Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

On Dedekind rings.

By Takeshi ISHIKAWA

(Received Aug. 7, 1958)

As an analogue of homological characterizations of Prüfer rings by functors Tor and $\otimes$, which was shown by A. Hattori,[1] we obtain the following theorem for Dedekind rings.

All notations and definitions in this note are the same as those in H. Cartan-S. Eilenberg.[2]

THEOREM. *For an integral domain Λ, the following conditions are equivalent*:

(a) *Λ is a Dedekind ring,*

(b) *Each divisible Λ-module is Λ-injective,*

(c) $\mathrm{Ext}_\Lambda^2(A, C)=0$ *for every pair of Λ-modules A and C,*

(d) $\mathrm{Ext}_\Lambda^1(X, C)=0$ *for every Λ-module X, if C is divisible,*

(e) $\mathrm{Hom}_\Lambda(A, C)$ *is divisible, if A is torsion-free and C is divisible.*

PROOF.

(a) $\Leftrightarrow$ (b): See H. A., VII, Prop. 5.1.

(b) $\Leftrightarrow$ (d): This is an immediate consequence of H. A., VI, 2.2 a.

(a) $\Leftrightarrow$ (c): Obvious from H. A., VI, 2.8.

(d) $\Rightarrow$ (e): Let A be torsion-free and C be devisible. For each $\lambda\in\Lambda$ consider the Λ-endomorphism $\lambda: A\rightarrow A$ given by $a\rightarrow\lambda a$. Then A is torsion-free if and only if $0\rightarrow A\xrightarrow{\lambda}A$ is exact for all $\lambda\neq 0$, and C is divisible if and only if $C\xrightarrow{\lambda}C\rightarrow 0$ is exact for all $\lambda\neq 0$. Then, from the exact sequence

$$0\rightarrow A\xrightarrow{\lambda}A\rightarrow A/\lambda A\rightarrow 0$$

we obtain the following exact sequence

$$\cdots\rightarrow\mathrm{Hom}_\Lambda(A, C)\xrightarrow{\mathrm{Hom}(\lambda,1)}\mathrm{Hom}_\Lambda(A, C)\rightarrow\mathrm{Ext}_\Lambda^1(A/\lambda A, C)\rightarrow\cdots.$$

By the assumption (d), $\mathrm{Ext}_\Lambda^1(A/\lambda A, C)=0$ and since the map $\mathrm{Hom}(\lambda, 1)$ coincides with $\lambda: \mathrm{Hom}_\Lambda(A, C)\rightarrow\mathrm{Hom}_\Lambda(A, C)$, this implies that $\mathrm{Hom}_\Lambda(A, C)$ is divisible.

(e) $\Rightarrow$ (b): Let C be devisible. We must show that C is injective under the assumption (e). Since Λ is an integral domain, each ideal I of Λ is torsion-free as a Λ-module. Thus $\mathrm{Hom}_\Lambda(I, C)$ is divisible and for each $\lambda\neq 0$, $\lambda\in\Lambda$ the exact sequence $C\xrightarrow{\lambda}C\rightarrow 0$ yields the exact sequence

1) A. Hattori, On Prüfer rings, J. Math. Soc. Japan, 9 (1957) 381-385.

2) H. Cartan, S. Eilenberg, Homological Algebra (cited as H. A.), 1956.

$$\mathrm{Hom}_\Lambda(I, C) \xrightarrow{\mathrm{Hom}(1,\lambda)} \mathrm{Hom}_\Lambda(I, C) \to 0 .$$

Thus, for each $f \in \mathrm{Hom}_\Lambda(I, C)$, there exists a homomorphism $g \in \mathrm{Hom}_\Lambda(I, C)$ such that $\mathrm{Hom}(1, \lambda)g = f$ i. e. $f\mu = \lambda g\mu$ for all $\mu \in I$. Since g is a Λ-homomorphism, choosing $\lambda \in I$,

$$f\mu = \lambda g\mu = g\lambda\mu = g\mu\lambda = \mu g\lambda, \quad g\lambda \in C .$$

Thus we find that for each ideal I of Λ and each Λ-homomorphism $f: I \to C$, there exists an element $c \in C$ such that $f\mu = \mu c$ for all $\mu \in I$. Thus C is injective from H. A., I, 3.2.

Tokyo Metropolitan University.