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Harmonic and Killing tensor fields in Riemannian spaces with boundary.

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In a previous paper [5], we have generalized some of the results on harmonic and Killing vector fields stated in Yano and Bochner [6] to the case of Riemannian spaces with boundary.

The purpose of the present paper is to do exactly the same thing for harmonic and Killing tensor fields. The study of harmonic tensor fields in such Riemannian spaces has been already started by Duff and Spencer [3], Conner [2] and Nakae [4].

§1. Fundamental formulas.

We consider a compact manifold M which is the closure of an open submanifold of an n-dimensional orientable Riemannian manifold V_n of class C^r $(r\geqq 2)$ with a positive definite metric $ds^2=g_{\mu\lambda}(\xi)d\xi^\mu d\xi^\lambda$ $(\kappa, \lambda, \cdots=1, 2, \cdots, n)$ and which is represented, in a neighborhood of each point on the boundary B of class C^r by $\xi^n\geqq 0$. It follows that B is an $(n-1)$-dimensional compact orientable submanifold (see, Chern [1]). The boundary B is represented locally also by

$$\xi^\kappa=\xi^\kappa(\eta^h) \qquad (h, i, j, \cdots=1, 2, \cdots, n-1) \tag{1}$$

in $U(P)\cap M$, $U(P)$ being a coordinate neighborhood in V_n of a point P on B.

We put $B_i^{\cdot\kappa}=\partial_i\xi^\kappa=\partial\xi^\kappa/\partial\eta^i$, $'g_{ji}=B_j^{\cdot\mu}B_i^{\cdot\lambda}g_{\mu\lambda}$ and denote by N^κ the unit normal to B such that N^κ and $B_1^{\cdot\kappa}, B_2^{\cdot\kappa}, \cdots, B_{n-1}^{\cdot\kappa}$ form the positive sense of M and by $\mathfrak{g}$ and $'\mathfrak{g}$ the determinants formed by $g_{\mu\lambda}$ and $'g_{ji}$ repectively.

Denoting by $'\nabla_i$ the covariant differentiation of van der Waerden-Bortolloti with respect to $'g_{ji}$ along B, equations of Gauss and those of Weingarten can be written respectively in the form

$$'\nabla_j B_i^{\cdot\kappa}=H_{ji}N^\kappa, \tag{2}$$

$$'\nabla_j N^\kappa=-H_j^{\cdot i}B_i^{\cdot\kappa}, \tag{3}$$

where H_{ji} is the second fundamental tensor of B with respect to the normal N^κ.

Now, Stokes' theorem states: We have, for an arbitrary vector field u^κ in M,

(4) $$\int_M \nabla_\lambda u^\lambda d\sigma = \int_B u^\lambda N_\lambda d'\sigma\,,$$

where ∇_λ denotes covariant differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$ formed by $g_{\mu\lambda}$ and

$$d\sigma = \sqrt{\mathfrak{g}}\, d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n\,, \qquad d'\sigma = \sqrt{'\mathfrak{g}}\, d\eta^1 \wedge d\eta^2 \wedge \cdots \wedge d\eta^{n-1}\,.$$

Consider a skew-symmetric tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ $(1 \leqq p \leqq n)$ and form

$$\begin{aligned}
&\nabla_\mu [v^\lambda{}_{\lambda_2\cdots\lambda_p}(\nabla_\lambda v^{\mu\lambda_2\cdots\lambda_p}) - v^\mu{}_{\lambda_2\cdots\lambda_p}(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})] \\
&\quad = K_{\mu\lambda} v^\mu{}_{\lambda_2\cdots\lambda_p} v^{\lambda\lambda_2\cdots\lambda_p} + \frac{p-1}{2} K_{\nu\mu\lambda\kappa} v^{\nu\mu}{}_{\lambda_3\cdots\lambda_p} v^{\lambda\kappa\lambda_3\cdots\lambda_p} \\
&\qquad + (\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) - (\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p}) \\
&\quad = F(v, v) + (\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) - (\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})\,,
\end{aligned}$$

where $K_{\mu\lambda}$ is the Ricci tensor, $K_{\nu\mu\lambda\kappa}$ the curvature tensor and

$$F(v, v) = K_{\mu\lambda} v^\mu{}_{\lambda_2\cdots\lambda_p} v^{\lambda\lambda_2\cdots\lambda_p} + \frac{p-1}{2} K_{\nu\mu\lambda\kappa} v^{\nu\mu}{}_{\lambda_3\cdots\lambda_p} v^{\lambda\kappa\lambda_3\cdots\lambda_p}\,.$$

Thus applying Stokes' theorem, we find

(6) $$\begin{aligned}
&\int_M [F(v, v) + (\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) - (\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})] d\sigma \\
&\quad = \int_B [v^\lambda{}_{\lambda_2\cdots\lambda_p}(\nabla_\lambda v^{\mu\lambda_2\cdots\lambda_p}) - v^\mu{}_{\lambda_2\cdots\lambda_p}(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})] N_\mu d'\sigma\,.
\end{aligned}$$

§ 2. Non-existence of harmonic or Killing tensors.

We now assume that the skew-symmetric tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ is tangential to B, that is,

(T) $$v^{\lambda_1\lambda_2\cdots\lambda_p} = B^{\lambda_1\lambda_2\cdots\lambda_p}_{i_1 i_2\cdots i_p}\, 'v^{i_1 i_2\cdots i_p} \quad \text{or} \quad v^{\lambda\lambda_2\cdots\lambda_p} N_\lambda = 0 \quad \text{on } B\,,$$

where

$$B^{\lambda_1\lambda_2\cdots\lambda_p}_{i_1 i_2\cdots i_p} = B_{i_1}^{\cdot\lambda_1} B_{i_2}^{\cdot\lambda_2} \cdots B_{i_p}^{\cdot\lambda_p}\,,$$

and $'v^{i_1 i_2\cdots i_p}$ is a skew-symmetric tensor field defined over B.

Differentiating $v_{\lambda\lambda_2\cdots\lambda_p} N^\lambda = 0$ along B, we obtain

$$B_j^{\cdot\mu}(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}) N^\lambda - v_{\lambda\lambda_2\cdots\lambda_p} H_j^{\cdot i} B_i^{\cdot\lambda} = 0$$

by virtue of the equations of Weingarten. Multiplying this by $B^{\lambda_2\lambda_3\cdots\lambda_p}_{i_2 i_3\cdots i_p}\, 'v^{j i_2\cdots i_p}$, we get

$(7)_T$ $$(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}) v^{\mu\lambda_2\cdots\lambda_p} N^\lambda = H_{ji}\, 'v^j{}_{i_2\cdots i_p}\, 'v^{i i_2\cdots i_p}\,,$$

where the subscript T (N) indicates that the formula or theorem concerns a vector or a tensor field tangential (normal) to B.

From (6), (T) and $(7)_T$, we find

$$(8)_T \qquad \int_M [F(v,v)+(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})-(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B H_{ji}\,'v^j{}_{i_2\cdots i_p}\,'v^{ii_2\cdots i_p} d'\sigma$$

for a tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ tangential to B.

This formula can be written in the following two forms:

$$(9)_T \qquad \int_M [F(v,v)+\frac{1}{p}(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}) - \frac{p+1}{p}(\nabla^{[\mu} v^{\lambda\lambda_2\cdots\lambda_p]})(\nabla_{[\mu} v_{\lambda\lambda_2\cdots\lambda_p]})-(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B H_{ji}\,'v^j{}_{i_2\cdots i_p}\,'v^{ii_2\cdots i_p} d'\sigma ,$$

$$(10)_T \qquad \int_M [F(v,v)-(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}) + \frac{1}{2}(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p}+\nabla^\lambda v^{\mu\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) -(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B H_{ji}\,'v^j{}_{i_2\cdots i_p}\,'v^{ii_2\cdots i_p} d'\sigma ,$$

where

$$(p+1)\nabla^{[\mu} v^{\lambda\lambda_2\cdots\lambda_p]} = \nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p} - \nabla^\lambda v^{\mu\lambda_2\cdots\lambda_p} - \nabla^{\lambda_2} v^{\lambda\mu\lambda_3\cdots\lambda_p} - \cdots - \nabla^{\lambda_p} v^{\lambda\lambda_2\cdots\lambda_{p-1}\mu} .$$

Harmonic and Killing tensor fields being respectively defined by

$$\nabla_{[\mu} v_{\lambda\lambda_2\cdots\lambda_p]}=0 , \qquad \nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p}=0$$

and

$$\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}=0 ,$$

we have, from $(9)_T$ and $(10)_T$,

THEOREM 1_T: *If, in an M with boundary B, the quadratic form $F(v,v)$ is positive (negative) definite and the second fundamental form is negative (positive) semi-definite, then there does not exist a harmonic (Killing) tensor field of order p $(1 \leqq p \leqq n-1)$ tangential to B other than the zero tensor.*

We next assume that the skew-symmetric tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ is normal to B, that is, the tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ satisfies

$$(N) \qquad v_{\lambda_1\lambda_2\cdots\lambda_p} B^{\lambda_1\lambda_2\cdots\lambda_p}_{i_1 i_2\cdots i_p}=0 .$$

Multiplying this by $B^{i_1 i_2\cdots i_p}_{\mu_1\mu_2\cdots\mu_p}$ and taking account of

$$B_i{}^{\cdot\lambda} B^i{}_{\cdot\mu} = A^\lambda_\mu - N^\lambda N_\mu ,$$

we find

$$(11)_N \qquad v_{\lambda_1\lambda_2\cdots\lambda_p}=N_{\lambda_1}(v_{\lambda\lambda_2\cdots\lambda_p}N^\lambda)+N_{\lambda_2}(v_{\lambda_1\lambda\lambda_3\cdots\lambda_p}N^\lambda)+\cdots+N_{\lambda_p}(v_{\lambda_1\lambda_2\cdots\lambda_{p-1}\lambda}N^\lambda) ,$$

where

$$B^{i_1 i_2\cdots i_p}_{\lambda_1\lambda_2\cdots\lambda_p} = B^{i_1}_{\cdot\lambda_1} B^{i_2}_{\cdot\lambda_2} \cdots B^{i_p}{}_{\lambda_p} \quad \text{and} \quad B^i{}_{\cdot\mu} = {}'g^{ih} g_{\mu\lambda} B_h{}^{\cdot\lambda} .$$

If we put

$(12)_N$ $$v_{\lambda\lambda_2\cdots\lambda_p}N^{\lambda}B_{i_2\cdots i_p}^{\lambda_2\cdots\lambda_p}='v_{i_2\cdots i_p}\,,$$

$'v_{i_2\cdots i_p}$ being skew-symmetric, we find

$(13)_N$ $$v_{\lambda\lambda_2\cdots\lambda_p}N^{\lambda}='v_{i_2\cdots i_p}B_{\lambda_2\cdots\lambda_p}^{i_2\cdots i_p}\,.$$

Differentiating (N) along B, we find

$$(\nabla_{\lambda}v_{\lambda_1\lambda_2\cdots\lambda_p})B_{ii_1\cdots i_p}^{\lambda\lambda_1\cdots\lambda_p}+v_{\lambda_1\lambda_2\cdots\lambda_p}H_{ii_1}N^{\lambda_1}B_{i_2\cdots i_p}^{\lambda_2\cdots\lambda_p}+\cdots+v_{\lambda_1\lambda_2\cdots\lambda_p}B_{i_1\cdots i_{p-1}}^{\lambda_1\cdots\lambda_{p-1}}H_{ii_p}N^{\lambda_p}=0$$

or

$$(\nabla_{\lambda}v_{\lambda_1\lambda_2\cdots\lambda_p})B_{ii_1\cdots i_p}^{\lambda\lambda_1\cdots\lambda_p}+H_{ii_1}'v_{i_2\cdots i_p}+\cdots+(-1)^{p-1}H_{ii_p}'v_{i_1\cdots i_{p-1}}=0\,.$$

Multiplying this by $'g^{ii_1}$ and taking account of

$$'g^{ii_1}B_{ii_1}^{\lambda\lambda_1}=g^{\lambda\lambda_1}-N^{\lambda}N^{\lambda_1}\,,$$

we find

$$(\nabla_{\lambda}v_{\lambda_1\lambda_2\cdots\lambda_p})(g^{\lambda\lambda_1}-N^{\lambda}N^{\lambda_1})B_{i_2\cdots i_p}^{\lambda_2\cdots\lambda_p}+H_{a}^{\cdot a}\,'v_{i_2\cdots i_p}-H_{i_2}^{\cdot a}\,'v_{ai_3\cdots i_p}-\cdots-H_{i_p}^{\cdot a}\,'v_{i_2\cdots a}=0\,.$$

Multiplying again this by $'v^{i_2\cdots i_p}$ and taking account of

$$B_{i_2\cdots i_p}^{\lambda_2\cdots\lambda_p}\,'v^{i_2\cdots i_p}=v^{\mu\lambda_2\cdots\lambda_p}N_{\mu}\,,$$

we obtain

$(14)_N$ $$(\nabla_{\lambda}v^{\lambda}{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_{\mu}-N^{\lambda}N^{\lambda_1}v^{\mu\lambda_2\cdots\lambda_p}N_{\mu}(\nabla_{\lambda}v_{\lambda_1\lambda_2\cdots\lambda_p})$$
$$=-H_{a}^{\cdot a}\,'v_{i_2\cdots i_p}\,'v^{i_2\cdots i_p}+(p-1)H_{ji}\,'v^{j}{}_{i_3\cdots i_p}\,'v^{ii_3\cdots i_p}\,.$$

Substituting

$$N^{\lambda}v^{\mu\lambda_2\cdots\lambda_p}N_{\mu}=v^{\lambda\lambda_2\cdots\lambda_p}-N^{\lambda_2}(v^{\lambda\mu\lambda_3\cdots\lambda_p}N_{\mu})-\cdots-N^{\lambda_p}(v^{\lambda\lambda_2\cdots\lambda_{p-1}\mu}N_{\mu})$$

obtained from $(11)_N$, we find

$(15)_N$ $$[v^{\lambda}{}_{\lambda_2\cdots\lambda_p}(\nabla_{\lambda}v^{\mu\lambda_2\cdots\lambda_p})-v^{\mu}{}_{\lambda_2\cdots\lambda_p}(\nabla_{\lambda}v^{\lambda\lambda_2\cdots\lambda_p})]N_{\mu}$$
$$=H_{a}^{\cdot a}\,'v_{i_2\cdots i_p}\,'v^{i_2\cdots i_p}-(p-1)H_{ji}\,'v^{j}{}_{i_3\cdots i_p}\,'v^{ii_3\cdots i_p}\,.$$

Thus, from (6), we have

$(16)_N$ $$\int_M[F(v,v)+(\nabla^{\mu}v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_{\lambda}v_{\mu\lambda_2\cdots\lambda_p})-(\nabla_{\mu}v^{\mu}{}_{\lambda_2\cdots\lambda_p})(\nabla_{\lambda}v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma$$
$$=\int_B[H_{a}^{\cdot a}\,'v_{i_2\cdots i_p}\,'v^{i_2\cdots i_p}-(p-1)H_{ji}\,'v^{j}{}_{i_3\cdots i_p}\,'v^{ii_3\cdots i_p}]d'\sigma$$

for a skew-symmetric tensor field $v^{\lambda_1\lambda_2\cdots\lambda_p}$ normal to B.

This formula can be written in the following two forms:

$(17)_N$ $$\int_M[F(v,v)+\frac{1}{p}(\nabla^{\mu}v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_{\mu}v_{\lambda\lambda_2\cdots\lambda_p})$$
$$-\frac{p+1}{p}(\nabla^{[\mu}v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]})-(\nabla_{\mu}v^{\mu}{}_{\lambda_2\cdots\lambda_p})(\nabla_{\lambda}v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma$$
$$=\int_B[H_{a}^{\cdot a}\,'v_{i_2\cdots i_p}\,'v^{i_2\cdots i_p}-(p-1)H_{ji}\,'v^{j}{}_{i_3\cdots i_p}\,'v^{ii_3\cdots i_p}]d'\sigma\,,$$

$$(18)_N \qquad \int_M [F(v,v)-(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}) + \frac{1}{2}(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p}+\nabla^\lambda v^{\mu\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) - (\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B [H_a^{\cdot a}{}'v_{i_2\cdots i_p}{}'v^{i_2\cdots i_p}-(p-1)H_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}]d'\sigma .$$

From $(17)_N$ and $(18)_N$, we have

THEOREM 2_N: *If, in an M with boundary B, the quadratic form $F(v,v)$ is positive (negative) definite and the quadratic form*

$$(19) \qquad H(v,v)=H_a^{\cdot a}{}'v_{i_2\cdots i_p}{}'v^{i_2\cdots i_p}-(p-1)H_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}$$

is negative (positive) semi-definite, then there does not exist a harmonic (Killing) tensor field of order p $(1\leqq p\leqq n-1)$ normal to B other than the zero tensor.

§ 3. Necessary and sufficient conditions for a tensor field to be harmonic or Killing tensor.

We apply Stokes' theorem to $\nabla_\lambda(v^{\kappa_1\kappa_2\cdots\kappa_p}v_{\kappa_1\kappa_2\cdots\kappa_p})$ and obtain

$$(20) \qquad \int_M [(g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(\nabla^\mu v^{\lambda\kappa_2\cdots\kappa_p})(\nabla_\mu v_{\lambda\kappa_2\cdots\kappa_p})]d\sigma = \int_B (\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda d'\sigma .$$

We now assume (T). Then from $(7)_T$, we obtain

$$(21)_T \qquad (\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda = pH_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p} -(p+1)(\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda$$

and

$$(22)_T \qquad (\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda = -H_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p} +(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda .$$

Thus from (20), $(21)_T$ and $(22)_T$, we obtain

$$(23)_T \qquad \int_M [(g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B [pH_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}-(p+1)(\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda]d'\sigma ,$$

$$(24)_T \qquad \int_M [(g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B [H_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}+(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda]d'\sigma .$$

Forming $(23)_T-(9)_T\times p$, we find

$$(25)_T \qquad \int_M [(\Delta v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(p+1)(\nabla^{[\mu}v^{\lambda\lambda_2\cdots\lambda_p]})(\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]}) + p(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = -(p+1)\int_B (\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda d'\sigma\,,$$

where

$$(26)_T \qquad \Delta v^{\kappa_1\kappa_2\cdots\kappa_p}=g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p}-\textstyle\sum_{i=1}^p K_\lambda{}^{\kappa_i}v^{\kappa_1\cdots\lambda\cdots\kappa_p} - \textstyle\sum_{j<i}^{1\cdots p} K^{\kappa_j\kappa_i}{}_{\mu\lambda}v^{\kappa_1\kappa_2\cdots\mu\cdots\lambda\cdots\kappa_p}\,.$$

Forming next $(24)_T+(10)_T$, we obtain

$$(27)_T \qquad \int_M [(\Box v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\cdots\kappa_p}+\frac{1}{2}(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p}+\nabla^\lambda v^{\mu\lambda_2\cdots\lambda_p}) (\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})-(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B (\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda d'\sigma\,,$$

where

$$(28)_T \qquad \Box v^{\kappa_1\kappa_2\cdots\kappa_p}=g^{\mu\lambda}\Delta_\mu\Delta_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p}+\frac{1}{p}\textstyle\sum_{i=1}^p K_\lambda{}^{\kappa_i}v^{\kappa_1\cdots\lambda\cdots\kappa_p} + \frac{1}{p}\textstyle\sum_{j<i}^{1\cdots p} K^{\kappa_j\kappa_i}{}_{\mu\lambda}v^{\kappa_1\cdots\mu\cdots\lambda\cdots\kappa_p}\,.$$

Since, for a harmonic tensor field $v^{\kappa_1\kappa_2\cdots\kappa_p}$, we can easily derive $\Delta v^{\kappa_1\kappa_2\cdots\kappa_p}=0$ from $\nabla_{[\mu}v_{\lambda_1\cdots\lambda_p]}=0$ and $\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p}=0$, we have, from $(25)_T$,

THEOREM 3_T: *A necessary and sufficient condition for a skew-symmetric tensor field* $v^{\kappa_1\kappa_2\cdots\kappa_p}$ $(1\leqq p\leqq n-1)$ *tangential to B to be harmonic is that*

$$(29)_T \qquad \begin{cases} \Delta v^{\kappa_1\kappa_2\cdots\kappa_p}=0 & \textit{in } M, \\ \nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]}v^{\mu\lambda_2\cdots\lambda_p}N^\lambda=0 & \textit{on } B. \end{cases}$$

Since, for a Killing tensor field $v^{\kappa_1\kappa_2\cdots\kappa_p}$, we can easily derive $\Box v^{\kappa_1\kappa_2\cdots\kappa_p}=0$ and $\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p}=0$ from $\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}=0$, we have, from $(27)_T$,

THEOREM 4_T: *A necessary and sufficient condition for a skew-symmetric tensor field* $v^{\kappa_1\kappa_2\cdots\kappa_p}$ $(1\leqq p\leqq n-1)$ *to be a Killing tensor field is that*

$$(30)_T \qquad \begin{cases} \Box v^{\kappa_1\kappa_2\cdots\kappa_p}=0, \quad \nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p}=0 & \textit{in } M, \\ (\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda=0 & \textit{on } B. \end{cases}$$

We next assume (*N*). Then from $(11)_N$ and $(14)_N$, we find

$$(31)_N \qquad (\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N^\lambda=p(\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_\mu + pH_a^{\cdot a}{}'v_{i_2\cdots i_p}{}'v^{i_2\cdots i_p}-p(p-1)H_{ji}{}'v^j{}_{i_3\cdots p}{}'v^{ii_3\cdots i_p}\,.$$

Substituting this into (20), we find

$$(32)_N \qquad \int_M [(g^{\mu\lambda}\nabla_\mu\nabla_\lambda v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p})]d\sigma$$

$$=\int_B [p(\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_\mu + pH_a^{\cdot a}{}' v_{i_2\cdots i_p}{}' v^{i_2\cdots i_p} - p(p-1)H_{ji}{}' v^j{}_{i_3\cdots i_p}{}' v^{ii_3\cdots i_p}]d'\sigma .$$

Forming $(32)_N-(17)_N\times p$, we find

$$\int_M [(\Delta v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+(p+1)(\nabla^{[\mu}v^{\lambda\lambda_2\cdots\lambda_p]})(\nabla_{[\mu}v_{\lambda\lambda_2\cdots\lambda_p]}) + p(\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma = \int_B p(\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_\mu d'\sigma . \tag{33$_N$}$$

Forming next $(32)_N+(18)_N$, we find

$$\int_M [(\Box v^{\kappa_1\kappa_2\cdots\kappa_p})v_{\kappa_1\kappa_2\cdots\kappa_p}+\frac{1}{2}(\nabla^\mu v^{\lambda\lambda_2\cdots\lambda_p}+\nabla^\lambda v^{\mu\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}) - (\nabla_\mu v^\mu{}_{\lambda_2\cdots\lambda_p})(\nabla_\lambda v^{\lambda\lambda_2\cdots\lambda_p})]d\sigma$$
$$=\int_B [p(\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_\mu + (p+1)H_a^{\cdot a}{}' v_{i_2\cdots i_p}{}' v^{i_2\cdots i_p} - (p^2-1)H_{ji}{}' v^j{}_{i_3\cdots i_p}{}' v^{ii_3\cdots i_p}]d'\sigma \tag{34$_N$}$$

From $(33)_N$, we have

THEOREM 5_N: *A necessary and sufficient condition for a skew-symmetric tensor field* $v^{\kappa_1\kappa_2\cdots\kappa_p}$ $(1\leqq p\leqq n-1)$ *normal to B to be harmonic is that*

$$\begin{cases} \Delta v^{\kappa_1\kappa_2\cdots\kappa_p}=0 & \textit{in } M, \\ (\nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p})v^{\mu\lambda_2\cdots\lambda_p}N_\mu=0 & \textit{on } B. \end{cases} \tag{35$_N$}$$

Suppose that $v^{\kappa_1\kappa_2\cdots\kappa_p}$ is a Killing tensor field normal to B:

$$\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p}=0 .$$

On taking account of $(11)_N$, from (N) and

$$B^{\mu\lambda\lambda_2\cdots\lambda_p}_{j\,i\,i_2\cdots i_p}(\nabla_\mu v_{\lambda\lambda_2\cdots\lambda_p}+\nabla_\lambda v_{\mu\lambda_2\cdots\lambda_p})=0 ,$$

we find

$$-H_{ji}{}' v_{i_2\cdots i_p}+H_{ji_2}{}' v_{ii_3\cdots i_p}+\cdots+H_{ji_p}{}' v_{i_2\cdots i_{p-1}i}$$
$$-H_{ji}{}' v_{i_2\cdots i_p}+H_{ii_2}{}' v_{ji_3\cdots i_p}+\cdots+H_{ii_p}{}' v_{i_2\cdots i_{p-1}j}=0 ,$$

from which, contracting by ${}'g^{ji}{}' v^{i_2\cdots i_p}$,

$$H_a^{\cdot a}{}' v_{i_2\cdots i_p}{}' v^{i_2\cdots i_p}-(p-1)H_{ji}{}' v^j{}_{i_3\cdots i_p}{}' v^{ii_3\cdots i_p}=0 . \tag{36$_N$}$$

Thus from $(34)_N$, we have

THEOREM 6_N: *A necessary and sufficient condition for a skew-symmetric tensor field* $v^{\kappa_1\kappa_2\cdots\kappa_p}$ $(1\leqq p\leqq n-1)$ *normal to B to be a Killing tensor field is that*

$$\begin{cases} \Box v^{\kappa_1\cdots\kappa_p}=0 , \quad \nabla_\lambda v^\lambda{}_{\lambda_2\cdots\lambda_p}=0 & \textit{in } M, \\ H_a^{\cdot a}{}' v_{i_2\cdots i_p}{}' v^{i_2\cdots i_p}-(p-1)H_{ji}{}' v^j{}_{i_3\cdots i_p}{}' v^{ii_3\cdots i_p}=0 & \textit{on } B. \end{cases} \tag{37$_N$}$$

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