *depending only on* k *such that*

$$c_1^{-1}\mathrm{N}y^{\frac{1}{d}}\leqq y^{(\nu)}\leqq c_1\mathrm{N}y^{\frac{1}{d}}\qquad(\nu=1,\cdots,d).$$

We will call $y\in Y$ reduced if $y^{\frac{1}{2}}=(y^{(1)\frac{1}{2}},\cdots,y^{(d)\frac{1}{2}})$ is strongly reduced. The set of all reduced y is $K^2=\{y^2;y\in K\}$, and for $y\in Y$ there exists an ε^2 determined uniquely in general, such that $\varepsilon^2 y\in K^2$. If y is reduced, then we have

$$c_1^{-2}\mathrm{N}y^{\frac{1}{d}}\leqq y^{(\nu)}\leqq c_1^2\mathrm{N}y^{\frac{1}{d}}.$$

Let $\mathfrak{a}$ be an ideal of k, then $\mathfrak{a}$ is a lattice in R^d of rank d. $\Pi_{\mathfrak{a}}$ denotes the set of all $\mathfrak{a}$-reduced x, namely the set of all x such that $\mathrm{S}(x+\alpha)^2\geqq \mathrm{S}x^2$ for every $\alpha\in\mathfrak{a}$. From Lemma 1, we see that $\Pi_{\mathfrak{a}}$ is a compact convex polyhedron, symmetric with respect to 0, and for every $x\in R^d$, there exists an $\alpha\in\mathfrak{a}$, determined uniquely in general, such that $x+\alpha\in\Pi_{\mathfrak{a}}$.

Let $A(\mathfrak{a})$ be the group of all affine transformations $\tau\to\varepsilon^2\tau+\alpha$ with $\varepsilon\in E$ and $\alpha\in\mathfrak{a}$. Matrices corresponding to these transformations are of the form

$$\begin{pmatrix}\varepsilon & \alpha\\ 0 & \varepsilon^{-1}\end{pmatrix}\qquad \varepsilon\in E,\quad \alpha\in\mathfrak{a},$$

so $A(\mathfrak{a})$ is a subgroup of Γ if $\mathfrak{a}\subset\mathfrak{o}$. We will call $\tau\in P$ $\mathfrak{a}$-reduced if $\mathrm{Im}\tau$ is reduced and $\mathrm{Re}\tau$ is $\mathfrak{a}$-reduced. $G(\mathfrak{a})$ denotes the set of all $\mathfrak{a}$-reduced τ, then we see that $G(\mathfrak{a})$ is a fundamental domain of $A(\mathfrak{a})$, bounded by a finite number of analytic surfaces in P.

Now we consider pairs $(\gamma,\delta)\neq(0,0)$ of elements $\gamma,\delta\in k$. Two pairs (γ,δ) and (γ',δ') are called associated if there exists $\varepsilon\in E$ with $\varepsilon\gamma=\gamma'$ and $\varepsilon\delta=\delta'$.

LEMMA 3. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of* k, *and* τ *a point* $\in P$. *For every* $c>0$, *there exist only a finite number of non-associated pairs* (γ,δ) *with* $\gamma\in\mathfrak{a}$ *and* $\delta\in\mathfrak{b}$ *such that*

$$\mathrm{N}|\gamma\tau+\delta|<c.$$

PROOF. Let (γ,δ) be a pair satisfying our assumptions. We choose $\varepsilon\in E$ so that $|\gamma\varepsilon\tau+\delta\varepsilon|$ is strongly reduced. From the Corollary of Lemma 2, we have

$$|\gamma'^{(\nu)}\tau^{(\nu)}+\delta'^{(\nu)}|<c_1c^{\frac{1}{d}}\qquad(\nu=1,\cdots,d)\tag{1}$$

for $(\gamma',\delta')=(\gamma\varepsilon,\delta\varepsilon)$. From $\mathrm{Im}\tau>0$, the set $\{\gamma\tau+\delta;\gamma\in\mathfrak{a},\delta\in\mathfrak{b}\}$ forms a lattice in C^d and the domain defined by $|\tau^{(\nu)}|<c_1c^{\frac{1}{d}}$ is obviously bounded. Hence there exist only a finite number of pairs (γ',δ') satisfying the inequalities (1).

LEMMA 4. *Let* $\mathfrak{a}$ *be an ideal of* k. *There exists a constant* $c_2>0$, *depending* *only on k, such that for every* $\tau\in P$ *with* $\mathrm{N}\,\mathrm{Im}\tau<c_2$ *there exists a pair* (γ,δ) *with* $\gamma\in\mathfrak{a}$ *and* $\delta\in\mathfrak{a}^{-1}$ *such that*

$$\mathrm{N}|\gamma\tau+\delta|<1.$$

PROOF. The quadratic form $\mathrm{S}|\omega|^2$ defines a euclidian metric on C^d. If τ is a point on P, then the set $\{\gamma\tau+\delta;\gamma\in\mathfrak{a},\delta\in\mathfrak{a}^{-1}\}$ forms a lattice in C^d of rank $2d$ whose determinant is equal to $\mathrm{N}\,\mathrm{Im}\tau\cdot\Delta$ where Δ is the absolute value of the discriminant of k. Now the set $\{\omega;\mathrm{S}|\omega|\leqq d\}$ is convex, compact and symmetric with respect to 0. Let c_3 be the volume of the set, then from Minkowski's theorem, we have a pair (γ,δ) with $\gamma\in\mathfrak{a}$ and $\delta\in\mathfrak{a}^{-1}$ such that $\mathrm{S}|\gamma\tau+\delta|<d$ if $c_3(\mathrm{N}\,\mathrm{Im}\tau\Delta)^{-1}>2^{2d}$. Since $\mathrm{N}|\gamma\tau+\delta|\leqq\mathrm{S}|\gamma\tau+\delta|d^{-1}$, $c_2=c_3\,2^{-2d}\Delta^{-1}$ has the required property.

Let $\mathfrak{a}_1,\cdots,\mathfrak{a}_h$ be a set of representatives of ideal classes of k such that i) each $\mathfrak{a}_i$ is integral, ii) $\mathfrak{a}_i$ has minimum norm among integral ideals in its class. We assume $\mathfrak{a}_1$ represents the principal class, then obviously we have $\mathfrak{a}_1=\mathfrak{o}$.

Let F_i be the set of all $\tau\in P$ such that

1) τ is $\mathfrak{a}_i^{-2}$ reduced,

2) $\mathrm{N}|\gamma\tau+\delta|\geqq1$ for all pairs (γ,δ) with $\gamma\in\mathfrak{a}_i$ and $\delta\in\mathfrak{a}_i^{-1}$.

Note that F_i is not empty. In fact assume that $y>0$ is reduced and $\mathrm{N}y>1$, then $\sqrt{-1}\,y=\tau$ is contained in F_i. For if $\gamma\in\mathfrak{a}_i$ and $\delta\in\mathfrak{a}_i^{-1}$, then we have $\mathrm{N}|\gamma\sqrt{-1}\,y+\delta|\geqq\mathrm{N}|\gamma|\mathrm{N}y>1$ provided $\gamma\neq0$. If $\gamma=0$, from the assumption ii) on $\mathfrak{a}_i$, we have $\mathrm{N}|\delta|\geqq1$ because we have $\delta\mathfrak{a}_i\sim\mathfrak{a}_i$ and $\mathrm{N}(\delta\mathfrak{a}_i)\geqq\mathrm{N}(\mathfrak{a}_i)$.

LEMMA 5. *There exist only a finite number of non-associated pairs* (γ,δ) *with* $\gamma\in\mathfrak{a}_i$ *and* $\delta\in\mathfrak{a}_i^{-1}$ *such that* $\mathrm{N}|\gamma\tau+\delta|=1$ *for some* $\tau\in F_i$.

PROOF. From Lemma 4, we have $\mathrm{N}\,\mathrm{Im}\tau\geqq c_2$, hence $\mathrm{N}|\gamma|$ is not greater than c_2^{-1}. Since $\gamma\in\mathfrak{a}_i$, there exist only a finite number of non-associated γ with $\mathrm{N}|\gamma|\leqq c_2^{-1}$. Let $\gamma_0=0,\gamma_1,\cdots,\gamma_\mu$ be representatives of such γ's. We have only to prove that there exist only a finite number of $\kappa\in\gamma_j^{-1}\mathfrak{a}_i^{-1}$ such that $\mathrm{N}|\tau+\kappa|\leqq\mathrm{N}|\gamma_j|^{-1}$ with $\tau\in F_i$ for $j=1,\cdots,\mu$. Since τ is $\mathfrak{a}_i^{-2}$-reduced, there exists a constant $c'>0$ with $|\mathrm{Re}\tau^{(\nu)}|<c'$. Since $|\tau^{(\nu)}+\kappa^{(\nu)}|\geqq c_2^{\frac{1}{d}}c_1^{-2}$, we have, for a suitable constant $c_4>0$,

$$|\tau^{(\nu)}+\kappa^{(\nu)}|<c_4\mathrm{N}|\gamma_j|^{-1}.$$

It follows that $|\kappa^{(\nu)}|<c_4\mathrm{N}|\gamma_j|^{-1}+c'$, hence there exist only a finite number of such κ.

From Lemma 5, we see that infinitely many inequalities 2) defining F_i may be replaced by a finite number among them. Therefore F_i is bounded by a finite number of analytic surfaces in P.

Let γ_i,δ_i be integers generating $\mathfrak{a}_i$. Then there exist $\alpha_i,\beta_i\in\mathfrak{a}_i^{-1}$ such that $\alpha_i\delta_i-\beta_i\gamma_i=1$. A_i denotes the transformation $\tau\to(\alpha_i\tau+\beta_i)/(\gamma_i\tau+\delta_i)$. For

$i=1$, we put $A_i = I$ (identity transformation). Now we will prove our main theorem.

THEOREM. *The set*

$$F = F_1 \cup A_2^{-1}F_2 \cup \cdots \cup A_h^{-1}F_h$$

is a fundamental domain of the group Γ.

PROOF. Let τ be a point on P. From Lemma 3, there exists a pair (γ, δ) of integers such that $\mathrm{N}|\gamma\tau+\delta| \leqq \mathrm{N}|\gamma'\tau+\delta'|$ for all $\gamma' \in \mathfrak{o}$, $\delta' \in \mathfrak{o}$, $(\gamma', \delta') \neq (0, 0)$. If $\mathrm{N}|\gamma\tau+\delta| \geqq 1$, then we can find a unit ε and an integer α such that $\varepsilon^2\tau + \alpha \in G(\mathfrak{o})$. Then $\tau' = \varepsilon^2\tau+\alpha$ is equivalent to τ with respect to Γ and τ' is a point on F_1. Assume that $\mathrm{N}|\gamma\tau+\delta| < 1$. Let $\mathfrak{a}$ be the ideal generated by γ and δ and $\mathfrak{a}_i$ the representative of its class. Then there exists an element $\kappa \in \mathfrak{a}_i^{-1}$ with $\mathfrak{a} = \kappa\mathfrak{a}_i$. We have $\kappa^{-1}\gamma$, $\kappa^{-1}\delta \in \mathfrak{a}_i$ and

$$\mathrm{N}|\kappa^{-1}\gamma\tau+\kappa^{-1}\delta| = \mathrm{N}|\kappa|^{-1}\mathrm{N}|\gamma\tau+\delta|\,.$$

From the assumption on γ, δ we have $\mathrm{N}|\kappa| \leqq 1$. On the other hand we have $\mathrm{N}\kappa \geqq 1$ from the assumption on $\mathfrak{a}_i$, hence we have $\mathrm{N}|\kappa| = 1$. Taking $\kappa^{-1}\gamma$ and $\kappa^{-1}\delta$ instead of γ and δ, we may assume that γ and δ generate one of $\mathfrak{a}_i$'s. Let $\bar{M}$ be a matrix in $\bar{\Gamma}$ such that

$$(\gamma_i, \delta_i)\bar{M} = (\gamma, \delta)$$

and put $\tau_1 = M\tau$ and $\tau_2 = A_i\tau_1$. We have

$$\mathrm{N}|\gamma_i\tau_1+\delta_i| \leqq \mathrm{N}|\gamma'\tau_1+\delta'|$$

for every pair of integers (γ', δ'), and for every (μ, ν) with $\mu \in \mathfrak{a}_i$ and $\nu \in \mathfrak{a}_i^{-1}$, we have

$$\begin{aligned}\mu\tau_2+\nu &= (\mu(\alpha_i\tau_1+\beta_i)+\nu(\gamma_i\tau_1+\delta_1))(\gamma_i\tau_1+\delta_i)^{-1}\\ &= (\mu'\tau_1+\nu')(\gamma_i\tau_1+\delta_i)^{-1}\end{aligned}$$

with $\mu' = \alpha_i\mu+\gamma_i\nu$, $\nu' = \beta_i\mu+\delta_i\nu \in \mathfrak{o}$. Hence we have

$$\mathrm{N}|\mu\tau_2+\nu| \geqq 1\,.$$

Let L be a transformation in $A(\mathfrak{a}_i^{-2})$ such that $L\tau_2 \in G(\mathfrak{a}_i^{-2})$. Then we have $L\tau_2 \in F_i$ and $A_i^{-1}LA_iM\tau \in A_i^{-1}F_i$. Since $A_i^{-1}A(\mathfrak{a}_i^{-2})A_i \subset \Gamma$ we have $A_i^{-1}LA_iM \in \Gamma$.

Next thing we have to prove is that if there exists an $M \neq I$ mapping a point $\tau \in F$ on a point $\tau' \in F$, then both τ and τ' lie on the boundary of F. First we assume that both τ and τ' lie on $A_i^{-1}F_i$. Put $\tau_1 = A_i\tau$ and $\tau_1' = A_i\tau'$. Put

$$\bar{A}_i\bar{M}\bar{A}_i^{-1} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

Then we have $\alpha, \delta \in \mathfrak{o}$, $\gamma \in \mathfrak{a}_i^2$ and $\beta \in \mathfrak{a}_i^{-2}$. If $\gamma = 0$, then α is a unit in k, and both τ_1 and $\tau_1' = \alpha^2\tau_1+\alpha\beta$ lie on F_i, hence they lie on the boundary on F_i.

Hence τ and τ' lie on the boundary of $A_i^{-1}F_i$. Assume that $\gamma\neq 0$. Then we have

$$\mathrm{N}\,\mathrm{Im}\tau_1'=\mathrm{N}\,\mathrm{Im}\tau_1(\mathrm{N}|\gamma\tau_1+\delta|)^{-2}\leqq \mathrm{N}\,\mathrm{Im}\tau_1$$

from $\tau_1\in F_i$, $\gamma\in\mathfrak{a}_i$ and $\delta\in\mathfrak{o}$. Similarly we have

$$\mathrm{N}\,\mathrm{Im}\tau_1=\mathrm{N}\,\mathrm{Im}\tau_1'(\mathrm{N}|-\gamma\tau_1'+\alpha|)^{-2}\leqq \mathrm{N}\,\mathrm{Im}\tau_1'.$$

Therefore we have $\mathrm{N}|\gamma\tau_1+\delta|=\mathrm{N}|-\gamma\tau_1'+\alpha|=1$, so we see that both τ_1 and τ_1' lie on the boundary of $A_i^{-1}F_i$.

Next we assume that $\tau\in A_i^{-1}F_i$ and $\tau'\in A_j^{-1}F_j$ with $i\neq j$. Put $\tau_1=A_i\tau$, $\tau_1'=A_j\tau'$ and

$$\bar{A}_j\bar{M}\bar{A}_i^{-1}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

Then we have $\alpha\in\mathfrak{a}_i\mathfrak{a}_j^{-1}$, $\beta\in\mathfrak{a}_i^{-1}\mathfrak{a}_j^{-1}$, $\gamma\in\mathfrak{a}_i\mathfrak{a}_j$ and $\delta\in\mathfrak{a}_i^{-1}\mathfrak{a}_j$. Furthermore we have $\alpha\delta-\beta\gamma=1$. By the same method as above, we have

$$\mathrm{N}|\gamma\tau_1+\delta|=\mathrm{N}|-\gamma\tau_1+\alpha|=1\,.$$

If $\gamma\neq 0$, then τ_1 and τ_1' lie on the boundaries of F_i and F_j respectively. Assume that $\gamma=0$, then we have $\alpha\delta=1$ with $\alpha\in\mathfrak{a}_i\mathfrak{a}_j^{-1}$ and $\delta\in\mathfrak{a}_j\mathfrak{a}_i^{-1}$. It follows that $\mathfrak{a}_i\sim\mathfrak{a}_j$. This contradicts the assumption $i\neq j$. Hence we have proved all of our assertion.

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