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On some relations concerning the operations P_α and S_α on classes of sets.

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Introduction.

As extensions of the σ-operation and δ-operation which appear in the theory of usual Borel sets, operations S_α and P_α were already considered in [1], [2] and [3] (cf. Def. 1). Especially in [1] and [2] a condition under which $P_\alpha S_\beta(K)$ is included in $S_\gamma P_\delta(K)$ for any class K of sets is obtained. Referring to these results, we have attempted to study the conditions under which some inequalities or equalities hold between $P_\delta S_\varepsilon$, $P_\alpha S_\beta P_\gamma$, $S_\alpha P_\beta S_\gamma$ etc.

In section 1 several definitions are given. We call the product of operations P_α, S_β etc. a monomial (cf. below Def. 1). In section 2 we shall give a method by means of which the comparison of $P_\delta S_\varepsilon$ with other monomials is fairly simplified and unified. This method is an extension of that used in [1] or [2]. In section 3, a condition for the inequality $S_\alpha P_\beta S_\gamma \leqq P_\delta S_\varepsilon$ or $S_\alpha P_\beta S_\gamma \leqq S_\delta P_\varepsilon$ is obtained. In section 4, we shall first study the condition for the inequality $P_\delta S_\varepsilon \leqq P_\alpha S_\beta P_\gamma$ and next the condition for the equality $P_\delta S_\varepsilon = P_\alpha S_\beta P_\gamma$.

These results are obtained without the generalized continuum hypothesis, but we have not succeeded to give without this hypothesis a condition under which the inequality $P_\delta S_\varepsilon \leqq S_\alpha P_\beta S_\gamma$ holds. Assuming this hypothesis, we shall give a condition for the above inequality in section 5. A condition for the equality $P_\delta S_\varepsilon = S_\alpha P_\beta S_\gamma$ to hold is obtained without the hypothesis.

Throughout this note, the symbol $\pi_\alpha(\beta)$ (cf. Def. 3) plays a main rôle. In section 6, we shall consider the behaviour of the value of $\pi_\alpha(\beta)$, especially we shall give a conditions under which we have $\pi_\alpha(\beta)=\beta$, $\pi_\alpha(\beta)=\beta+1$ or $\pi_\alpha(\beta)\geqq\beta+2$.

§ 1. Definitions.

1. The following definition of the operation S_α (resp. P_α) is given in [1], [2] and [3].

DEFINITION 1. *Let K be any class of sets, and α an ordinal number. $S_\alpha(K)$ (resp. $P_\alpha(K)$) is the class of all sets which are expressed as the unions (resp.*

intersections) of less than $\aleph_\alpha$ *sets contained in* K*; that is, a set* X *is contained in* $S_\alpha(K)$ *(resp.* $P_\alpha(K)$*) if and only if, for a certain non-void set* A *of indices, we have* $X=\bigcup_{a\in A} X_a$ *(resp.* $X=\bigcap_{a\in A} X_a$*), where* $X_a \in K$ *and* $0<\overline{\overline{A}}<\aleph_\alpha$.

Thus we have operations S_α and P_α which operate on a class K of sets and yield a class of sets, and so naturally we have their product operations $P_\alpha S_\beta$, $S_\alpha P_\beta S_\gamma$ etc. defined by $P_\alpha S_\beta(K)=P_\alpha(S_\beta(K))$ etc. We call a product of S_α's and P_β's a *monomial*, and the number of its factors S_α and P_β the *degree* of the monomial.

Now we introduce an order relation between these operations on classes.

DEFINITION 2. *Let* F *and* G *be two operations which operate on a class of sets and yield a class of sets. We define*

(a) $F\leqq G$, *if for any non-void class* K *of sets, we have* $F(K)\subset G(K)$,

(d) $F=G$, *if we have* $F\leqq G$ *and* $G\leqq F$,

(c) $F<G$, *if we have* $F\leqq G$ *but not* $G\leqq F$.

2. We denote the identical operation on classes of sets by I; that is, $I(K)=K$ for any class K of sets. We say that an operation F is *positive* if $I\leqq F$; F is *increasing* if $K\subset H$, where K and H are classes of sets, implies $F(K)\subset F(H)$; and finally F is *intrinsic*, if for any class K of sets, the sets in $F(K)$ are included in the union of all sets contained in K. Then obviously any monomial is positive, increasing and intrinsic.

The following lemma and Theorem A is stated in [**3**].

LEMMA 1. *Let* F, G *and* H *be operations on classes.*

(a) $F\leqq G$ *implies* $FH\leqq GH$.

(b) *If* H *is increasing,* $F\leqq G$ *implies* $HF\leqq HG$.

(c) *If* F *is positive then* $G\leqq FG$.

(d) *If* F *is positive and* G *is increasing then* $G\leqq GF$.

Let $\mathrm{cf}(\alpha)$ be the index of the least initial ordinal number which is cofinal to ω_α.

THEOREM A. (a) *If* $\alpha<\beta$, *then* $S_\alpha<S_\beta$ *and* $P_\alpha<P_\beta$.

(b) *If* $\beta<\alpha$, *then* $S_\alpha S_\beta=S_\alpha$ *and* $P_\alpha P_\beta=P_\alpha$.

(c) *If* $\alpha\leqq \mathrm{cf}(\beta)$, *then* $S_\alpha S_\beta=S_\beta$ *and* $P_\alpha P_\beta=P_\beta$.

(d) *If* $\mathrm{cf}(\beta)<\alpha\leqq\beta+1$, *then* $S_\alpha S_\beta=S_{\beta+1}$ *and* $P_\alpha P_\beta=P_{\beta+1}$.

REMARK According to (b), (c) and (d) of Theorem A, a condition under which we have $S_\alpha S_\beta=S_\gamma$ or $P_\alpha P_\beta=P_\gamma$ is $\gamma=\beta+1$ in the case $\mathrm{cf}(\beta)<\alpha\leqq\beta+1$, and $\gamma=\max(\alpha,\beta)$ otherwise.

3. Now we shall define a function $\pi_\alpha(\beta)$ of two variables α and β of ordinal numbers, which takes an ordinal number as its value. We shall describe in the sequel conditions under which an inequality or an equality between two forms from among $S_\delta P_\varepsilon$, $P_\delta S_\varepsilon$, $P_\alpha S_\beta P_\gamma$ etc. holds, in terms of comparison of the magnitudes of the indices α, β, γ etc., the cofinal type cf(α), and the function $\pi_\alpha(\beta)$.

Definition 3. *$\pi_\alpha(\beta)$ denotes the least ordinal number γ such that for any aggregate of cardinal numbers $\mathfrak{m}_a$; $a \in A$, A being a set of indices, $0 < \overline{\overline{A}} < \aleph_\alpha$ and $\mathfrak{m}_a < \aleph_\beta$ for any $a \in A$ imply $\prod_{a \in A} \mathfrak{m}_a < \aleph_\gamma$.*

In [1] and [2] a condition for the inequality $P_\alpha S_\beta \leqq S_\gamma P_\delta$ was already obtained as follows:

Theorem B. *For the inequality $P_\alpha S_\beta \leqq S_\gamma P_\delta$ to hold, it is necessary and sufficient that we have $\alpha \leqq \delta$ and $\pi_\alpha(\beta) \leqq \gamma$.*

The most essential part in the proof of Theorem B is to prove the necessity of the second inequality $\pi_\alpha(\beta) \leqq \gamma$. By a slight modification of the method used to prove it in [1] or [2], we obtain a universal method to decide whether a monomial is greater than $P_\delta S_\varepsilon$ or not. We shall explain this method in the next section.

§ 2. Discriminative systems.

1. The following definition is an extension of the definition of Hausdorff operation mentioned in [4].

Definition 4. *An operation Φ by which a sequence (not necessarily countable) $\{X_\lambda; \lambda < \mu\}$ of sets X_λ, μ being an ordinal number, corresponds to a set $X = \Phi\{X_\lambda\}$ is called a Hausdorff operation if the correspondence is as follows: $X = \Phi\{X_\lambda\} = \bigcap_{\nu \in \Delta} \bigcup_{\lambda \in \nu} X_\lambda$, where ν is a subsequence of the sequence of ordinal numbers λ less than μ and Δ is an aggregate of such subsequences ν.*

Let $\mathfrak{h}$ be an aggregate of Hausdorff operations and K be a class of sets. Let $\mathfrak{h}(K)$ denote the class of all sets X which can be expressed in the form $X = \Phi\{X_\lambda\}$ where $\Phi \in \mathfrak{h}$ and $X_\lambda \in K$ for any λ. Thus $\mathfrak{h}$ is regarded as an operation by which a class of sets corresponds to a class of sets. We call any aggregate of Hausdorff operations an *integral operation* regarding it as such an operation on classes.

Of course $P_\alpha S_\beta$ is an integral operation, and by the well-known completely distributive property between the operations of unions and intersections, it is easily seen that any monomial is also an integral operation.

The following lemmas are easy to see, and so we omit the proofs.

Lemma 2. *Any integral operation is positive, increasing and intrinsic.*

Lemma 3. *Let Φ be a Hausdorff operation, and Z be a given set. Put $Y_\lambda = X_\lambda \cap Z$ for any λ, then we have $\Phi\{Y_\lambda\} = \Phi\{X_\lambda\} \cap Z$.*

Lemma 4. *Let $\mathfrak{h}$ be an integral operation, K a class of set and Z a given set. Put $H = \{Y; Y = X \cap Z, X \in K\}$, then we have $\mathfrak{h}(H) = \{U; U = V \cap Z, V \in \mathfrak{h}(K)\}$.*

Now we shall see

LEMMA 5. *Let $\mathfrak{k}$ and $\mathfrak{h}$ be integral operations. If $\mathfrak{k} \not\leqq \mathfrak{h}$, then there exists a class H of sets and a non-void set E such that $E \in \mathfrak{k}(H)$ but not $E \in \mathfrak{h}(H)$.*

PROOF. $\mathfrak{k} \not\leqq \mathfrak{h}$ implies the existence of a class H' of sets and a set E' such that $E' \in \mathfrak{k}(H')$ but not $E' \in \mathfrak{h}(H')$. If E' is void, then put $H = \{X \cup \{t\}; X \in H'\}$ and $E = \{t\}$ where t is an arbitrary element. By the provious lemma we can easily see that H and E satisfy the condition of this lemma.

2. DEFINITION 5. *Let A be any set with $\overline{\overline{A}} < \aleph_\alpha$ and ψ_a an ordinal number, assigned to every $a \in A$, such that $\psi_a < \omega_\beta$. The set $\mathfrak{S} = \{\psi_a; a \in A\}$ is called a discriminative system of $P_\alpha S_\beta$. $E_\mathfrak{S}$ denotes the set of all functions φ which map each index $a \in A$ to an ordinal number $\varphi(a)$ less than ψ_a. Let $\Psi^*_{a,\eta}$, where $a \in A$ and $\eta < \psi_a$, be the set of all functions φ in $E_\mathfrak{S}$ with $\varphi(a) = \eta$. Put $A_a = A - \{a\}$ and $\Psi_{a,\eta} = \Psi^*_{a,\eta} \cup A_a$. We denote $K_\mathfrak{S} = \{\Psi_{a,\eta}; a \in A, \eta < \psi_a\}$.*

We assume that $E_\mathfrak{S}$ has no element in common with A, which is not an essential restriction.

The following two lemmas follow from the definition of discriminative systems.

LEMMA 6. *If $E \in P_\alpha S_\beta(K)$, then we can find a discriminative system $\mathfrak{S} = \{\psi_a; a \in A\}$ of $P_\alpha S_\beta$ such that $E = \bigcap_{a \in A} \bigcup_{\eta < \psi_a} X_{a,\eta}$ where $X_{a,\eta} \in K$.*

LEMMA 7. *If $\mathfrak{S}$ is a discriminative system of $P_\alpha S_\beta$ then we have $E_\mathfrak{S} \in P_\alpha S_\beta(K_\mathfrak{S})$.*

3. $F \leqq G$, where F and G are operations on classes of sets, means that for *any* class K of sets, *every* set in $F(K)$ is contained in $G(K)$. But in the case when $F = P_\alpha S_\beta$ and G is an integral operation, Theorem 1 shows that we can ascertain $F \leqq G$ by proving that a *certain* set in $F(K)$ is contained in $G(K)$ for a *certain* kind of classes K of sets.

THEOREM 1. *Let $\mathfrak{h}$ be any integral operation on classes of sets. The necessary and sufficient condition for $P_\alpha S_\beta \leqq \mathfrak{h}$ is that, for any discriminative system $\mathfrak{S}$ of $P_\alpha S_\beta$, the set $E_\mathfrak{S}$ is contained in the class $\mathfrak{h}(K_\mathfrak{S})$.*

PROOF. By Lemma 7, the necessity of this condition is obvious. Now assume that the condition is satisfied. If E is a set contained in $P_\alpha S_\beta(K)$, where K is a class of sets, then by Lemma 6, we have $E = \bigcap_{a \in A} \bigcup_{\eta < \psi_a} X_{a,\eta}$ where $X_{a,\eta} \in K$ and $\mathfrak{S} = \{\psi_a; a \in A\}$ is a discriminative system of $P_\alpha S_\beta$. By the assumption, $E_\mathfrak{S}$ is contained in $\mathfrak{h}(K_\mathfrak{S})$ and hence there exists a Hausdorff operation $\Phi \in \mathfrak{h}$ and we have $E_\mathfrak{S} = \Phi\{\Psi_{a(\lambda),\eta(\lambda)}\} = \bigcap_{\nu \in \Delta} \bigcup_{\lambda \in \nu} \Psi_{a(\lambda),\eta(\lambda)}$ where $\Psi_{a(\lambda),\eta(\lambda)} \in K_\mathfrak{S}$, ν is a subsequence of the sequence $\{\lambda; \lambda < \mu\}$ of ordinal numbers and Δ is an aggregate of such subsequences ν. Now we shall show $E = \bigcap_{\nu \in \Delta} \bigcup_{\lambda \in \nu} X_{a(\lambda),\eta(\lambda)}$.

Since $E_\mathfrak{S}$ contains no element b in A, for any element b in A there exists a $\nu(b) \in \Delta$ such that $\bigcup_{\lambda \in \nu(b)} \Psi_{a(\lambda),\eta(\lambda)}$ does not contain b. By the construction of

$\Psi_{a,\eta}$, $a(\lambda)$ is constantly equal to b for $\lambda\in\nu(b)$. Since, for any $\varphi\in E_{\mathfrak{S}}$, there exists one and only one $\eta<\psi_b$ such that $\varphi\in\Psi_{b,\eta}$, we have $\{\eta(\lambda); \lambda\in\nu(b)\}=\{\eta; \eta<\psi_b\}$. Now since $\{\nu(b); b\in A\}\subset\varDelta$, we have

$$\bigcap_{\nu\in\varDelta}\bigcup_{\lambda\in\nu}X_{a(\lambda),\eta(\lambda)}\subset\bigcap_{b\in A}\bigcup_{\lambda\in\nu(b)}X_{a(\lambda),\eta(\lambda)}=\bigcap_{b\in A}\bigcup_{\eta<\psi_b}X_{b,\eta}=E.$$

On the other hand let e be an element in E, then for any $a\in A$ there exists a $\eta=\varphi(a)<\psi_a$ such that $e\in X_{a,\varphi(a)}$. φ is a member of $E_{\mathfrak{S}}$ and hence for any $\nu\in\varDelta$ there exists a $\lambda\in\nu$ such that $\varphi\in\Psi_{a(\lambda),\eta(\lambda)}$, but this implies $\varphi(a(\lambda))=\eta(\lambda)$, and hence $e\in X_{a(\lambda),\eta(\lambda)}$. Hence $e\in\bigcap_{\nu\in\varDelta}\bigcup_{\lambda\in\nu}X_{a(\lambda),\eta(\lambda)}$ and we have $E\subset\bigcap_{\nu\in\varDelta}\bigcup_{\lambda\in\nu}X_{a(\lambda),\eta(\lambda)}$.

Thus we have proved $E=\bigcap_{\nu\in\varDelta}\bigcup_{\lambda\in\nu}X_{a(\lambda),\eta(\lambda)}=\Phi\{X_{a(\lambda),\eta(\lambda)}\}$, and since $\Phi\in\mathfrak{h}_r$ we have $E\in\mathfrak{h}(K)$, completing the proof.

The necessity of the inequality $\pi_\alpha(\beta)\leqq\gamma$ in Theorem B is a direct corollary of this theorem. Indeed, if $P_\alpha S_\beta\leqq S_\gamma P_\delta$, then for any discriminative system $\mathfrak{S}$ of $P_\alpha S_\beta$, we have $E_{\mathfrak{S}}\in S_\gamma P_\delta(K_{\mathfrak{S}})$, and hence $E_{\mathfrak{S}}=\bigcup_{\nu\in\varDelta}\bigcap_{\lambda\in\varLambda_\nu}\Psi_{a(\lambda),\eta(\lambda)}$, where $\Psi_{a(\lambda),\eta(\lambda)}\in K_{\mathfrak{S}}$, $\varDelta$ and $\varLambda_\nu$ are sets of indices, and $\overline{\overline{\varDelta}}<\aleph_\gamma$ and $\overline{\overline{\varLambda_\nu}}<\aleph_\delta$ for any $\nu\in\varDelta$. Since $\bigcap_{\lambda\in\varLambda_\nu}\Psi_{a(\lambda),\eta(\lambda)}$ is included in $E_{\mathfrak{S}}$ for any $\nu\in\varDelta$, it contains no element in common with A, and hence $a(\lambda)$, where $\lambda\in\varLambda_\nu$, must range over all A. Hence $\bigcap_{\lambda\in\varLambda_\nu}\Psi_{a(\lambda),\eta(\lambda)}$ contains at most one function in $E_{\mathfrak{S}}$, and hence the power of $\varDelta$ must be equal to or greater than that of $E_{\mathfrak{S}}$, which can be equal to or greater than any cardinal number less than $\aleph_{\pi_\alpha(\beta)}$. From this the necessity of $\pi_\alpha(\beta)\leqq\gamma$ in Theorem B follows immediately.

§ 3. $S_\alpha P_\beta S_\gamma\leqq P_\delta S_\varepsilon$ and $S_\alpha P_\beta S_\gamma\leqq S_\delta P_\varepsilon$.

1. In this section we shall study the conditions under which the inequality $S_\alpha P_\beta S_\gamma\leqq P_\delta S_\varepsilon$ or $S_\alpha P_\beta S_\gamma\leqq S_\delta P_\varepsilon$ holds.

DEFINITION 6. *Let $F(S_{\alpha_1},\cdots,S_{\alpha_m}; P_{\beta_1},\cdots,P_{\beta_n})$ be a monomial, then the monomial $F(P_{\alpha_1},\cdots,P_{\alpha_m}; S_{\beta_1},\cdots,S_{\beta_n})$ is called the dual of the former.* (*For example, $P_\alpha S_\beta$ is the dual of $S_\alpha P_\beta$.*) *An equality $F_1=G_1$ or an inequality $F_1\leqq G_1$ is called the dual of an equality $F_2=G_2$ or an inequality $F_2\leqq G_2$ respectively, if F_2 is the dual of F_1 and G_2 is the dual of G_1.*

Of course the dual of the dual of a monomial, an equality or an inequality is the same as the original one. By the usual proof with the duality principle we can see

LEMMA 8. *If an equality or an inequality holds between two monomials, then the dual of it also holds.*

DEFINITION 7. *Let $F(S_{\alpha_1},\cdots,S_{\alpha_m}; P_{\beta_1},\cdots,P_{\beta_n})$ be a monomial. We say that* *the monomial* $F(S_{\alpha_1},\cdots,S_{\alpha_m}; I,\cdots,I)$, *where* I *is the identical operation on classes (see* § 1, 2), *is the S-component of F, and the monomial* $F(I,\cdots,I; P_{\beta_1},\cdots,P_{\beta_n})$ *is the P-component of F.*

LEMMA 9. *Let F and G be monomials. If* $F\leqq G$, *then the S-component (resp. the P-component) of G is greater than, or equal to, the S-component (resp. P-component) of F.*

PROOF. Assume that the S-component of $F(S_{\alpha_1},\cdots,S_{\alpha_m}; P_{\beta_1},\cdots,P_{\beta_n})$ is not less than, nor equal to, the S-component of $G(S_{\gamma_1},\cdots,S_{\gamma_s}; P_{\delta_1},\cdots,P_{\delta_t})$. By the successive application of the formulae in Theorem A, the S-components of F and G are reduced to some single operations S_ξ and S_η respectively, where the assumption implies $\xi>\eta$. Let E be a set whose power is $\aleph_\eta$, and let K be the class of all sets each of which consists of a single element in E. Then $S_\xi(K)$ contains E, but $S_\eta(K)$ does not contain E. Since F is greater than, or equal to, its S-component, E is also contained in $F(K)$. On the other hand, we can easily prove that E is not contained in the class $G(S_{\gamma_1},\cdots,S_{\gamma_s}; P_{\delta_1},\cdots,P_{\delta_i}, I, I,\cdots,I)(K)$ $(i\leqq t)$ by the induction with respect to the suffix i, and then putting $i=t$ we can see that $G(K)$ does not contain the set E, and we have $F\not\leqq G$ which proves the lemma.

We can discuss about the P-component similarly.

2. By Lemma 8, the condition on the ordinal numbers of indices $\alpha,\beta,\cdots$ for the inequality $S_\alpha P_\beta S_\gamma\leqq P_\delta S_\varepsilon$ is the same as the condition for the inequality $P_\alpha S_\beta P_\gamma\leqq S_\delta P_\varepsilon$ to hold.

THEOREM 2. *A necessary and sufficient condition for the inequality* $S_\alpha P_\beta S_\gamma \leqq P_\delta S_\varepsilon$ *(or its dual* $P_\alpha S_\beta P_\gamma\leqq S_\delta P_\varepsilon$*) is that we have*

$$\pi_\alpha(\beta)\leqq\delta\,, \tag{1}$$

and

$$\left.\begin{array}{ll} \textit{if } \mathrm{cf}(\gamma)<\alpha\leqq\gamma+1 & \textit{then } \gamma<\varepsilon \\ \textit{otherwise} & \max(\alpha,\gamma)\leqq\varepsilon\,. \end{array}\right\} \tag{2}$$

PROOF. Since $S_\alpha P_\beta\leqq S_\alpha P_\beta S_\gamma$, the condition $\pi_\alpha(\beta)\leqq\delta$ is necessary by Theorem B, while (2) is the necessary and sufficient condition for the inequality $S_\alpha S_\gamma\leqq S_\varepsilon$ (see Remark below Theorem A). If (2) holds, then we have $S_\alpha S_\gamma\leqq S_\varepsilon$, and by Lemma 1, we have $P_\delta S_\alpha S_\gamma\leqq P_\delta S_\varepsilon$, while (1) implies $S_\alpha P_\beta\leqq P_\delta S_\alpha$, and hence we have $S_\alpha P_\beta S_\gamma\leqq P_\delta S_\alpha S_\gamma\leqq P_\delta S_\varepsilon$, which completes the proof.

3. In order to study the condition for the inequality $S_\alpha P_\beta S_\gamma\leqq S_\delta P_\varepsilon$ we shall prepare some preliminary lemmas.

LEMMA 10. *Let F be a monomial. We have* $F(K\cup\{\phi\})=F(K)\cup\{\phi\}$, *where K is a class of sets and* $\{\phi\}$ *is the class which consists of only the void set* ϕ.

PROOF. Since F is increasing and positive, we have $F(K\cup\{\phi\})\supset F(K)\cup\{\phi\}$, while it is easy to see that we have $P_\alpha(K\cup\{\phi\})\subset P_\alpha(K)\cup\{\phi\}$ and $S_\alpha(K\cup\{\phi\})$

$\subset S_\alpha(K)\cup\{\phi\}$. Hence by induction with respect to the degree of the monomial F, we can see $F(K\cup\{\phi\})\subset F(K)\cup\{\phi\}$ and the proof is completed.

We say that two classes K and H of sets are mutually *independent* if any set in K has no element in common with any set in H.

LEMMA 11. *Let K and H be mutually independent classes of sets. Let $\mathfrak{X}$ denote the union of all sets in K, and let F be a monomial. Then $E\in F(K\cup H)$ implies $E\frown\mathfrak{X}\in F(K)\cup\{\phi\}$.*

PROOF. Since $\{Y;\ Y=X\frown\mathfrak{X},\ X\in K\cup H\}=K\cup\{\phi\}$, it follows from Lemma 4 that $F(K\cup\{\phi\})$ contains $E\frown\mathfrak{X}$. Hence $E\frown\mathfrak{X}$ is contained in $F(K)\cup\{\phi\}$ by previous lemma and the proof is completed.

LEMMA 12. *Let F and G be monomials and let δ be an infinite limit ordinal number. If $\mathrm{cf}(\delta)<\alpha$ and $F\not\leqq S_\lambda G$ for any $\lambda<\delta$, then $S_\alpha F\not\leqq S_\delta G$.*

PROOF. Let Λ be a cofinal subsequence of the seqcence of ordinal numbers less than δ such that $\overline{\overline{\Lambda}}=\aleph_{\mathrm{cf}(\delta)}$. By the assumption and Lemma 5, there exist a class K_λ and a non-void set E_λ for any $\lambda<\delta$ such that $E_\lambda\in F(K_\lambda)$ but not $E_\lambda\in S_\lambda G(K_\lambda)$. Naturally we may assume that those classes K_λ are mutually independent. Put $K=\bigcup_{\lambda\in\Lambda}K_\lambda$ and $E=\bigcup_{\lambda\in\Lambda}E_\lambda$, then since $\overline{\overline{\Lambda}}=\aleph_{\mathrm{cf}(\delta)}<\aleph_\alpha$, E is contained in $S_\alpha F(K)$. Suppose that E is contained in $S_\delta G(K)$, then $E=\bigcup_{a\in A}X_a$ where $\overline{\overline{A}}<\aleph_\delta$ and $X_a\in G(K)$. But since the sequence Λ is cofinal to δ, there exists a $\lambda\in\Lambda$ such that $\overline{\overline{A}}<\aleph_\lambda$. Hence $E\in S_\lambda G(K)$. Let H be the class which is the union of all classes K_ν $(\nu\in\Lambda)$ except K_λ, then H and K_λ are mutually independent and $K=H\cup K_\lambda$. Let $\mathfrak{X}$ be the union of all sets in K_λ, then $E\frown\mathfrak{X}$ is contained in $S_\lambda G(K_\lambda)\cup\{\phi\}$ by Lemma 11. But since $E\frown\mathfrak{X}=E_\lambda$ and it is not void, E_λ is contained in $S_\lambda G(K_\lambda)$ which is a contradiction. Hence $E\notin S_\delta G(K)$ and $S_\alpha F\not\leqq S_\delta G$, which proves the lemma.

4. THEOREM 3. *For the inequality $S_\alpha P_\beta S_\gamma\leqq S_\delta P_\varepsilon$ (or its dual $P_\alpha S_\beta P_\gamma\leqq P_\delta S_\varepsilon$) to hold, it is necessary and sufficient that we have $\beta\leqq\varepsilon$ and*

$$\left.\begin{array}{ll} \textit{if } \mathrm{cf}(\pi_\beta(\gamma))<\alpha\leqq\pi_\beta(\gamma)+1 & \textit{then } \pi_\beta(\gamma)<\delta\,, \\ \textit{otherwise} & \max(\pi_\beta(\gamma),\alpha)\leqq\delta\,. \end{array}\right\} \quad (3)$$

PROOF. Condition (3) is necessary and sufficient for the inequality $S_\alpha S_{\pi_\beta(\gamma)}\leqq S_\delta$ (see Remark below Theorem A). Hence under condition (3) we have $S_\alpha S_{\pi_\beta(\gamma)}P_\varepsilon\leqq S_\delta P_\varepsilon$ by Lemma 1 (a). By Theorem B, $\beta\leqq\varepsilon$ implies $P_\beta S_\gamma\leqq S_{\pi_\beta(\gamma)}P_\varepsilon$ and we have $S_\alpha P_\beta S_\gamma\leqq S_\alpha S_{\pi_\beta(\gamma)}P_\varepsilon\leqq S_\delta P_\varepsilon$ which proves the sufficiency of these conditions.

By Lemma 9, $\beta\leqq\varepsilon$ and $\alpha\leqq\delta$ are obviously necessary, and since $P_\beta S_\gamma\leqq S_\alpha P_\beta S_\gamma\leqq S_\delta P_\varepsilon$, the condition $\pi_\beta(\gamma)\leqq\delta$ is also necessary. Now we shall prove that in the case when $\mathrm{cf}(\pi_\beta(\gamma))<\alpha$, condition $\pi_\beta(\gamma)<\delta$ is also necessary.

Assume $\alpha \leqq \delta$, $\mathrm{cf}(\pi_\beta(\gamma)) < \alpha$ and $\pi_\beta(\gamma) = \delta$, then we have $\mathrm{cf}(\delta) < \alpha \leqq \delta$. Hence δ is an infinite limit number. If $\lambda < \delta = \pi_\beta(\gamma)$, then by Theorem B, we have $P_\beta S_\gamma \nleqq S_\lambda P_\varepsilon$. Hence, by Lemma 12, we have $S_\alpha P_\beta S_\gamma \nleqq S_\delta P_\varepsilon$, and the proof is completed.

§ 4. $P_\delta S_\varepsilon \leqq P_\alpha S_\beta P_\gamma$ and $P_\delta S_\varepsilon = P_\alpha S_\beta P_\gamma$.

1. It is easily seen that the function $\pi_\alpha(\beta)$ is continuous with respect to the argument α; that is, when α is a limit number, we have $\lim\limits_{\lambda \uparrow \alpha} \pi_\lambda(\beta) = \pi_\alpha(\beta)$. But in general it is not continuous with respect to the argument β. For example, put $\tilde{\omega}(0) = 0$, $\aleph_{\tilde{\omega}(n+1)} = 2^{\aleph_{\tilde{\omega}(n)}}$ for finite ordinal number n and $\aleph_{\tilde{\omega}(\omega_0)} = \sum\limits_{u<\omega_0} \aleph_{\tilde{\omega}(n)}$. Then according to König's theorem and $\aleph_{\tilde{\omega}(n+1)} > \aleph_{\tilde{\omega}(n)}$, we have $\aleph_{\tilde{\omega}(\omega_0)} = \sum\limits_{n<\omega_0} \aleph_{\tilde{\omega}(n)} < \prod\limits_{n<\omega_0} \aleph_{\tilde{\omega}(n+1)} < \aleph_{\pi_1(\tilde{\omega}(\omega_0))}$, while since $(\aleph_{\tilde{\omega}(n)})^{\aleph_0} = \aleph_{\tilde{\omega}(n)}$ for $0 < n < \omega_0$ we have $\pi_1(\tilde{\omega}(n+1)) = \tilde{\omega}(n+1)$ and hence $\lim\limits_{n \uparrow \omega_0} \pi_1(\tilde{\omega}(n)) = \tilde{\omega}(\omega_0)$. Therefore we have $\lim\limits_{\lambda \uparrow \tilde{\omega}(\omega_0)} \pi_1(\lambda) = \lim\limits_{n \uparrow \omega_0} \pi_1(\tilde{\omega}(n)) < \pi_1(\tilde{\omega}(\omega_0))$.

DEFINITION 8. *We put* $l\pi_\delta(\varepsilon) = \pi_\delta(\varepsilon)$ *if either* ε *is an isolated number or* $\varepsilon = 0$, *and* $l\pi_\delta(\varepsilon) = \lim\limits_{\lambda \uparrow \varepsilon} \pi_\delta(\lambda)$ *if* ε *is an infinite limit number.*

LEMMA 13. $l\pi_\delta(\varepsilon)$ *is the least ordinal number* μ *such that for any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, $\mathfrak{m} < \aleph_\varepsilon$ *and* $\mathfrak{n} < \aleph_\delta$ *imply* $\mathfrak{m}^{\mathfrak{n}} < \aleph_\mu$.

PROOF. The case when ε is an isolated number is trivial. When ε is an infinite limit number, we have $l\pi_\delta(\varepsilon) \leqq \mu$ if and only if $\pi_\delta(\lambda) \leqq \mu$ for every isolated number $\lambda < \varepsilon$, and from this our statement follows immediately. When $\delta = \varepsilon = 0$, then $l\pi_0(0) = \pi_0(0) = 0$, and our statement is trivial. Now assume that $\varepsilon = 0$ and $0 < \delta$. $l\pi_\delta(0) = \pi_\delta(0)$ is the least ordinal number μ such that for any set $\{n_a; a \in A\}$ of finite cardinal numbers n_a, $\overline{\overline{A}} < \aleph_\delta$ implies $\prod\limits_{a \in A} n_a < \aleph_\mu$. Let n be any finite cardinal number. We may assume that $n \leqq n_a$ for any $a \in A$ and $\aleph_0 \leqq \overline{\overline{A}}$, since $0 < \delta$. Now we have $n^{\overline{\overline{A}}} \leqq \prod\limits_{a \in A} n_a \leqq \aleph_0^{\overline{\overline{A}}} \leqq (n^{\aleph_0})^{\overline{\overline{A}}} = n^{\overline{\overline{A}}}$. Hence μ is the least ordinal number such that $n < \aleph_0$ and $\overline{\overline{A}} < \aleph_\delta$ imply $n^{\overline{\overline{A}}} < \aleph_\mu$.

2. Let K be a class of sets, and let $P(K)$ denote the class of all sets which are represented as intersections of an arbitrary number of sets in K. The operation P, as well as the products PS_α, $P_\alpha S_\beta P$ etc., are integral operations, and we have $P_\alpha < P$ for any ordinal number α.

Let A be a set. A family $\{A_\xi; \xi \in \Xi\}$ of subsets A_ξ of A is called a *covering* of A, if $\bigcup\limits_{\xi \in \Xi} A_\xi = A$. The power of the family Ξ is called the *power of this covering.*

LEMMA 14. *We have $P_\delta S_\varepsilon \not\leqq P_\alpha S_\beta P$, if there is a discriminative system $\mathfrak{S} = \{\psi_a ; a \in A\}$ of $P_\delta S_\varepsilon$, which satisfies the folowing condition.*

(D) *For any covering $\{A_\xi ; \xi \in \Xi\}$ of A with a power less than $\aleph_\alpha$, there exists a $\xi \in \Xi$ such that $\prod_{a \in A_\xi} \overline{\overline{\psi_a}} \geqq \aleph_\beta$.*

PROOF. Since $E_\mathfrak{S}$ is contained in $P_\delta S_\varepsilon(K_\mathfrak{S})$, it is sufficient to show that $E_\mathfrak{S}$ is not contained in $P_\alpha S_\beta P(K_\mathfrak{S})$. Assume on the contrary that $E_\mathfrak{S}$ is contained in $P_\alpha S_\beta P(K_\mathfrak{S})$, then $E_\mathfrak{S} = \bigcap_{\xi \in \Xi} X_\xi$, where $\overline{\overline{\Xi}} < \aleph_\alpha$ and $X_\xi \in S_\beta P(K_\mathfrak{S})$. Put $A_\xi = A - X_\xi$, then, since $E_\mathfrak{S}$ contains no element in A, $\{A_\xi ; \xi \in \Xi\}$ is a covering of A with a power less than $\aleph_\alpha$. Hence there exists a ξ in Ξ with $\prod_{a \in A_\xi} \overline{\overline{\psi_a}} \geqq \aleph_\beta$. Let $E_{\mathfrak{S},\xi}$ be the set of all functions φ in $E_\mathfrak{S}$ such that $\varphi(a) = 1$ for any $a \in A - A_\xi$, then $\prod_{a \in A_\xi} \overline{\overline{\psi_a}} \geqq \aleph_\beta$ implies $\overline{\overline{E_{\mathfrak{S},\xi}}} \geqq \aleph_\beta$. Now $X_\xi = \bigcup_{b \in B} Y_b$ where $\overline{\overline{B}} < \aleph_\beta$ and Y_b are intersections of sets $\Psi_{a,\eta}$ in a subclasses of $K_\mathfrak{S}$. But since $\Psi_{a,\eta}$ and $\Psi_{a,\zeta}$ intersect only when $\eta = \zeta$, and Y_b, being a subset of X_ξ, contains no element in A_ξ, all functions in $Y_b \cap E_\mathfrak{S}$ take a definite value for each $a \in A_\xi$. Hence Y_b contains at most one function in $E_{\mathfrak{S},\xi}$. Since $\overline{\overline{B}} < \aleph_\beta \leqq \overline{\overline{E_{\mathfrak{S},\xi}}}$, $X_\xi = \bigcup_{b \in B} Y_b$ can not contain $\overline{\overline{E_{\mathfrak{S},\xi}}}$ and hence $E_\mathfrak{S}$ is not contained in X_ξ, which is a contradiction, and the proof is completed.

3. THEOREM 4. *For the inequality*

$$P_\delta S_\varepsilon \leqq P_\alpha S_\beta P_\gamma \text{ (or its dual } S_\delta P_\varepsilon \leqq S_\alpha P_\beta S_\gamma) \tag{4}$$

to hold, it is necessary and sufficient that one of the following conditions (i), (ii) *and* (iii) *holds.*

(i) $\delta \leqq \alpha$ *and* $\varepsilon \leqq \beta$,

(ii) $\delta^* \leqq \gamma$ *and* $\pi_{\delta^*}(\varepsilon) \leqq \beta$,

(iii) $\delta^* \leqq \gamma$, $\mathrm{cf}(\varepsilon) < \alpha$ *and* $l\pi_{\delta^*}(\varepsilon) \leqq \beta$,

where

$$\delta^* = \delta \textit{ if either } \delta \textit{ is a limit number or } \delta = \delta' + 1 \textit{ and } \alpha \leqq \mathrm{cf}(\delta') ,$$

and $\delta^* = \delta'$ *if* $\delta = \delta' + 1$ *and* $\mathrm{cf}(\delta') < \alpha$.

REMARK. When $\alpha < \delta$, δ^* is the least ordinal number with $P_\delta \leqq P_\alpha P_{\delta^*}$ (see Remark below Theorem A).

PROOF OF THEOREM 4. First we shall show that these conditions are sufficient. Condition (i) is trivial. If condition (ii) is satisfied, then we have $P_\delta S_\varepsilon \leqq P_\alpha P_{\delta^*} S_\varepsilon \leqq P_\alpha S_\beta P_\gamma$. Now assume that condition (iii) is satisfied. But if $l\pi_{\delta^*}(\varepsilon) = \pi_{\delta^*}(\varepsilon)$, then condition (iii) is reduced to condition (ii). Hence we may assume that ε is an infinite limit number. Put $\lambda = \mathrm{cf}(\varepsilon) + 1$, then, since λ is an isolated number and $\lambda \leqq \alpha$, we have $P_\alpha P_\lambda = P_\alpha$. Now we shall show that the inequality $P_{\delta^*} S_\varepsilon \leqq P_\lambda S_\beta P_\gamma$ holds. For this purpose, it is sufficient to show that for any discriminative system $\mathfrak{S}=\{\psi_a\,;a\in A\}$ of $P_{\delta*}S_\varepsilon$, $E_{\mathfrak{S}}$ is contained in $P_\lambda S_\beta P_\gamma(K_{\mathfrak{S}})$. But if there exists an ordinal number $\mu<\varepsilon$ such that $\psi_a<\omega_\mu$ for any $a\in A$, then $\mathfrak{S}$ is a discriminative system of $P_{\delta*}S_\mu$. Since $\pi_{\delta*}(\mu)\leqq l\pi_{\delta*}(\varepsilon)$, we have $P_{\delta*}S_\mu\leqq S_\beta P_\gamma\leqq P_\lambda S_\beta P_\gamma$ and hence $E_{\mathfrak{S}}\in P_\lambda S_\beta P_\gamma(K_{\mathfrak{S}})$. If $\operatorname{l.u.b.}_{a\in A}\psi_a=\omega_\varepsilon$, then, since ε is an infinite limit number, we can take a strictly increasing sequence $\{\sigma_\rho\,;\rho<\omega_{\mathrm{cf}(\varepsilon)}\}$ of ordinal numbers σ_ρ cofinal to ε. Put $\Xi_0=\{a\,;\psi_a<\omega_{\sigma_1}\}$, $\Xi_\rho=\{a\,;\omega_{\sigma_\rho}\leqq\psi_a<\omega_{\sigma_{\rho+1}}\}$ for $0<\rho<\omega_{\mathrm{cf}(\varepsilon)}$, and $Y_\rho=\bigcap_{a\in\Xi_\rho}\bigcup_{\eta<\psi_a}\Psi_{a,\eta}$, then we have $E_{\mathfrak{S}}=\bigcap_{\rho<\omega_{\mathrm{cf}(\varepsilon)}}Y_\rho$. But since $\{\psi_a\,;a\in\Xi_\rho\}$ is a discriminative system of $P_{\delta*}S_{\sigma_{\rho+1}}$ and $\pi_{\delta*}(\sigma_{\rho+1})\leqq l\pi_{\delta*}(\varepsilon)\leqq\beta$, we have $P_{\delta*}S_{\sigma_{\rho+1}}\leqq S_\beta P_\gamma$ and hence $Y_\rho\in S_\beta P_\gamma(K_{\mathfrak{S}})$. Furthermore, since $\mathrm{cf}(\varepsilon)<\lambda$, we have $E_{\mathfrak{S}}\in P_\lambda S_\beta P_\gamma(K_{\mathfrak{S}})$. Hence we have $P_{\delta*}S_\varepsilon\leqq P_\lambda S_\beta P_\gamma$, and so $P_\delta S_\varepsilon\leqq P_\alpha P_{\delta*}S_\varepsilon\leqq P_\alpha P_\lambda S_\beta P_\gamma=P_\alpha S_\beta P_\gamma$ by $P_\alpha P_\lambda=P_\alpha$, and this proves the sufficiency of condition (iii).

Next we shall prove the necessity of the conditions. First $\varepsilon\leqq\beta$ is obviously necessary, and if $\alpha<\delta$, then $\delta^*\leqq\gamma$ is also necessary. Indeed, if $\alpha<\delta$ and $\gamma<\delta^*$, then, since both α and γ are less than δ, $P_\delta\leqq P_\alpha P_\gamma$ implies $\delta=\gamma+1$ and $\mathrm{cf}(\gamma)<\alpha$. But by the definition of δ^*, this implies $\delta^*=\gamma$ in contradiction to the assumption $\gamma<\delta^*$.

Now we shall consider the case $\alpha<\delta$, and split it into two cases (a) $\alpha<\delta^*$ and (b) $\alpha=\delta^*<\delta$.

Case (a) $\alpha<\delta^*$. Here we remark

(I) If $\delta^*\neq 0$, then, for any ordinal number τ' less than δ^*, there exists an ordinal number τ such that $\tau'\leqq\tau<\delta^*$ and $\alpha\leqq\mathrm{cf}(\tau)$.

Indeed, if δ^* is an isolated number, then put $\delta^*=\tau+1$. Since $\alpha<\delta^*=\mathrm{cf}(\delta^*)$, we have $\delta=\delta^*$ by the definition of δ^*, and hence we have $\alpha\leqq\mathrm{cf}(\tau)$. If δ^* is a limit number, then let τ be the next successor of $\max(\tau',\alpha)$ and we have $\alpha<\tau=\mathrm{cf}(\tau)$.

Next we shall show

(II) $l\pi_{\delta*}(\varepsilon)\leqq\beta$ is necessary for (4).

Indeed, if $\delta^*=0$, then obviously we have $l\pi_{\delta*}(\varepsilon)=\varepsilon$. Hence $l\pi_{\delta*}(\varepsilon)=\varepsilon\leqq\beta$ is necessary. Assume $\delta^*\neq 0$. $\beta<l\pi_{\delta*}(\varepsilon)$ implies the existence of cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ with $\mathfrak{m}<\aleph_\varepsilon$, $\mathfrak{n}<\aleph_{\delta*}$ and $\mathfrak{m}^{\mathfrak{n}}\geqq\aleph_\beta$. Since $\delta^*>0$, we may assume that $\mathfrak{n}$ is not finite and hence $\mathfrak{n}=\aleph_{\tau'}$ for some ordinal number $\tau'<\delta^*$. By previous remark (I), there exists an ordinal number τ with $\tau'\leqq\tau<\delta^*$ and $\alpha\leqq\mathrm{cf}(\tau)$. Let ψ be an ordinal number whose power is equal to $\mathfrak{m}$ and put $\psi_a=\psi$ for any $a\in A$ where A is a set with $\overline{\overline{A}}=\aleph_\tau$, then since $\psi_a<\omega_\varepsilon$ and $\tau<\delta^*\leqq\delta$, $\mathfrak{S}=\{\psi_a\,;a\in A\}$ is a discriminative system of $P_\delta S_\varepsilon$. Let $\{A_\xi\,;\xi\in\Xi\}$ be a covering of A with a power less than $\aleph_\alpha$, then since $\alpha\leqq\mathrm{cf}(\tau)$, there exists a $\xi\in\Xi$ with $\overline{\overline{A_\xi}}=\aleph_\tau$. Hence we have $\prod_{a\in A_\xi}\overline{\psi_a}=\mathfrak{m}^{\aleph_\tau}\geqq\aleph_\beta$ and the system $\mathfrak{S}$ satisfies the condition (D) in Lemma 14. Furthermore,

(III) If $\alpha \leqq \mathrm{cf}(\varepsilon)$, then $\pi_{\delta^*}(\varepsilon) \leqq \beta$ is necessary for (4).

Indeed, if $\beta < \pi_{\delta^*}(\varepsilon)$, then we have either $\beta < l\pi_{\delta^*}(\varepsilon)$ or $l\pi_{\delta^*}(\varepsilon) \leqq \beta < \pi_{\delta^*}(\varepsilon)$. But in the former case we have already seen that (4) does not hold. Hence we shall consider only the latter case and so let ε be an infinite limit number. $\beta < \pi_{\delta^*}(\varepsilon)$ implies the existence of a set of cardinal numbers $\mathfrak{m}_a, a \in A$ such that $\overline{\overline{A}} < \aleph_{\delta^*}$, $\mathfrak{m}_a < \aleph_\varepsilon$ for any $a \in A$ and $\prod_{a \in A} \mathfrak{m}_a \geqq \aleph_\beta$. Here the set $\{\mathfrak{m}_a ; a \in A\}$ of cardinal numbers is cofinal to $\aleph_\varepsilon$, or otherwise there would be a cardinal number $\mathfrak{m} < \aleph_\varepsilon$ which is greater than $\mathfrak{m}_a$ for any $a \in A$. Hence we have $\aleph_\beta \leqq \mathfrak{m}^{\overline{\overline{A}}}$ which implies $\beta < l\pi_{\delta^*}(\varepsilon)$ in contradiction to the assumption. Hence the power of A is not finite and, putting $A = \aleph_{\tau'}$, we have $\mathrm{cf}(\varepsilon) \leqq \tau' < \delta^*$. Now by the remark (I) above, there exists an ordinal number τ with $\tau' \leqq \tau < \delta^*$ and $\alpha \leqq \mathrm{cf}(\tau)$. $\prod_{a \in A} \mathfrak{m}_a \geqq \aleph_\beta$ implies $\aleph_\varepsilon^{\aleph_\tau} \geqq \aleph_\beta$.

Let $\{\mathfrak{m}_\lambda ; \lambda < \omega_{\mathrm{cf}(\varepsilon)}\}$ be a strictly increasing sequence of cardinal numbers $\mathfrak{m}_\lambda$ less than $\aleph_\varepsilon$ and cofinal to $\aleph_\varepsilon$. Let B be a set with $\overline{\overline{B}} = \aleph_\tau$. Choose an ordinal number $\psi_{(b,\lambda)}$ whose power is equal to $\mathfrak{m}_\lambda$ for any $b \in B$ and $\lambda < \omega_{\mathrm{cf}(\varepsilon)}$, and put $\Lambda = \{(b, \lambda) ; b \in B, \lambda < \omega_{\mathrm{cf}(\varepsilon)}\}$, then by $\mathrm{cf}(\varepsilon) < \delta^* \leqq \delta$ and $\tau < \delta^* \leqq \delta$, we have $\overline{\overline{\Lambda}} < \aleph_\delta$, and the system $\mathfrak{S} = \{\psi_{(b,\lambda)} ; (b, \lambda) \in \Lambda\}$ is a discriminative system of $P_\delta S_\varepsilon$. We shall show that this system satisfies condition (D) in Lemma 14.

Let $\{\Lambda_\xi ; \xi \in \Xi\}$ be any covering of Λ with $\overline{\overline{\Xi}} < \aleph_\alpha$. Put $B_{\xi,\lambda} = \{b ; (b, \lambda) \in \Lambda_\xi\}$, $\Delta_\xi = \{\lambda ; \overline{\overline{B_{\xi,\lambda}}} = \aleph_\tau\}$ and $\lambda_\xi^* = \underset{\lambda \in \Delta_\xi}{\mathrm{l.u.b.}}\, \lambda$. Assume that $\lambda_\xi^* < \omega_{\mathrm{cf}(\varepsilon)}$ for any $\xi \in \Xi$, then since $\alpha \leqq \mathrm{cf}(\varepsilon)$ and $\overline{\overline{\Xi}} < \aleph_\alpha$, the set $\{\lambda_\xi^* ; \xi \in \Xi\}$ is not cofinal to $\omega_{\mathrm{cf}(\varepsilon)}$. Hence $\lambda^* = \underset{\xi \in \Xi}{\mathrm{l.u.b.}}\, \lambda_\xi^* < \omega_{\mathrm{cf}(\varepsilon)}$. Put $\kappa = \lambda^* + 1$, then we have $\kappa < \omega_{\mathrm{cf}(\varepsilon)}$, and $\overline{\overline{B_{\xi,\kappa}}} < \aleph_\tau$ for any $\xi \in \Xi$. Since $\overline{\overline{\Xi}} < \aleph_\alpha \leqq \aleph_{\mathrm{cf}(\tau)}$, we have $\sum_{\xi \in \Xi} \overline{\overline{B_{\xi,\kappa}}} < \aleph_\tau = \overline{\overline{B}}$. But since $\{\Lambda_\xi ; \xi \in \Xi\}$ is a covering of Λ, any (b, κ) with $b \in B$ lies in some Λ_ξ, and hence $\bigcup_{\xi \in \Xi} B_{\xi,\kappa} = B$ which is a contradiction. Hence there exists a $\xi \in \Xi$ with $\lambda_\xi^* = \omega_{\mathrm{cf}(\varepsilon)}$, and the set $\Delta_\xi = \{\lambda ; \overline{\overline{B_{\xi,\lambda}}} = \aleph_\tau\}$ is cofinal to $\omega_{\mathrm{cf}(\varepsilon)}$. Hence we have $\sum_{\lambda \in \Delta_\xi} \mathfrak{m}_\lambda = \aleph_\varepsilon$, and for any $\lambda \in \Delta_\xi$ there exist $\aleph_\tau$ elements $b \in B$ such that $(b, \lambda) \in \Lambda_\xi$. Hence we have $\prod_{(b,\lambda) \in \Lambda_\xi} \overline{\psi_{(b,\lambda)}} = (\prod_{\lambda \in \Delta_\xi} \mathfrak{m}_\lambda)^{\aleph_\tau} \geqq (\sum_{\lambda \in \Delta_\xi} \mathfrak{m}_\lambda)^{\aleph_\tau} = \aleph_\varepsilon^{\aleph_\tau} \geqq \aleph_\beta$. Hence the system $\mathfrak{S}$ satisfies condition (D) in Lemma 14, and inequality (4) does not hold.

Now we have proved that in the case (a) $\alpha < \delta^*$ it is necessary for (4) that either (ii) or (iii) is satisfied. Next we shall show that these conditions are also necessary for (4) in the case

(b) $\alpha = \delta^* < \delta$.

In this case, by the definition of δ^*, we have $\delta = \delta^* + 1 = \alpha + 1$ and $\alpha > \mathrm{cf}(\delta^*) = \mathrm{cf}(\alpha)$. Hence α is an infinite limit number. Let λ be any ordinal number with $\mathrm{cf}(\alpha)<\lambda<\alpha$. First we shall show that, if neither (ii) nor (iii) is satisfied, then the inequality $P_\delta S_\varepsilon \leqq P_\lambda S_\beta P_\gamma$ does not hold.

Indeed, if we replace λ for α in this theorem, δ^* will remain unaltered. Condition (ii) is independent of the value of α as far as δ^* is unaltered. If condition (iii)′ $\delta^*<\gamma$, $\mathrm{cf}(\varepsilon)<\lambda$ and $l\pi_{\delta_*}(\varepsilon)\leqq\beta$, is satisfied, then since $\lambda<\alpha$, condition (iii) will be satisfied by α, in contradiction to our assumption. Hence now neither condition (ii) nor (iii)′ is satisfied. Furthermore we have $\lambda<\delta^*$ and hence we are taken back to case (a). Therefore $P_\delta S_\varepsilon \not\leqq P_\lambda S_\beta P_\gamma$ for any $\mathrm{cf}(\alpha)<\lambda<\alpha$ and so also for any $\lambda<\alpha$.

Hence by Lemma 12, we have $P_\alpha P_\delta S_\varepsilon \not\leqq P_\alpha S_\beta P_\gamma$. But since $P_\alpha P_\delta = P_\alpha P_{\alpha+1} = P_{\alpha+1} = P_\delta$, we have $P_\delta S_\varepsilon \not\leqq P_\alpha S_\beta P_\gamma$, and the whole proof of Theorem 4 is completed.

4. By the definitions of $\pi_\alpha(\beta)$ and $l\pi_\alpha(\beta)$, we can easily see

LEMMA 15. *$\pi_\alpha(\beta)$ and $l\pi_\alpha(\beta)$ are not less than* $\max(\alpha,\beta)$.

Next we shall show

LEMMA 16. *If β is a limit number and $\mathrm{cf}(\beta)<\alpha$, then $\pi_\alpha(\beta)\geqq\beta+2$.*

PROOF. Let $\{\mathfrak{m}_\lambda;\lambda<\omega_{\mathrm{cf}(\beta)}\}$ be a strictly increasing sequence of cardinal numbers cofinal to $\aleph_\beta$. According to König's theorem we have $\aleph_\beta=\sum_{\lambda<\omega_{\mathrm{cf}(\beta)}}\mathfrak{m}_\lambda < \prod_{\lambda<\omega_{\mathrm{cf}(\beta)}}\mathfrak{m}_\lambda$, while by the definition of $\pi_\alpha(\beta)$ we have $\prod_{\lambda<\omega_{\mathrm{cf}(\beta)}}\mathfrak{m}_\lambda<\aleph_{\pi_\alpha(\beta)}$. Hence we have $\pi_\alpha(\beta)\geqq\beta+2$.

THEOREM 5. *For the equality*

$$P_\delta S_\varepsilon = P_\alpha S_\beta P_\gamma \qquad (\textit{or its dual } S_\delta P_\varepsilon = S_\alpha P_\beta S_\gamma) \tag{5}$$

to hold, it is necessary and sufficient that one of the following conditions (i), (ii) *and* (iii) *holds.*

(i) $\alpha=\delta$, $\beta=\varepsilon$ *and* $\pi_\beta(\gamma)<\delta$,

(ii) $\beta=\varepsilon$ *and* $\alpha=\delta=\mathrm{cf}(\alpha)=\pi_\beta(\gamma)$,

(iii) $\alpha<\beta=\gamma=\delta=\varepsilon=\pi_\beta(\beta)$[1].

PROOF. From Theorem A and B the sufficiency of these conditions follows immediately. Now we shall show the necessity of these conditions. First comparing the S-components of the both sides of (5), we have $\varepsilon=\beta$. If further $\alpha=\delta$, then according to Theorem 3, $P_\alpha S_\beta \geqq P_\alpha S_\beta P_\gamma$ implies $\pi_\beta(\gamma)\leqq\alpha$. If especially $\pi_\beta(\gamma)=\alpha$, then we have $\alpha\leqq\mathrm{cf}(\pi_\beta(\gamma))\leqq\pi_\beta(\gamma)=\alpha$. Hence equality (5) and the condition $\alpha=\delta$ lead to either (i) or (ii).

By the comparison of P-components of both sides of (5), we have $\alpha\leqq\delta$. If especially $\alpha<\delta$, then we have $\delta^*\leqq\gamma$, where δ^* is the ordinal number defined in Theorem 4. Further under the condition $\alpha<\delta$ the equality

1) The ordinal number β with $\beta=\pi_\beta(\beta)$ is a so-called strongly inaccessible number.

$$P_\delta S_\beta = P_\alpha S_\beta P_\gamma \tag{6}$$

implies $\pi_\beta(\gamma) \leqq \delta$ by Theorem 3. On the other hand, by Theorem 4 and Lemma 15, (6) implies $l\pi_{\delta^*}(\beta) = \beta$. Hence we have

$$\delta^* \leqq l\pi_{\delta^*}(\beta) = \beta \leqq \pi_\beta(\gamma) \leqq \delta\,. \tag{7}$$

Assume $\delta = \delta^* + 1$, then by the definition of δ^*, we have $\mathrm{cf}(\delta^*) < \alpha \leqq \delta^*$. Hence by (7), we have $\mathrm{cf}(\delta^*) < \delta^* \leqq \beta$, and hence by Lemma 16, we have $\delta^* + 2 \leqq \pi_\beta(\delta^*) \leqq \pi_\beta(\gamma) \leqq \delta = \delta^* + 1$ which is impossible. Hence if $\alpha < \delta$, then equality (6) implies $\delta = \delta^*$, but $\delta = \delta^* \leqq \gamma \leqq \pi_\beta(\gamma)$ and (7) lead to (iii) immediately.

§ 5. $P_\delta S_\varepsilon \leqq S_\alpha P_\beta S_\gamma$ and $P_\delta S_\varepsilon = S_\alpha P_\beta S_\gamma$.

1. First we shall show a lemma which corresponds to Lemma 14 for inequality (4) or to Theorem 2 of [**2**] for the equality $P_\alpha S_\beta \leqq S_\gamma P_\delta$,

Definition 9. *$[K_{\mathfrak{S}}]_\gamma$ denotes the class of subsets of $E_{\mathfrak{S}}$ (cf. Definition 5) such that*

(K) *A subset X of $E_{\mathfrak{S}}$ is contained in $[K_{\mathfrak{S}}]_\gamma$ if and only if, for any element a of A, the power of the set $X(a) = \{\varphi(a)\,;\, \varphi \in X\}$ is less than $\aleph_\gamma$.*

Lemma 17. *The inequality*

$$P_\delta S_\varepsilon \leqq S_\alpha P_\beta S_\gamma \text{ (or its dual } S_\delta P_\varepsilon \leqq P_\alpha S_\beta P_\gamma) \tag{8}$$

holds if and only if $\delta \leqq \beta$ and for any discriminative system $\mathfrak{S}$ of $P_\delta S_\varepsilon$, $E_{\mathfrak{S}}$ is contained in $S_\alpha([K_{\mathfrak{S}}]_\gamma)$.

Proof Since the expression $E_{\mathfrak{S}} = \bigcup_{b \in B} X_b$, $\overline{\overline{B}} < \aleph_\alpha$, $X_b \in [K_{\mathfrak{S}}]_\gamma$ implies $E_{\mathfrak{S}} = \bigcup_{b\in B} \bigcap_{a \in A} \bigcup_{\eta \in X_b(a)} \Psi_{a,\eta}$ where $\Psi_{a,\eta} \in K_{\mathfrak{S}}$, the sufficiency of this condition follows from Theorem 1, and the necessity of $\delta \leqq \beta$ follows from Lemma 9.

Obviously $E_{\mathfrak{S}} \in S_\alpha P_\beta S_\gamma(K_{\mathfrak{S}})$ implies $E_{\mathfrak{S}} \in S_\alpha P S_\gamma(K_{\mathfrak{S}})$ where P is the operation defined in § 4,2. Now we shall show that if $E_{\mathfrak{S}}$ is contained in $S_\alpha P S_\gamma(K_{\mathfrak{S}})$ for a discriminative system $\mathfrak{S}$ of $P_\delta S_\varepsilon$, then $E_{\mathfrak{S}}$ is contained in $S_\alpha([K_{\mathfrak{S}}]_\gamma)$.

$E_{\mathfrak{S}} = \bigcup_{b \in B} X_b$ where $X_b \in PS_\gamma(K_{\mathfrak{S}})$ and $\overline{\overline{B}} < \aleph_\alpha$. Since $E_{\mathfrak{S}}$ is disjoint with A, no X_b contains an element in A. Now we have $X_b = \bigcap_{c \in C_b} Y_{b,c}$ where $Y_{b,c} \in S_\gamma(K_{\mathfrak{S}})$. Let a be any element in A, then since X_b does not contain a, there exists an element $c(a)$ in C_b such that $Y_{b,c(a)}$ does not contain a. If we put $X_b' = \bigcap_{a \in A} Y_{b,c(a)}$, then $X_b' \supset X_b$ and $X_b' \cap A = \phi$. Hence $\bigcup_{b \in B} X_b' = E_{\mathfrak{S}}$. Now $Y_{b,c(a)} = \bigcup_{d \in D_{b,a}} \Psi_{a(d),\eta(d)}$ where $\overline{\overline{D}}_{b,a} < \aleph_\gamma$ and $\Psi_{a(d),\eta(d)} \in K_{\mathfrak{S}}$. Since $Y_{b,c(a)}$ does not contain a, $a(d)$ is constantly equal to a for $d \in D_{b,a}$. If we put $\chi_b(a) = \{\eta(d)\,;\, d \in D_{b,a}\}$, then $\chi_b(a)$ is a subset of $\{\eta\,;\, \eta < \psi_a\}$ and $\overline{\overline{\chi_b(a)}} < \aleph_\gamma$. Since $\varphi(a) = \eta$ for any $\varphi \in E_{\mathfrak{S}} \cap \Psi_{a,\eta}$, we have $\chi_b(a) = \{\varphi(a)\,;\, \varphi \in Y_{b,c(a)} \cap E_{\mathfrak{S}}\}$ and $Y_{b,c(a)}$ consists of all elements in A except a and functions φ in $E_{\mathfrak{S}}$ such that $\varphi(a)\in\chi_b(a)$, and X_b' consists of all functions φ in $E_{\mathfrak{S}}$ such that $\varphi(a)\in\chi_b(a)$ for any $a\in A$. Therefore $X_b'\in[K_{\mathfrak{S}}]_\gamma$, and $E_{\mathfrak{S}}\in S_\alpha([K_{\mathfrak{S}}]_\gamma)$, and the necessity of the condition is proved.

The symbol $\pi_\alpha(\beta)$ can be defined, as it is easily seen, as the least ordinal number γ with $P_\alpha S_\beta \leqq S_\gamma P$. Similarly there exists a least ordinal number α, which we shall denote by $\rho_{\delta,\varepsilon}(\gamma)$, such that for any discriminative system $\mathfrak{S}$ of $P_\delta S_\varepsilon$, the set $E_{\mathfrak{S}}$ is contained in $S_\alpha([K_{\mathfrak{S}}]_\gamma)$. If we use the symbol $\rho_{\delta,\varepsilon}(\gamma)$, then Lemma 17 can be stated as follows.

LEMMA 17′. *We have* (8) *if and only if* $\delta\leqq\beta$ *and* $\rho_{\delta,\varepsilon}(\gamma)\leqq\alpha$.

It would be desirable to determine the value $\rho_{\delta,\varepsilon}(\gamma)$ in terms of $\pi_\delta(\varepsilon)$ and cf(δ) etc., but this seems to be a very difficult problem.

2. Now we shall discuss about several conditions which are sufficient or necessary for inequality (8).

LEMMA 18. *It is sufficient for* (8) *that one of the following conditions* (i), (ii), (iii) *or* (iv) *is satisfied:*

(i) $\delta\leqq\beta$ *and* $\varepsilon\leqq\gamma$,

(ii) $\delta\leqq\beta$ *and* $\pi_\delta(\varepsilon)\leqq\alpha$,

(iii) $\delta\leqq\beta$, $\varepsilon=\gamma+1$ *and* $\pi_\delta(\mathrm{cf}(\gamma)+1)\leqq\alpha$,

(iv) $\delta\leqq\beta$, $\varepsilon=\gamma+1$ *and* $\delta\leqq\mathrm{cf}(\gamma)<\alpha$.

PROOF. The sufficiency of (i), (ii) or (iii) is an immediate consequence of Theorem A and Theorem B. For example, if (iii) is satisfied, then we have

$$P_\delta S_\varepsilon = P_\delta S_{\gamma+1} = P_\delta S_{\mathrm{cf}(\gamma)+1} S_\gamma \leqq S_{\pi_\delta(\mathrm{cf}(\gamma)+1)} P_\delta S_\gamma \leqq S_\alpha P_\beta S_\gamma .$$

Now assume that condition (iv) is satisfied. Let $\mathfrak{S}=\{\psi_a; a\in A\}$ be a discriminative system of $P_\delta S_\varepsilon$. Rearranging the order of ordinal numbers less than ψ_a, we may assume that each ψ_a is at most equal to the initial ordinal number ω_γ. Let $\Delta=\{\mu_\nu; \nu<\omega_{\mathrm{cf}(\gamma)}\}$ be a strictly increasing sequence of ordinal numbers cofinal to ω_γ. Let X_γ be the set of all functions φ in $E_{\mathfrak{S}}$ such that $\varphi(a)<\mu_\nu$ for any $a\in A$, then $X_\nu\in[K_{\mathfrak{S}}]_\gamma$. On the other hand, since $\varphi(a)<\omega_\gamma$ for any $\varphi\in E_{\mathfrak{S}}$ and $a\in A$, and since $\overline{\overline{A}}<\aleph_\delta\leqq\aleph_{\mathrm{cf}(\gamma)}$, the set $\{\varphi(a); a\in A\}$ of ordinal numbers is not cofinal to ω_γ, and hence there exists an ordinal number $\mu_\nu\in\Delta$ with $\varphi(a)<\mu_\nu$ for any $a\in A$ and hence φ is contained in X_ν. Hence $E_{\mathfrak{S}}=\bigcup_{\nu<\omega_{\mathrm{cf}(\gamma)}} X_\nu$ and since $\mathrm{cf}(\gamma)<\alpha$, $E_{\mathfrak{S}}$ is contained in $S_\alpha([K_{\mathfrak{S}}]_\gamma)$.

LEMMA 19. *If* $\gamma<\varepsilon$, *then the following conditions* (a), (b) *and* (c) *are necessary for* (8)*:*

(a) *either* $\varepsilon\leqq\alpha$, *or* $\varepsilon=\gamma+1$ *and* $\mathrm{cf}(\gamma)<\alpha$,

(b) *either* $\pi_\delta(\varepsilon)\leqq\alpha$, *or* $\pi_\delta(\varepsilon)=\pi_\delta(\gamma)$,

or $\pi_\delta(\varepsilon)=\pi_\delta(\gamma)+1$ *and* $\mathrm{cf}(\pi_\delta(\gamma))<\alpha$,

(c) $\delta\leqq\alpha$, *and especially if* δ *is an isolated number, then* $\delta<\alpha$.

PROOF. The necessity of (a) is easily proved by comparing the S-components of both sides of (8). Next we shall show the necessity of (b). If $\max(\alpha,\pi_\delta(\gamma))<\pi_\delta(\varepsilon)$, then there exists a discriminative system $\mathfrak{S}$ of $P_\delta S_\varepsilon$ such that $\overline{\overline{E_\mathfrak{S}}}\geqq\max(\aleph_\alpha,\aleph_{\pi_\delta(\gamma)})$. If (8) holds, then by Lemma 17, we have $E_\mathfrak{S}=\bigcup_{b\in B}X_b$ where $\overline{\overline{B}}<\aleph_\alpha$ and $X_b\in[K_\mathfrak{S}]_\gamma$. By the definition of $[K_\mathfrak{S}]_\gamma$, every set X_b in $[K_\mathfrak{S}]_\gamma$ has a power less than $\aleph_{\pi_\delta(\gamma)}$. Hence we have necessarily $\overline{\overline{E_\mathfrak{S}}}=\aleph_{\pi_\delta(\gamma)}$ and $\overline{\overline{B}}\geqq\aleph_{\mathrm{cf}(\pi_\delta(\gamma))}$, and this implies $\pi_\delta(\varepsilon)=\pi_\delta(\gamma)+1$ and $\mathrm{cf}(\pi_\delta(\gamma))<\alpha$.

Next we shall show the necessity of (c). First let δ be an isolated number and put $\delta=\delta'+1$. Let A be a set with $\overline{\overline{A}}=\aleph_{\delta'}$, and put $\psi_a=\omega_\gamma$ for any $a\in A$. then $\mathfrak{S}=\{\psi_a\,;\,a\in A\}$ is a discriminative system of $P_\delta S_\varepsilon$. Assume $\alpha\leqq\delta$ and yet $E_\mathfrak{S}\in S_\alpha([K_\mathfrak{S}]_\gamma)$, then $E_\mathfrak{S}=\bigcup_{b\in B}X_b$ where $\overline{\overline{B}}<\aleph_\alpha$ and $X_b\in[K_\mathfrak{S}]_\gamma$. Since $\overline{\overline{B}}$ is at most equal to $\overline{\overline{A}}=\aleph_{\delta'}$, we can set up a one-to-one correspondence $f(b)=a$ between all elements b of B and elements a of A. Now since the power of the set $X_b(a)=\{\varphi(a)\,;\,\varphi\in X_b\}$ is less than the power $\aleph_\gamma$ of ψ_a, an ordinal number $\nu(b)$ less than $\psi_{f(b)}$, which is not contained in $X_b(f(b))$, can be selected for any $b\in B$. Put $\theta(a)=\nu(b)$ if $a=f(b)$ and $\theta(a)=1$ if $a\neq f(b)$ for any $b\in B$. Since $\theta\in E_\mathfrak{S}$, θ is contained in some X_b but since the ordinal number $\theta(f(b))=\nu(b)$ is not contained in $X_b(f(b))$, X_b does not contain θ, which is a contradiction. Hence if δ is an isolated number, then $\alpha\leqq\delta$ implies $P_\delta S_\varepsilon\not\leqq S_\alpha P_\beta S_\gamma$.

Next, let δ be a limit number, and assume $\alpha<\delta$, then as we have seen above, $P_{\alpha+1}S_\varepsilon\not\leqq S_\alpha P_\beta S_\gamma$. Since $\alpha+1<\delta$, we have $P_\delta S_\varepsilon\not\leqq S_\alpha P_\beta S_\gamma$ and the necessity of (c) is proved.

3. There remains yet a gap between the necessary conditions and the sufficient conditions for (8); we could not solve the problem of finding a necessary and sufficient condition for (8), which is essentially the same as that of determining the value $\rho_{\delta,\varepsilon}(\gamma)$ as stated below Lemma 17.

As A. Koźniewski and A. Lindenbaum noticed in [**2**], the value of $\pi_\alpha(\beta)$ is completely determined under the generalized continuum hypothesis as follows.

LEMMA 20. *Under the generalized continuum hypothesis,*

$$\begin{aligned}\pi_\alpha(\beta)&=\beta && \text{if either } \beta=\beta'+1 \text{ and } \alpha\leqq\mathrm{cf}(\beta'),\\ & && \text{or } \beta \text{ is a limit number and } \alpha\leqq\mathrm{cf}(\beta),\\ &=\beta+1 && \text{if } \beta=\beta'+1 \text{ and } \mathrm{cf}(\beta')<\alpha\leqq\beta,\end{aligned}$$

$= \beta+2$ *if* β *is a limit number and* $\mathrm{cf}(\beta)<\alpha\leqq\beta$,
$= \alpha$ *if* α *is a limit number and* $\beta<\alpha$,
$= \alpha+1$ *if* α *is an isolated number and* $\beta<\alpha$.

Hereafter, until we complete the proof of Theorem 6, we assume the generalized continuum hypothesis.

LEMMA 21. *If* $\gamma<\varepsilon$ *and* $\lambda<\delta$, *then* (8) *implies* $\pi_\delta(\lambda)\leqq\alpha$.

PROOF. If $\lambda<\delta$, then we have $\pi_\delta(\lambda)=\delta$ or $\pi_\delta(\lambda)=\delta+1$ according as δ is a limit number or δ is an isolated number respectively. But, by Lemma 19 (c), $\gamma<\varepsilon$ and (8) imply $\delta\leqq\alpha$ or $\delta<\alpha$ according as δ is a limit number or δ is an isolated number respectively. Hence in either case we have $\pi_\delta(\lambda)\leqq\alpha$.

THEOREM 6. *We have* (8) *if and only if one of the following conditions* (i), (ii) *and* (iii) *is satisfied:*

(i) $\delta\leqq\beta$ *and* $\varepsilon\leqq\gamma$,

(ii) $\delta\leqq\beta$, *and* $\pi_\delta(\varepsilon)\leqq\alpha$,

(iii) $\delta\leqq\beta$, $\varepsilon=\gamma+1$ *and* $\max(\delta^+, \mathrm{cf}(\gamma)+1)\leqq\alpha$,

where $\delta^+=\delta$ *if* δ *is a limit number and* $\delta^+=\delta+1$ *if* δ *is an isolated number.*

REMARK. By Lemma 20, it is easily seen that under the generalized continuum hypothesis, condition (iii) above is equivalent to condition (iii) in Lemma 18, and condition (iv) in Lemma 18 is a special case of this condition (iii).

PROOF OF THEOREM 6. The sufficiency of these conditions (i), (ii) and (iii) is already proved. The necessity of $\delta\leqq\beta$ is obvious. Hereafter we assume $\delta\leqq\beta$, $\gamma<\varepsilon$ and $\alpha<\pi_\delta(\varepsilon)$, and we shall show that inequality (8) implies (iii).

First, by Lemma 21, $\alpha<\pi_\delta(\varepsilon)$ implies $\delta\leqq\varepsilon$, and by Lemma 19, (8) and $\alpha<\pi_\delta(\varepsilon)$ imply that we have

$$\begin{aligned} &\text{either} && \pi_\delta(\varepsilon)=\pi_\delta(\gamma) \\ &\text{or} && \pi_\delta(\varepsilon)=\pi_\delta(\gamma)+1 \text{ and } \mathrm{cf}(\pi_\delta(\gamma))<\alpha. \end{aligned} \tag{9}$$

Especially the latter case of (9) occurs only when $\pi_\delta(\gamma)$ is a limit number, or otherwise we have $\pi_\delta(\gamma)=\mathrm{cf}(\pi_\delta(\gamma))<\alpha<\pi_\delta(\varepsilon)$, and $\pi_\delta(\varepsilon)=\pi_\delta(\gamma)+1$ is impossible.

Next we shall show $\delta\leqq\gamma$. Indeed if $\gamma<\delta$, then, by Lemma 21, we have $\pi_\delta(\gamma)\leqq\alpha<\pi_\delta(\varepsilon)$. Hence we are in the latter case of (9), and since $\pi_\delta(\gamma)$ is a limit number, we have $\pi_\delta(\gamma)=\delta=\alpha$ where $\mathrm{cf}(\delta)<\delta=\alpha$. Now since $\delta\leqq\varepsilon$, we have either $\delta=\varepsilon$, $\delta+1=\varepsilon$ or $\delta+2\leqq\varepsilon$, but in either case, by Lemma 20 and $\mathrm{cf}(\delta)<\delta$, we have $\pi_\delta(\varepsilon)\geqq\delta+2$ in contradiction to $\pi_\delta(\varepsilon)=\pi_\delta(\gamma)+1$.

Next we shall show $\varepsilon=\gamma+1$. Indeed if $\gamma+2\leqq\varepsilon$ then, by Lemma 19 (a), we have $\varepsilon\leqq\alpha$, and hence $\pi_\delta(\gamma)\leqq\gamma+2\leqq\varepsilon\leqq\alpha<\pi_\delta(\varepsilon)$. Hence we are in the latter case of (9) and $\pi_\delta(\gamma)=\gamma+2$. But then $\pi_\delta(\gamma)$ is not a limit number, which is a contradiction.

Now we have $\varepsilon=\gamma+1$. By Lemma 19 (c) and $\gamma<\varepsilon$, $\delta^+\leqq\alpha$ is necessary. If $\delta\leqq \mathrm{cf}(\gamma)$ then by Lemma 20, we have $\alpha<\pi_\delta(\varepsilon)=\varepsilon$ and hence, by Lemma 19 (a), we have $\mathrm{cf}(\gamma)<\alpha$. If $\mathrm{cf}(\gamma)<\delta$, then we have also $\mathrm{cf}(\gamma)<\delta\leqq\alpha$, and the whole proof of this theorem is completed.

4. In Theorem 6, we assumed the generalized continuum hypothesis, but in determining the condition for $P_\delta S_\varepsilon=S_\alpha P_\beta S_\gamma$, we need not assume this hypothesis.

THEOREM 7. *For the equality $P_\delta S_\varepsilon=S_\alpha P_\beta S_\gamma$ (or its dual $S_\delta P_\varepsilon=P_\alpha S_\beta P_\gamma$) to hold, it is necessary and sufficient that one of the following conditions* (i), (ii) *and* (iii) *is satisfied:*

(i) $\delta=\beta,\ \varepsilon=\gamma,\ \pi_\alpha(\beta)=\beta$ *and* $\alpha\leqq\mathrm{cf}(\gamma)$,

(ii) $\gamma<\alpha=\beta=\delta=\varepsilon=\pi_\delta(\alpha)$,

(iii) $\mathrm{cf}(\gamma)<\alpha=\beta=\delta=\pi_\alpha(\alpha)$ *and* $\varepsilon=\gamma+1$.

PROOF. Sufficiency follows immediately from Theorem A and Theorem B. $\delta=\beta$ is obviously necessary. Now by the comparison of S-components of both sides of this equality, we have either (A) $\varepsilon=\gamma$ and $\alpha\leqq\mathrm{cf}(\gamma)$, or (B) $\gamma<\alpha=\varepsilon$, or (C) $\varepsilon=\gamma+1$ and $\mathrm{cf}(\gamma)<\alpha\leqq\gamma$. But by Theorem 2, we have $\pi_\alpha(\beta)\leqq\delta$, and hence (i) follows from (A). By Lemma 19 (c), we have $\delta\leqq\alpha$. Hence we have $\alpha\leqq\pi_\alpha(\beta)\leqq\delta=\beta\leqq\alpha$, and hence (ii) or (iii) follows from (B) or (C) respectively.

§ 6. The value of $\pi_\alpha(\beta)$.

In this section we shall discuss about the value of the function $\pi_\alpha(\beta)$ and especially give the condition under which it is equal to β or $\beta+1$.

DEFINITION 10. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers. $\mathfrak{n}^{\overset{\mathfrak{m}}{\sqcup}}$ denotes the least cardinal number $\mathfrak{x}$ such that $\mathfrak{m}'<\mathfrak{m}$ implies $\mathfrak{n}^{\mathfrak{m}'}<\mathfrak{x}$.*

Let α and β be ordinal numbers. $\beta^{\overset{\alpha}{\sqcup}}$ denotes the ordinal number γ with $\aleph_\gamma=\aleph_\beta^{\overset{\aleph_\alpha}{\sqcup}}$.

The definition of $\mathfrak{n}^{\overset{\mathfrak{m}}{\sqcup}}$ is similar to the definition of the symbol $\mathfrak{n}^{\overset{\mathfrak{m}}{\smile}}=\sum_{\mathfrak{m}'<\mathfrak{m}}\mathfrak{n}^{\mathfrak{m}'}$ (cf. [**3**, Def. 4]). Indeed, $\mathfrak{n}^{\overset{\mathfrak{m}}{\smile}}$ is the least cardinal number $\mathfrak{x}$ such that $\mathfrak{m}'<\mathfrak{m}$ implies $\mathfrak{n}^{\mathfrak{m}'}\leqq\mathfrak{x}$. Hence $\mathfrak{n}^{\overset{\mathfrak{m}}{\sqcup}}$ is equal to the next succeeding cardinal number to $\mathfrak{n}^{\overset{\mathfrak{m}}{\smile}}$ or equal to $\mathfrak{n}^{\overset{\mathfrak{m}}{\smile}}$ itself according as the family $\{\mathfrak{n}^{\mathfrak{m}'};\ \mathfrak{m}'<\mathfrak{m}\}$ of cardinal numbers has a maximum number in it or not respectively. Especially if $\mathfrak{m}=\aleph_{\alpha+1}$, then $\mathfrak{n}^{\overset{\mathfrak{m}}{\sqcup}}$ is greater than $\mathfrak{n}^{\overset{\mathfrak{m}}{\smile}}$.

LEMMA 22. *$\pi_\alpha(\beta)$ is equal to either β or $\beta^{\overset{\alpha}{\sqcup}}$.*

PROOF. By the definitions of $\pi_\alpha(\beta)$ and $\beta^{\overset{\alpha}{\sqcup}}$, we have obviously $\beta\leqq\pi_\alpha(\beta)\leqq\beta^{\overset{\alpha}{\sqcup}}$. If $\beta<\pi_\alpha(\beta)$, then there is a set $\{\mathfrak{m}_a;\ a\in A\}$ of cardinal numbers with

$\overline{\overline{A}} < \aleph_\alpha$, $\mathfrak{m}_a < \aleph_\beta$ for any $a \in A$ and $\prod_{a\in A} \mathfrak{m}_a \geqq \aleph_\beta$. Let B be any set whose power $\mathfrak{n}$ is less than $\aleph_\alpha$, then the power of the set product $A\times B$ is also less than $\aleph_\alpha$. Put $\mathfrak{m}_{(a,b)} = \mathfrak{m}_a$ for any $(a, b) \in A \times B$, then we have $\aleph_{\pi_\alpha(\beta)} > \prod_{(a,b)\in A\times B} \mathfrak{m}_{(a,b)} = (\prod_{a\in A} \mathfrak{m}_a)^{\mathfrak{n}} \geqq \aleph_\beta{}^{\mathfrak{n}}$. Here $\mathfrak{n}$ is any cardinal number less than $\aleph_\alpha$, hence we have $\beta^{\overset{\alpha}{\sqcup}} \leqq \pi_\alpha(\beta)$ and therefore $\beta^{\overset{\alpha}{\sqcup}} = \pi_\alpha(\beta)$, completing the proof.

DEFINITION 11. *Let β be an ordinal number.* $\mathrm{p}(\beta)$ *denotes the least ordinal number γ with* $\aleph_\beta{}^{\aleph_\gamma} > \aleph_\beta$ (cf. [**3**, Def. 3]).

$\mathrm{q}(\beta)$ *denotes the least ordinal number γ such that there exists a cardinal number $\mathfrak{m}$ less than $\aleph_\beta$ with* $\mathfrak{m}^{\aleph_\gamma} \geqq \aleph_\beta$.

The symbol $\mathrm{q}(\beta)$ is closely related to $\mathrm{p}(\beta)$. For example, it is easily seen that $\mathrm{q}(\beta+1) = \mathrm{p}(\beta)$ and when β is an infinite limit number we have $\mathrm{q}(\beta) \geqq \overline{\lim_{\lambda\uparrow\beta}}\, \mathrm{p}(\gamma)$.

LEMMA 23. $\pi_\alpha(\beta) = \beta$ *if and only if* $\alpha \leqq \min(\mathrm{cf}(\beta), \mathrm{q}(\beta))$.

PROOF. If $\mathrm{cf}(\beta) < \alpha$ and β is a limit number, then we have $\beta+2 \leqq \pi_\alpha(\beta)$ by Lemma 16. If $\mathrm{cf}(\beta) < \alpha$ and β is an isolated number, then $\mathrm{cf}(\beta) = \beta$ and obviously we have $\beta < \alpha \leqq \pi_\alpha(\beta)$. If $\mathrm{q}(\beta) < \alpha$, then there exists a cardinal number $\mathfrak{m}$ less than $\aleph_\beta$ with $\mathfrak{m}^{\aleph_{\mathrm{q}(\beta)}} \geqq \aleph_\beta$ which implies $\pi_\alpha(\beta) > \beta$.

Now assume $\alpha \leqq \min(\mathrm{cf}(\beta), \mathrm{q}(\beta))$. Let $\{\mathfrak{m}_a ; a \in A\}$ be a family of cardinal numbers such that $\overline{\overline{A}} < \aleph_\alpha$ and $\mathfrak{m}_a < \aleph_\beta$ for any $a \in A$. Then $\alpha \leqq \mathrm{cf}(\beta)$ implies that the least upper bound $\mathfrak{m}$ of the cardinal numbers $\mathfrak{m}_a$ is less than $\aleph_\beta$. Furthermore $\mathrm{q}(\beta) \geqq \alpha$ implies $\mathfrak{m}^{\overline{\overline{A}}} < \aleph_\beta$. Hence we have $\prod_{a\in A} \mathfrak{m}_a \leqq \mathfrak{m}^{\overline{\overline{A}}} < \aleph_\beta$, and hence $\pi_\alpha(\beta) \leqq \beta$, completing the proof.

LEMMA 24. *If* $\mathrm{q}(\beta) < \alpha \leqq \mathrm{p}(\beta)$, *then we have* $\pi_\alpha(\beta) = \beta+1$.

PROOF. By Lemma 23, $\mathrm{q}(\beta) < \alpha$ implies $\beta < \pi_\alpha(\beta)$, but $\alpha \leqq \mathrm{p}(\beta)$ implies $\aleph_\beta^{\overset{\aleph_\alpha}{\sqcup}} = \aleph_\beta$ (see [**3**, Lemma 6]). Hence by Lemma 22 and the remark below Definition 10, we have $\pi_\alpha(\beta) = \beta^{\overset{\alpha}{\sqcup}} = \beta+1$.

LEMMA 25. *If* $\max(\mathrm{p}(\beta), \mathrm{q}(\beta)) < \alpha$, *then* $\beta+2 \leqq \pi_\alpha(\beta)$.

POOF. $\mathrm{q}(\beta) < \alpha$ implies $\pi_\alpha(\beta) = \beta^{\overset{\alpha}{\sqcup}} > \beta$. Furthermore, since $\alpha > \mathrm{p}(\beta)$, we have $\aleph_\beta^{\overset{\aleph_\alpha}{\sqcup}} > \aleph_\beta^{\aleph_{\mathrm{p}(\beta)}} \geqq \aleph_{\beta+1}$ which implies $\beta^{\overset{\alpha}{\sqcup}} > \beta+1$, completing the proof.

LEMMA 26. *If* $\mathrm{cf}(\beta) < \alpha$, *then* $\pi_\alpha(\beta) \geqq \beta+2$.

PROOF. We have already seen in Lemma 16 that this statement is valid when β is a limit number. If β is an isolated number and $\beta = \beta'+1$, then $\pi_\alpha(\beta) = \beta'^{\overset{\alpha}{\sqcup}}$. Moreover, by $\beta = \mathrm{cf}(\beta) < \alpha$, we have $\aleph_{\beta'}^{\overset{\aleph_\alpha}{\sqcup}} > \aleph_{\beta'}^{\aleph_\beta} = 2^{\aleph_\beta} \geqq \aleph_{\beta+1}$. Hence we have $\pi_\alpha(\beta) > \beta+1$.

It is easily seen that α and β satisfy one and only one of conditions which appear in Lemma 23, 24, 25 and 26. Hence we have

THEOREM 8. *$\pi_\alpha(\beta)$ takes the following values.*

$$
\begin{aligned}
\pi_\alpha(\beta) &= \beta, && \textit{if and only if } \alpha \leqq \min(\mathrm{cf}(\beta), \mathrm{q}(\beta)), \\
&= \beta^{\overset{\alpha}{\sqcup}} = \beta+1, && \textit{if and only if } \mathrm{q}(\beta) < \alpha \leqq \mathrm{p}(\beta), \\
&= \beta^{\overset{\alpha}{\sqcup}} \geqq \beta+2, && \textit{if and only if either } \alpha > \mathrm{cf}(\beta) \\
& && \textit{or } \alpha > \max(\mathrm{p}(\beta), \mathrm{q}(\beta)).
\end{aligned}
$$

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