Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

Cauchy's theorem in Banach spaces.

By Yoshimichi MIBU

(Received June 27, 1958)
(Revised Aug. 6, 1958)

It is not too much to say that Cauchy's theorem acts a role as a starting point of the whole theory of complex-valued regular functions of a complex variable. And, as far as we know, this theorem was extended only to the following two cases. First, this was extended to the case of regular functions defined on the complex plane and having its values in a complex Banach space. Next, R. Lorch proved that Cauchy's theorem is valid for regular functions (see Definition 3 bellow) in a commutative Banach algebra. The purpose of the present paper is to extend Cauchy's theorem so as to include both of the above extentions (see Corollary 2).

DEFINITION 1. Let X and Y be two complex Banach spaces, and let a mapping $y=f(x)$ from X to Y be defined on an open set $D\subseteqq X$ and suppose that for every $x\in D$ and $h\in X$ the quotient $[f(x+\lambda h)-f(x)]/\lambda$, which is defined for sufficiently small $|\lambda|$, tends to a unique limit as $\lambda\rightarrow 0$. We then say that $f(x)$ is *G-differentiable* in D, and write $\delta f(x;h)=\lim_{\lambda\rightarrow 0}[f(x+\lambda h)-f(x)]/\lambda$. If further $\delta f(x;h)$ is a continuous function of h for any fixed $x\in D$, then $f(x)$ is said to be *F-differentiable* in D (see [**1**, pp. 72 and 73]).

REMARK 1. If $f(x)$ is G-differentiable in D, then it is proved that (i) $\delta f(x;\alpha h)=\alpha\delta f(x;h)$, where α is a complex number, (ii) $\delta f(x;h_1+h_2)=\delta f(x;h_1)+\delta f(x;h_2)$ (see [**1**, pp. 72 and 73]). Hence if $f(x)$ is F-differentiable, we see that $\delta f(x;\cdot)$ for fixed x in D is a bounded linear operator from X to Y.

ASSUMPTION (A). Let X, Y and Z be three complex Banach spaces. Suppose that to each pair (x,y), $x\in X$ and $y\in Y$, there corresponds an element z (denoted by $x\circ y$ or $y\circ x$) in Z and the following conditions are satisfied:

(1°) $(x_1+x_2)\circ y=x_1\circ y+x_2\circ y$, $x\circ(y_1+y_2)=x\circ y_1+x\circ y_2$.

(2°) $\lambda x\circ y=\lambda(x\circ y)=x\circ\lambda y$, where λ is a complex number.

(3°) $\|x\circ y\|\leqq\|x\|\cdot\|y\|$.

DEFINITION 2. Suppose that Assumption (A) is satisfied. Let $f(x)$ be a continuous function defined on a rectifiable curve Γ in X and having its values in Y. (The concept of rectifiable curve in a Banach space X can be defined quite similarly as in the usual case.) Then we can define the integral $\int_\Gamma f(x)\circ dx$ as follows:

Let Γ be parametrized by $x=x(t)$, $0 \leqq t \leqq 1$. Then it is easily seen that the sum $\sum_{i=1}^{n} f(x(t_i')) \circ [x(t_i)-x(t_{i-1})]$, where $0=t_0 < t_1 < \cdots < t_n = 1$ and $t_{i-1} \leqq t_i' \leqq t_i$, $i=1, 2, \cdots, n$, tends to a definite limit J as $\max_i (t_i - t_{i-1}) \to 0$. We denote this limit J as $\int_\Gamma f(x) \circ dx$.

Remark 2. It is easily seen that

$$\left\| \int_\Gamma f(x) \circ dx \right\| \leqq M l(\Gamma)\,, \tag{1}$$

where $M = \sup_{x \in \Gamma} \|f(x)\|$ and $l(\Gamma)$ is the length of the curve Γ.

Theorem (Cauchy). *Let X, Y and Z be three complex Banach spaces satisfying Assumption* (A). *Let $f(x)$ be a function defined on an open subset $D \subseteqq X$ and having its values in Y. Furthermore, we assume that*

(i) *D is a convex set,*

(ii) *$f(x)$ is F-differentiable in D,*

(iii) *For every $x \in D$ and every $h, k \in X$*

$$\delta f(x\,;h) \circ k = \delta f(x\,;k) \circ h\,.$$

Then we have

$$\int_\Gamma f(x) \circ dx = 0 \tag{2}$$

for any rectifiable closed curve Γ contained in D.

Proof. It is easily seen that for every $\varepsilon < 0$ there exists a polygon K in D such that $\left\| \int_\Gamma f(x) \circ dx - \int_K f(x) \circ dx \right\| < \varepsilon$. Hence, in order to prove (2), it is sufficient to show that for every polygon K contained in D the following equality holds:

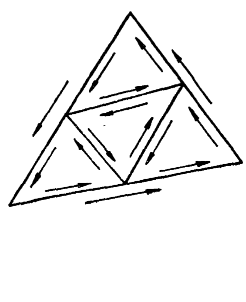
$$\int_K f(x) \circ dx = 0 \tag{3}$$

By using the condition (i) of the present theorem we see easily that for the proof of (3) it is sufficient to show that the equality

$$\int_\varDelta f(x) \circ dx = 0 \tag{4}$$

holds for every triangle $\varDelta$ contained in D.

Hence we shall prove (4). We can devide $\varDelta$ into four mutually congruent triangles $\varDelta_1, \varDelta_2, \varDelta_3$ and $\varDelta_4$ by joining mid points of all the sides of $\varDelta$. The orientation of each triangle is designated by the left diagram.



Setting $J = \int_\varDelta f(x) \circ dx$, $J_i = \int_{\varDelta_i} f(x) \circ dx$, $i=1, 2, 3, 4$, we have clearly $J=\sum_{i=1}^{4} J_i$; hence $\|J\| \leqq \sum_{i=1}^{4} \|J_i\|$. Consequently there exists at least one Δ' $(=\Delta_i)$ such that

$$\|J\| \leqq 4 \left\| \int_{\Delta'} f(x) \circ dx \right\|.$$

By the same reasoning we have a triangle Δ'' which is one-fourth of Δ', such that

$$\left\| \int_{\Delta'} f(x) \circ dx \right\| \leqq 4 \left\| \int_{\Delta''} f(x) \circ dx \right\|.$$

Continuing this process we obtain a sequence of triangles

$$\Delta, \Delta', \Delta'', \cdots, \Delta^{(n)}, \cdots \tag{5}$$

such that $\Delta^{(n+1)} \subseteqq \Delta^{(n)}$ $(n=1,2,\cdots)$ and

$$\|J\| \leqq 4^n \left\| \int_{\Delta^{(n)}} f(x) \circ dx \right\|, \quad n=1,2,\cdots. \tag{6}$$

Let L be the plane which contains the triangle Δ and $\overline{\Delta^{(n)}}$ the closed domain bounded by $\Delta^{(n)}$ in L. As is easily seen, $\bigcap_{n=1}^{\infty} \overline{\Delta^{(n)}}$ consists of a single point c in $L \cap D$. We set

$$f(c+h)=f(c)+\delta f(c\,; h)+\tau(h). \tag{7}$$

Then for every $\varepsilon > 0$ there exists a $\sigma > 0$ such that

$$\|h\| < \sigma \quad \text{implies} \quad \|\tau(h)\| \leqq \varepsilon \|h\|, \qquad \text{(see [1, p. 82])}. \tag{8}$$

We select a natural number N such that

$$l/2^N < \sigma, \tag{9}$$

where l is the length of the perimeter of Δ.

Since the length of the perimeter of $\Delta^{(N)}$ is clearly $l/2^N$, we have

$$\|x-c\| \leqq l/2^N < \sigma \quad \text{for every} \quad x \in \Delta^{(n)}. \tag{10}$$

We set $f(x)=f(c+(x-c))=f(c)+\delta f(c\,; x-c)+\tau(x-c)$. Then

$$\int_{\Delta^{(N)}} f(x) \circ dx = \int_{\Delta^{(N)}} f(c) \circ dx + \int_{\Delta^{(N)}} \delta f(c\,; x-c) \circ dx + \int_{\Delta^{(N)}} \tau(x-c) \circ dx. \tag{11}$$

Evidently $\int_{\Delta^{(N)}} f(c) \circ dx = 0$. And by using relations (1), (8) and (10) we have

$$\left\| \int_{\Delta^{(N)}} \tau(x-c) \circ dx \right\| \leqq \varepsilon \cdot l/2^N \cdot l/2^N = \varepsilon l^2/4^N.$$

Hence by relations (6) and (11) we easily see that

$$\|J\| \leqq 4^N \int_{\Delta^{(N)}} \delta f(c\,; x-c) \circ dx + \varepsilon l^2. \tag{12}$$

If the equality

$$\int_{\Delta^{(n)}} \delta f(c\,;x-c)\circ dx=0 \tag{13}$$

holds for every n, then we have clearly $J=0$. Hence for the proof of (4) it is sufficient to show (13).

Let

$$p_1=c+\lambda_1 h+\mu_1 k,\quad p_2=c+\lambda_2 h+\mu_2 k,\quad p_3=c+\lambda_3 h+\mu_3 k\,, \tag{14}$$
$$(\text{where } h,k\in X)$$

be the vertices of the triangle $\Delta^{(n)}$. And let the segment $\overrightarrow{p_1p_2}$ be parametrized by

$$\begin{aligned} x(t)&=c+\lambda_1 h+\mu_1 k+[(\lambda_2-\lambda_1)h+(\mu_2-\mu_1)k]t \\ &=c+[\lambda_2 t+\lambda_1(1-t)]h+[\mu_2 t+\mu_1(1-t)]k,\quad 0\leqq t\leqq 1\,. \end{aligned} \tag{15}$$

Hence by the condition (ii) of Remark 1 we have

$$\begin{aligned} \int_{\overrightarrow{p_1p_2}} \delta f(c\,;x-c)\circ dx&=\int_{\overrightarrow{p_1p_2}} \delta f(c\,;[\lambda_2 t+\lambda_1(1-t)]h+[\mu_2 t+\mu_1(1-t)]k)\circ dx \\ &=\int_{\overrightarrow{p_1p_2}} [\lambda_2 t+\lambda_1(1-t)]\delta f(c\,;h)\circ dx+\int_{\overrightarrow{p_1p_2}} [\mu_2 t+\mu_1(1-t)]\delta f(c\,;k)\circ dx\,. \end{aligned} \tag{16}$$

On the other hand, from (15) we have $dx=(\lambda_2-\lambda_1)h\,dt+(\mu_2-\mu_1)k\,dt$. Hence

$$\begin{aligned} &\int_{\overrightarrow{p_1p_2}} [\lambda_2 t+\lambda_1(1-t)]\delta f(c\,;h)\circ dx \\ &\quad=\int_0^1 [\lambda_2 t+\lambda_1(1-t)]\delta f(c\,;h)\circ[(\lambda_2-\lambda_1)h+(\mu_2-\mu_1)k]dt \\ &\quad=\int_0^1 (\lambda_2-\lambda_1)[\lambda_2 t+\lambda_1(1-t)][\delta f(c\,;h)\circ h]dt \\ &\qquad+\int_0^1 (\mu_2-\mu_1)[\lambda_2 t+\lambda_1(1-t)][\delta f(c\,;h)\circ k]dt \\ &\quad=(\lambda_2-\lambda_1)(\lambda_2+\lambda_1)/2\cdot[\delta f(c\,;h)\circ h]+(\mu_2-\mu_1)(\lambda_2+\lambda_1)/2\cdot[\delta f(c\,;h)\circ k] \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\int_{\overrightarrow{p_1p_2}} [\mu_2 t+\mu_1(1-t)]\delta f(c\,;k)\circ dx \\ &\quad=\int_0^1 [\mu_2 t+\mu_1(1-t)]\delta f(c\,;k)\circ[(\lambda_2-\lambda_1)h+(\mu_2-\mu_1)k]dt \\ &\quad=\int_0^1 (\lambda_2-\lambda_1)[\mu_2 t+\mu_1(1-t)][\delta f(c\,;k)\circ h]dt \\ &\qquad+\int_0^1 (\mu_2-\mu_1)[\mu_2 t+\mu_1(1-t)][\delta f(c\,;k)\circ k]dt \\ &\quad=(\lambda_2-\lambda_1)(\mu_2+\mu_1)/2\cdot[\delta f(c\,;k)\circ h]+(\mu_2-\mu_1)(\mu_2+\mu_1)/2\cdot[\delta f(c\,;k)\circ k]\,. \end{aligned} \tag{18}$$

Consequently, from (16), (17) and (18) we have

$$\int_{\overrightarrow{p_1p_2}} \delta f(c\,;x-c)\circ dx=(\lambda_2^2-\lambda_1^2)[\delta f(c\,;h)\circ h]/2+(\mu_2-\mu_1)(\lambda_2+\lambda_1)[\delta f(c\,;h)\circ k]/2$$
$$+(\mu_2+\mu_1)(\lambda_2-\lambda_1)[\delta f(c\,;k)\circ h]/2+(\mu_2^2-\mu_1^2)[\delta f(c\,;k)\circ k]/2\,. \tag{19}$$

Similarly

$$\int_{\overrightarrow{p_2p_3}} \delta f(c\,;x-c)\circ dx=(\lambda_3^2-\lambda_2^2)[\delta f(c\,;h)\circ h]/2+(\mu_3-\mu_2)(\lambda_3+\lambda_2)[\delta f(c\,;h)\circ k]/2$$
$$+(\mu_3+\mu_2)(\lambda_3-\lambda_2)[\delta f(c\,;k)\circ h]/2+(\mu_3^2-\mu_2^2)[\delta f(c\,;k)\circ k]/2\,, \tag{20}$$

and

$$\int_{\overrightarrow{p_3p_1}} \delta f(c\,;x-c)\circ dx=(\lambda_1^2-\lambda_3^2)[\delta f(c\,;h)\circ h]/2+(\mu_1-\mu_3)(\lambda_1+\lambda_3)[\delta f(c\,;h)\circ k]/2$$
$$+(\mu_1+\mu_3)(\lambda_1-\lambda_3)[\delta f(c\,;k)\circ h]/2+(\mu_1^2-\mu_3^2)[\delta f(c\,;k)\circ k]/2\,. \tag{21}$$

Hence

$$\begin{aligned}
\int_{\Delta^{(n)}} \delta f(c\,;x-c)\circ dx &= \int_{\overrightarrow{p_1p_2}}+\int_{\overrightarrow{p_2p_3}}+\int_{\overrightarrow{p_3p_1}}\\
&=[\delta f(c\,;h)\circ h](\lambda_2^2-\lambda_1^2+\lambda_3^2-\lambda_2^2+\lambda_1^2-\lambda_3^2)/2\\
&\quad+[\delta f(c\,;h)\circ k][(\mu_2-\mu_1)(\lambda_2+\lambda_1)+(\mu_3-\mu_2)(\lambda_3+\lambda_2)+(\mu_1-\mu_3)(\lambda_1+\lambda_3)]/2\\
&\quad+[\delta f(c\,;k)\circ h][(\mu_2+\mu_1)(\lambda_2-\lambda_1)+(\mu_3+\mu_2)(\lambda_3-\lambda_2)+(\mu_1+\mu_3)(\lambda_1-\lambda_3)]/2\\
&\quad+[\delta f(c\,;k)\circ k](\mu_2^2-\mu_1^2+\mu_3^2-\mu_2^2+\mu_1^2-\mu_3^2)/2\\
&=[\delta f(c\,;h)\circ k](\mu_2\lambda_1-\mu_1\lambda_2+\mu_3\lambda_2-\mu_2\lambda_3+\mu_1\lambda_3-\mu_3\lambda_1)/2\\
&\quad+[\delta f(c\,;k)\circ h](-\mu_2\lambda_1+\mu_1\lambda_2-\mu_3\lambda_2+\mu_2\lambda_3-\mu_1\lambda_3+\mu_3\lambda_1)/2\\
&=0. \qquad \text{(See the condition (iii) of the Theorem.)}
\end{aligned}$$

Hence our theorem is completely proved.

COROLLARY 1. *Suppose that Assumption* (A) *is satisfied. Let $f(x)$ be a function defined on an open set $D\subseteqq X$ and having its values in Y. Let Γ be a rectifiable closed curve contained in D. If the conditions* (ii) *and* (iii) *of the above theorem are satisfied, then we have*

$$\int_\Gamma f(x)\circ dx=\int_{\Gamma_1} f(x)\circ dx \tag{22}$$

for any rectifiable closed curve $\Gamma_1\subseteqq D$ which is homotopic to Γ in D. And hence if Γ is homotopic to a point in D, we have clearly

$$\int_\Gamma f(x)\circ dx=0\,. \tag{23}$$

PROOF. Let Γ and Γ_1 be parametrized by $x=x(t)$, $0\leqq t\leqq 1$, and $x=x_1(t)$, $0\leqq t\leqq 1$, respectively. Since Γ is homotopic to Γ_1 in D, there is a function of two real variables, $F(s,t)$, continuous on the unit square $\{0\leqq s\leqq 1,\ 0\leqq t\leqq 1\}$ with respect to both variables and such that $F(0,t)=x(t)$, $F(1,t)=x_1(t)$ and $F(s,0)=F(s,1)$, $0\leqq s\leqq 1$, and moreover $F(s,t)$ is in D for every (s,t) in the unit square. We take a natural number n and set

(24) $$P_{i,j}^{(n)}=F(i/n, j/n), \quad i=0,1,2,\cdots n, \quad j=0,1,2,\cdots n\,.$$

For every i $(0\leqq i\leqq n)$ let $K_i^{(n)}$ be the polygon consisting of the line segments joining the points $P_{i,0}^{(n)}, P_{i,1}^{(n)},\cdots, P_{i,n}^{(n)}$ in this order. And let $Q_{i,j}^{(n)}$ $(0\leqq i\leqq n-1,$ $0\leqq j\leqq n-1)$ be the polygon consisting of the line segments joining the points $P_{i,j}^{(n)}, P_{i,j+1}^{(n)}, P_{i+1,j+1}^{(n)}, P_{i+1,j}^{(n)}, P_{i,j}^{(n)}$ in this order. If n is sufficiently large, it is easily seen that there exists an open convex set $D_{i,j}^{(n)}\subseteqq D$ which contains $Q_{i,j}^{(n)}$. Hence by the preceding theorem we have

(25) $$\int_{Q_{i,j}^{(n)}} f(x)\circ dx=0, \quad j=0,1,\cdots, n-1, \quad i=0,1,\cdots, n-1\,.$$

Adding (25) for $j=0,1,\cdots, n-1$ we obtain

(26) $$\int_{K_i^{(n)}} f(x)\circ dx=\int_{K_{i+1}^{(n)}} f(x)\circ dx, \quad i=0,1,\cdots, n-1\,.$$

From (26) we have clearly

(27) $$\int_{K_0^{(n)}} f(x)\circ dx=\int_{K_n^{(n)}} f(x)\circ dx\,.$$

Letting $n\to\infty$, we obtain directly (22).

Definition 3 (Lorch). Let $\mathfrak{B}$ be a commutative Banach algebra with a unit element. A function $f(x)$ whose domain D (open) and range R are in $\mathfrak{B}$ is said to *have a derivative* $f'(x_0)$ *at* $x=x_0$ if for each $\varepsilon>0$ we can choose such a $\delta>0$ that for all h in $\mathfrak{B}$ with $\|h\|<\delta$ it holds

$$\|f(x_0+h)-f(x_0)-hf'(x_0)\|<\varepsilon\|h\|\,.$$

If $f(x)$ has a derivative everywhere in D, then it is regular in D.

Let $\mathfrak{B}$ be a commutative Banach algebra with a unit element. In Assumption (A) if we set $X=Y=Z=\mathfrak{B}$ and define $x\circ y=xy$ for $x\in X=\mathfrak{B}$ and $y\in Y=\mathfrak{B}$, the conditions (1°), (2°) and (3°) are obviously satisfied. Let $f(x)$ be a regular function (see Definition 3 above) defined on an open subset $D\subseteqq\mathfrak{B}$. Since $\delta f(c\,;h)=\lim_{\lambda\to 0}[f(c+\lambda h)-f(c)]/\lambda=\lim_{\lambda\to 0}[\lambda hf'(c)+\varepsilon]/\lambda=hf'(c), f(x)$ is F-differentiable in D and $\delta f(c\,;h)\circ k=hkf'(c)=\delta f(c\,;k)\circ h$. Thus $f(x)$ satisfies the conditions (ii) and (iii) of our theorem. Hence by Corollary 1 we have the following

Corollary 2 (Theorem of Lorch). *Let $\mathfrak{B}$ be a commutative Banach algebra with a unit element and $f(x)$ a regular function defined on an open subset $D\subseteqq\mathfrak{B}$. If a rectifiable closed curve $\Gamma\subseteqq D$ is homotopic to another rectifiable closed cvrve $\Gamma_1\subseteqq D$ in D, then we have*

(28) $$\int_\Gamma f(x)dx=\int_{\Gamma_1} f(x)dx\,.$$

In particular, if Γ is homotopic to a single point in D, then clearly

$$\int_\Gamma f(x)dx=0. \tag{29}$$

(*When D is a convex set, any closed curve contained in D is homotopic to a point in D and hence the above relation* (29) *holds for any rectifiable closed curve $\Gamma \subseteqq D$.*)

Let X and Z be two complex Banach spaces and $\mathfrak{E}(X,Z)$ the Banach space of all bounded linear operators from X to Z. If we set $X=X$, $Y=\mathfrak{E}(X,Z)$ and $Z=Z$ and define $x\circ T=T(x)$ for $x\in X$ and $T\in Y=\mathfrak{E}(X,Z)$, then the conditions (1°), (2°) and (3°) of the assumption (A) are obviously satisfied. Let $f(x)$ be a F-differentiable function defined on an open set $D\subseteqq X$ and having its values in Z. By Remark 1 $\delta f(x;h)$ can be regarded as a bounded linear operator from X to Z for any fixed $x\in D$. Thus, $\delta f(x;\cdot)$ is a mapping from D to $Y=\mathfrak{E}(X,Z)$. On the other hand, it is proved that $\delta f(x;\cdot)$ is a continuous mapping from D to $Y=\mathfrak{E}(X,Z)$ (see [**1**, p. 82]). Hence, by Definition 2 we can define the integral $\int_\Gamma \delta f(x;\cdot)\circ dx$ for any rectificable curve $\Gamma\subseteqq D$. We shall denote this integral by $\int_\Gamma \delta f(x;dx)$. It is not difficult to show that $\delta f(x;\cdot)$, considered as a mapping from $D\subseteqq X$ to $Y=\mathfrak{E}(X,Z)$, satisfies the conditions (ii) and (iii) of our theorem. Hence by our theorem we get immediately the following.

COROLLARY 3. *Let $f(x)$ be a F-differentiable function defined on an open convex subset D of a complex Banach space X and having its values in a complex Banach space Z. Then we have*

$$\int_\Gamma \delta f(x;dx)=0 \tag{30}$$

for any rectiable closed curve Γ contained in D.

References

[1] E. Hille, Functional analysis and semi-groups, New York, 1946.
[2] E. R. Lorch, The theory of analytic functions in normed abelian vector rings, Trans. Amer. Math. Soc., **54** (1943), 414-425.