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Hyperbolic transfinite diameter and some theorems on analytic functions in an annulus.

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1. Preliminaries.

Let a, b be two points in $|z|<1$, then the hyperbolic metric $[a, b]$ is defined by

(1) $$[a, b]=\left|\frac{a-b}{1-\bar{a}b}\right|.$$

Let E be a bounded closed set, contained entirely in $|z|<1$, such that E and $|z|=1$ bound a connected domain D_0.

By introducing the hyperbolic metric (1) in $|z|<1$, Tsuji ([15], [16]) defined a potential of positive mass distribution on E and a hyperbolic transfinite diameter of E, and obtained some results analogous to those of Frostman [2] and de la Vallée-Poussin [17] in the theory of logarithmic potential and also to those of Pólya and Szegö [12] in the theory of transfinite diameter.

We summarize the results obtained by Tsuji as follows:

(i) Let $d\nu(a)\geqq 0$ be a positive mass distributed on E of total mass 1 and consider

(2) $$I(\nu)=\iint_E \log\frac{1}{[a, b]}\,d\nu(a)d\nu(b)\,,\qquad \nu(E)=1\,,$$

(3) $$V=\inf_{\nu} I(\nu)\,,\quad \infty\geqq V>0\,.$$

Then there exists $\mu\geqq 0$, such that

(4) $$I(\mu)=\iint_E \log\frac{1}{[a, b]}\,d\mu(a)d\mu(b)=V\,,\quad \mu(E)=1\,.$$

(ii) For the potential of the mass distribution $d\mu(a)$ on E

(5) $$u(z)=\int_E \log\frac{1}{[z, a]}\,d\mu(a)=\int_E \log\left|\frac{1-\bar{a}z}{z-a}\right|d\mu(a)\,,$$

we have, similarly to the result of Frostman,

(6) $$\sup_{|z|<1}\{u(z)\}=V$$

and

(6′) $$u(z)=V \qquad \text{on } E\,,$$

except on a set of capacity zero.

(iii) Define the hyperbolic transfinite diameter, similarly to the definition by Fekete [1], as follows:

Let

$$d_n = d_n(E) = \sqrt[\binom{n}{2}]{\underset{p_i \in E, p_j \in E}{\mathrm{Max.}} \prod_{i<j}^{n} [p_i, p_j]} \,, \tag{7}$$

where $p_1, p_2, \cdots, p_n$ are any n points in E, then the sequence $\{d_n\}$ is monotone decreasing and there exists

$$\lim_{n\to\infty} d_n(E) = \tau(E) \,. \tag{8}$$

Hereafter we shall call $\tau(E)$ the *hyperbolic transfinite diameter* of E.

In the special case where the set E is a continuum, there holds a relation

$$\tau(E) = \frac{1}{\mathrm{M}(D_0)} \,, \tag{9}$$

$\mathrm{M}(D_0)$ being a Riemann modulus of the ring domain D_0 bounded by $|z|=1$ and E ([15], [16]).

(iv) Define the hyperbolic capacity $C(E)$ of E, similarly to the definition of capacity in the usual potential theory, as follows:

$$C(E) = e^{-V} \,.$$

Then we have

$$\tau(E) = C(E) \,. \tag{10}$$

Recently the present author [9] has proved some theorems on values omitted by functions belonging to a certain class of functions which are regular and univalent in an annulus $Q<|z|<1$, by the method of symmetrization due to Pólya and Szegö [13].

The object of this paper is to extend the above theorems to the case of some classes of the analytic functions which are *not necessarily univalent* in an annulus.

The method used here is analogous to the method of logarithmic transfinite diameter utilized by Hayman [6]. The above results on the hyperbolic transfinite diameter will be effectively utilized in this paper instead of the logarithmic transfinite diameter.

2. An application of the hyperbolic transfinite diameter.

In this section we deduce a fundamental theorem which will play an important rôle in the sequel.

For the purpose we take an annulus in the z-plane

$$D:\ Q<|z|<1 \quad (Q>0)$$

as a doubly-connected basic domain. We now consider a class $\mathfrak{F}$ of the functions $w=f(z)$ which are single-valued and analytic (not necessarily univalent) in D, and whose ranges of values D_f for D lie in the domain $|w|<1$ and have the boundary component $|w|=1$, which is the image of outer circle $|z|=1$ of D, among other boundary components.

For simplicity we denote by p the number of rotations of $w=f(z)$ along $|z|=1$, i.e.,

$$\Delta \arg f(z)=\mathrm{Im}\left\{\oint_{|z|=1}\frac{df(z)}{f(z)}\right\}=2\pi p. \tag{11}$$

Then we have the following

THEOREM 1. *Suppose that a function $w=f(z)\in\mathfrak{F}$ and let E_f be the complement of D_f with respect to the open circular disk $|w|<1$ in the w-plane. Then the following inequality holds for the hyperbolic transfinite diameter of E_f:*

$$\tau(E_f)\leqq Q. \tag{12}$$

The equality sign holds if and only if $w=f(z)$ is univalent and so maps univalently the annulus D onto the ring-domain D_f.

PROOF. We first restrict ourselves to the case $V<\infty$. For the proof we use the equilibrium potential described in (ii) of Sec. 1:

$$u(w)=\int_{E_f}\log\frac{1}{[w, a]}\,d\mu(a),\quad \mu(E_f)=1, \tag{13}$$

in the w-plane. From (ii)

$$u(w)=V \quad \text{on } E_f,$$

except a capacity zero. Furthermore, obviously

$$u(w)=0 \quad \text{on } |w|=1.$$

By the maximum principle for harmonic functions [11], we have

$$0<u(w)<V \tag{14}$$

in D_f.

Next, we consider the function

$$U(z)=\frac{u\{f(z)\}}{V}-\frac{\log\dfrac{1}{|z|}}{\log\dfrac{1}{Q}} \tag{15}$$

which is harmonic in the annulus D and satisfies the conditions:

(i) $U(z)=0$ on $|z|=1$,

(ii) $\overline{\lim_{z\to Qe^{i\theta}}}\, U(z)\leqq 0$ at every point $Qe^{i\theta}$ on $|z|=Q$,

(iii) $U(z)$ is bounded from above in D.

By applying again the maximum principle to the function $U(z)$, we have either $U(z)<0$ or $U(z)\equiv 0$ in the annulus D. Since $U(z)$ is also harmonic on $|z|=1$ and $\partial U/\partial n_z \leqq 0$ there, we have

$$(16) \qquad 0 \geqq \frac{1}{2\pi}\int_{|z|=1} \frac{\partial U(z)}{\partial n_z}\, ds_z = \frac{1}{2\pi V}\int_{|z|=1} \frac{\partial u\{f(z)\}}{\partial n_z}\, d\theta$$

$$-\frac{1}{2\pi \log \dfrac{1}{Q}}\int_{|z|=1} \frac{\partial}{\partial n_z} \log \frac{1}{|z|}\, ds_z \qquad (z=e^{i\theta}),$$

where ds_z denotes the arc-element along $|z|=1$ and $\partial/\partial n_z$ the differentiation performed with respect to the inward-drawn normal on $|z|=1$. On $|z|=1$, there obviously holds

$$(17) \qquad \frac{\partial u}{\partial n_z} = \frac{\partial u}{\partial n_w}\left|\frac{dw}{dz}\right|$$

$$= \frac{\partial u}{\partial n_w}\frac{|z|}{|w|}\left|\frac{dw}{dz}\right| = \frac{\partial u}{\partial n_w}\left|\frac{d\log w}{d\log z}\right| = \frac{\partial u}{\partial n_w}\frac{d\varphi}{d\theta},$$

$\partial/\partial n_w$ denoting the differentiation performed with respect to the inward-drawn normal on $|w|=1$ $(w=e^{i\varphi})$. Accordingly, the first integral of the right-hand side of (16) is equal to

$$(18) \qquad \frac{p}{2\pi V}\int_{|w|=1} \frac{\partial u(w)}{\partial n_w}\, d\varphi = \frac{p}{2\pi V}\int_{E_f}\left\{\int_{|w|=1} \frac{\partial}{\partial n_w} \log\left|\frac{1-\bar{a}w}{w-a}\right| d\varphi\right\} d\mu(a)$$

$$= \frac{p}{2\pi V}\int_{E_f}\left\{\int_{|w|=1} d\arg(w-a)\right\} d\mu(a) = \frac{p}{V},$$

by virtue of the Cauchy-Riemann equation and $\mu(E_f)=1$. Thus we obtain from (16) and (18)

$$0 \geqq \frac{p}{V} - \frac{1}{\log \dfrac{1}{Q}}, \quad e^V \geqq \left(\frac{1}{Q}\right)^p,$$

$$(19) \qquad \tau(E_f) = e^{-V} \leqq Q^p \leqq Q \qquad (\text{by (iv)}).$$

Considering the inequalities (16) and (19), the equality sign of (12) holds only if $\partial U(z)/\partial n_z = 0$ everywhere on $|z|=1$ and $p=1$, i.e.,

$$(20) \qquad \Delta \arg f(z) = 2\pi.$$

Then, if we put

$$W(z) = U(z) + iV(z),$$

$V(z)$ being a conjugate harmonic function of $U(z)$, $W(z)$ is obviously single-valued and analytic in the annulus $1-\varepsilon \leqq |z| \leqq 1$ for any $\varepsilon > 0$. Furthermore $U(z)=0$ and $V(z)=\text{const.}$ on $|z|=1$, since $\partial U(z)/\partial n_z = 0$ there. Therefore $W(z) = \text{const.}$ in the annulus $1-\varepsilon \leqq |z| \leqq 1$ and so in the annulus D. Thus we obtain

$$U(z)=\text{const.}=0 \quad \text{in } D,$$

$$\frac{u\{f(z)\}}{V}=\frac{\log \frac{1}{|z|}}{\log \frac{1}{Q}} \quad \text{in } D.$$

Accordingly,

$$\frac{u\{f(z)\}}{V} \to 0 \quad \text{as} \quad z \to e^{i\theta} \qquad (0 \leqq \theta < 2\pi),$$

$$\frac{u\{f(z)\}}{V} \to 1 \quad \text{as} \quad z \to Qe^{i\theta} \qquad (0 \leqq \theta < 2\pi).$$

Thus we conclude that $w=f(z)$ must always approach the boundary of D_f (either $|w|=1$ or E_f) as $|z| \to 1$ or $|z| \to Q$.

Here we utilize the following lemma due to Heins quoted in Hayman's paper [6]:

Lemma. *Suppose that $f(z)$ is meromorphic in a domain D, that the values which $f(z)$ takes in D lie in a domain D_f, and that as z tends to the boundary of D in any manner $f(z)$ always approaches the boundary of D_f. Then $f(z)$ takes every value of D_f an equal finite number of times in D.*

In our case we may apply the lemma with $Q<|z|<1$ for D, and see that $f(z)$ takes every value in D_f exactly once, since $f(z)$ satisfies the condition (20), i.e. $f(z)$ is univalent in D.

Conversely, let $f(z)$ be univalent in the annulus D, then D_f is a ring-domain and $\tau(E_f)=1/\mathrm{M}(D_f)=Q$ from (iii) of Sec. 1 and by the invariant property of Riemann modulus under any univalent mapping. Thus the equality sign holds in (12).

It is trivial to consider the case $V=\infty$, since in this case $\tau(E_f)=0$ from (10). q. e. d.

Remark 1. Although the inequality (12) is also obtainable from the principle of harmonic measure [11], it seems to be somewhat complicated to investigate the case of equality of (12), because for the present it must be assumed that the boundary of D_f is composed of a finite number of closed Jordan curves.

Remark 2. In the above theorem, the domain D_f is not always doubly-connected. We denote by $\tilde{D}$ a ring-domain having the outer boundary component $|w|=1$ and containing D_f, and denote by $\tilde{E}$ the complement of $\tilde{D}$ with respect to the disk $|w|<1$. Schiffer ([14], [10]) proved that

$$\frac{1}{Q} \leqq \mathrm{M}(\tilde{D}), \tag{21}$$

under the essentially same assumptions as those of our theorem. The inequality (21) is equivalent to

$$\tau(\tilde{E}) \leqq Q, \tag{22}$$

in terms of the hyperbolic transfinite diameter. Since the set function $\tau(E)$ is monotone increasing and $\tilde{E} \subset E_f$,

$$\tau(\tilde{E}) \leqq \tau(E_f).$$

Therefore Schiffer's theorem follows from our result (12) as an immediate consequence.

3. A fundamental theorem on the range of values of $f(z) \in \mathfrak{F}$.

For preparations we introduce some notations necessary for the discussions in this section.

We denote the ring-domain bounded by $|w|=1$ and the rectilinear slit $<P, +\infty>$ $(P>1)$ by G_P which is called the Grötzsch's extremal domain [4] and whose Riemann modulus is denoted by $\Phi(P)$. It is well-known that $\Phi(P)$ is a strictly increasing function of P. Let $Q(<1)$ be a positive number such that

$$\frac{1}{Q} = \Phi(P), \tag{23}$$

then the annulus $1<|z|<Q^{-1}$ can be mapped onto G_P in the w-plane, in such a way that the circle $|w|=1$ and the slit $<P, +\infty>$ correspond to $|z|=1$ and $|z|=Q^{-1}$ respectively, by the so-called Grötzsch's extremal function $w=F(z; Q^{-1})$ $(F(1; Q^{-1})=1)$. It can be explicitly represented in terms of the elliptic functions [8]. Obviously, by the reflection principle, the function $w=F(z; Q^{-1})$ univalently maps the annulus $Q<|z|<1$ onto the ring-domain G'_P bounded by $|w|=1$ and the rectilinear slit $<0, 1/P>$.

After the above preparations, we deduce a fundamental theorem on the range of values taken by the function $f(z) \in \mathfrak{F}$ in the annulus D:

THEOREM 2. *Let e be a closed set of real numbers which is contained in the interval $0 \leqq x < 1$ and whose Lebesgue measure is at least $1/P$ $(1/Q = \Phi(P))$. For each $x \in e$, let be associated a closed set of points $C(x)$ contained entirely in the circle $|w|<1$ in the w-plane such that, if w_1, w_2 be any points on $C(x_1), C(x_2)$ respectively, then the inequality*

$$[w_1, w_2] \geqq [x_1, x_2] \tag{24}$$

always holds. Then the range of values D_f taken by $f(z) \in \mathfrak{F}$ in the annulus D contains at least one of the sets $\{C(x)\}$, except possibly when e is an interval of length $1/P$ and $w=f(z)$ takes the form

$$f(z) \equiv S\{F(z; Q^{-1})\} \in \mathfrak{F}, \tag{25}$$

S denoting any linear transformation mapping the unit circular disk $|w| \leqq 1$ onto

itself.

PROOF. To prove Theorem 2, suppose that D_f contains none of the sets $\{C(x)\}$ completely, so that E_f contains at least one point on each set $C(x)$. We shall have to show that $f(z)$ takes the form (25) and to do this we show that $\tau(E_f)\geqq Q$.

Let a, b be the lower and upper bounds for x in e, and for any x in e let $l(x)$ be the measure of the part of e in the interval $<a, x>$. Thus $l(a)=0$, $l(b)\geqq 1/P$. Let e^* denote the interval on the real ξ-axis $0\leqq\xi\leqq l(b)$, then we shall show that $\tau(E_f)\geqq\tau(e^*)$. To do this we recall the definition (7) of $d_n(E_f)$, i. e., the hyperbolic diameter of order n. Now let $\xi_1, \xi_2,\cdots, \xi_n$ be points of e^* such that

$$\{d_n(e^*)\}^{\frac{n(n-1)}{2}}=\prod_{1\leqq i<j\leqq n}[\xi_i, \xi_j]. \tag{26}$$

Since $l(x)$ is continuous and only changes at x, if x is a limit point of e, and since e is closed, we can find x_i $(i=1, 2,\cdots, n)$ in e such that

$$l(x_i)=\xi_i$$

and hence using (24) a point w_i on each set $C(x_i)$ and in E_f, such that

$$[w_i, w_j]\geqq[x_i, x_j]\geqq[\xi_i, \xi_j],$$

using the obvious inequality

$$\left|\frac{x_i-x_j}{1-x_ix_j}\right|\geqq\left|\frac{\xi_i-\xi_j}{1-\xi_i\xi_j}\right|.$$

Thus there are points $w_1, w_2,\cdots, w_n$ of E_f such that

$$\prod_{1\leqq i<j\leqq n}[w_i, w_j]\geqq\prod_{1\leqq i<j\leqq n}[\xi_i, \xi_j].$$

Using this and (26) we easily obtain

$$\{d_n(E_f)\}^{\frac{n(n-1)}{2}}\geqq\prod_{1\leqq i<j\leqq n}[\xi_i, \xi_j]=\{d_n(e^*)\}^{\frac{n(n-1)}{2}},$$

so that

$$d_n(E_f)\geqq d_n(e^*),$$

and hence, as $n\to\infty$,

$$\tau(E_f)\geqq\tau(e^*). \tag{27}$$

Now e^* is an interval of length $l(b)$ $(\geqq 1/P)$ on the positive real axis and the hyperbolic transfinite diameter of such an interval is the reciprocal of Riemann modulus of the ring-domain bounded by $|w|=1$ and the slit $<0, l(b)>$ $(l(b)\geqq 1/P)$. Therefore we have

$$\tau(e^*)\geqq Q, \tag{28}$$

using the monotonic dependence of Riemann modulus upon the respective ring-domain. From (27) and (28), we obtain

$$\tau(E_f) \geqq Q. \tag{29}$$

On the other hand, by Theorem 1, there holds

$$\tau(E_f) \leqq Q,$$

with the equality sign only if $f(z)$ is univalent in the annulus D. Thus we have

$$\tau(E_f) = \tau(e^*) = Q, \qquad (1/Q = \Phi(P)).$$

Accordingly the Lebesgue measure of e^* and therefore that of e are equal to $1/P$, and $f(z)$ is univalent in D. Then we must have $b-a \geqq 1/P$, and hence

$$[a, b] = \left| \frac{a-b}{1-ab} \right| \geqq \frac{1}{P},$$

with the equality sign only if e is an interval of length $1/P$. From (24) there exist two points w_1 and w_2 belonging to E_f such that

$$[w_1, w_2] \geqq \frac{1}{P}, \quad w_1 \in C(a), \quad w_2 \in C(b).$$

Using a linear transformation S carrying w_1 and w_2 into 0 and w_2' respectively, we obtain

$$[w_1, w_2] = [0, w_2'] = |w_2'| \geqq \frac{1}{P},$$

for a value w_2' omitted by $S\{f(z)\}$ in D.

On the other hand, by Grötzsch's theorem [4] (or author's lemma [9]), there holds for the univalent function $S\{f(z)\}$

$$[0, w_2'] = |w_2'| \leqq \frac{1}{P},$$

with the equality sign only if the function $S\{f(z)\}$ univalently maps the annulus D onto a ring-domain bounded by the circle $|w|=1$ and the slit $<0, 1/P>$ $(1/Q = \Phi(P))$, except a rotation about the origin $w=0$. Thus we obtain

$$f(z) \equiv S^{-1}\{F(z; Q^{-1})\},$$

S^{-1} denoting the inverse transformation of S. q. e. d.

4. A consequence of Theorem 2.

We can clearly show a number of applications of Theorem 2 by giving various particular forms to the set $C(x)$. We may, for instance, take for $C(x)$ the circle about the origin $w=0$ passing through a point x on the positive real axis. It is obvious that the assumption (24) is satisfied. Thus we can prove the following

THEOREM 3. *Suppose that* $f(z)\in\mathfrak{F}$ *and* $f(z)\neq 0$ *in* D. *Then we have the inequality*

$$r_f \leqq \frac{1}{P} \quad \left(\frac{1}{Q}=\Phi(P)\right), \tag{30}$$

r_f *being the lower bound of radii of the circles* $|w|=r$ *contained completely in the range of values* D_f. *The equality sign holds in* (30) *if and only if*

$$f(z)\equiv e^{i\theta}F(z\,;Q^{-1}),$$

where θ *is a real number.*

PROOF. We take the closed interval $<0, 1/P>$ ($1/Q=\Phi(P)$) and the circle $|w|=x$ ($x\in e$) for the set e and for the set $C(x)$, respectively, as described above, in Theorem 2. Then the inequality

$$r_f < \frac{1}{P} \tag{31}$$

follows immediately from Theorem 2, unless the function $f(z)$ takes the form (25).

For the case where $f(z)\equiv S\{F(z\,;Q^{-1})\}$ ($f(z)\neq 0$ in D), the complement E_f of D_f is a closed rectilinear segment containing the origin $w=0$. Then some detailed discussion leads us to the inequality (31), except possibly for the case $f(z)\equiv e^{i\theta}F(z\,;Q^{-1})$ (θ: real).

Conversely, if $f(z)\equiv e^{i\theta}F(z\,;Q^{-1})$, then there holds the equality sign in (30). q. e. d.

Here we newly take an annulus in the z-plane

$$D:\ 1<|z|<R \tag{32}$$

as a doubly-connected basic domain and consider a class $\mathfrak{R}$ of the functions $w=f(z)$ which are single-valued and analytic (not necessarily univalent) in D, and whose ranges of values D_f for D lie in the domain $|w|>1$ and have the boundary component $|w|=1$, as the image of inner circle $|z|=1$ of D, among other boundary ones. Then we immediately obtain the following

THEOREM 4. *Suppose that* $w=f(z)\in\mathfrak{R}$ *and let* d *be the shortest distance from the origin* $w=0$ *to the outer boundary component of* D_f. *Then it holds that*

$$d\geqq P \quad (R=\Phi(P)). \tag{33}$$

The equality sign holds if and only if

$$w=f(z)\equiv e^{i\theta}F(z\,;R), \quad (\theta:\ \text{real}),$$

$F(z\,;R)$ *being the Grötzsch's extremal function described in Sec.* 3.

PROOF. Applying Theorem 3 to the function $1/f(1/z)$, we can easily prove the theorem. q. e. d.

REMARK. Recently Hayman ([5], [6]) has generalized the Koebe's 1/4–Theorem on the class of univalent functions to a class of regular functions in the unit circle $|z|<1$. The above Theorem 4 is regarded as a generalization of Hayman's result to the case where the basic domain is an annulus $1<|z|<R$.

5. Theorem on a class $\mathfrak{M}$ of meromorphic functions.

In this section we consider a somewhat wider class $\mathfrak{M}$ of the functions $w=f(z)$ which are single-valued and meromorphic (not necessarily univalent) in D, and whose ranges of values D_f for D lie in the domain $|w|>1$ and have the boundary component $|w|=1$, as the image of inner circle $|z|=1$ of D, among other boundary ones. Obviously there holds $\mathfrak{F}\subset\mathfrak{M}$. For such a class of functions we have the following

THEOREM 5. *Let $1<a<b\leqq+\infty$. Then the necessary and sufficient condition that for every $f(z)\in\mathfrak{M}$, D_f contains some circle $|w|=r$ with $a<r<b$ is that*

$$\left[\frac{1}{a}, \frac{1}{b}\right]>\frac{1}{P} \qquad (R=\Phi(P)). \tag{34}$$

PROOF. *Necessity.* Suppose that

$$\left[\frac{1}{a}, \frac{1}{b}\right]\leqq\frac{1}{P}.$$

Then we have

$$a\geqq\frac{1+\dfrac{1}{bP}}{\dfrac{1}{P}+\dfrac{1}{b}}. \tag{35}$$

On the other hand, we consider the composite function of the Grötzsch's extremal function $\zeta=F(z;R)$ and the linear fractional function $w=w(\zeta)=\left(\zeta+\frac{1}{b}\right)\Big/\left(1+\frac{\zeta}{b}\right)$. The function

$$w=w\{F(z;R)\}\in\mathfrak{M}$$

univalently maps the annulus $1<|z|<R$ onto the ring-domain bounded by $|w|=1$ and the rectilinear segment $\langle a', b\rangle$, where

$$a'=\frac{1+\dfrac{1}{bP}}{\dfrac{1}{P}+\dfrac{1}{b}}. \tag{36}$$

Therefore this function takes no real value w for which $a\leqq w\leqq b$, since $a'\leqq a$ from (35) and (36).

Sufficiency. Suppose that for some function $f(z)$ the range of values D_f

contains none of circles $|w|=r$ $(a<r<b)$. Performing a linear transformation S to the function $f(z)$, if necessary, we consider the function $1/S\{f(1/z)\}$ in the annulus $1/R<|z|<1$, where S transforms a and b into a' and ∞ respectively, such that

$$\left[\frac{1}{a}, \frac{1}{b}\right]=\left[\frac{1}{a'}, 0\right]=\frac{1}{a'}.$$

In Theorem 2 we may take for the set $C(x)$ the image of the circle $|w|=r$ $(a\leqq r\leqq b)$ under the transformation TS, T being the reciprocal transformation, and take for the set e of x the interval $<0, 1/a'>$. Thus we obtain from this theorem that

$$\frac{1}{a'}\leqq\frac{1}{P},$$

and accordingly

$$\left[\frac{1}{a}, \frac{1}{b}\right]\leqq\frac{1}{P}.$$

Therefore, if (34) is satisfied, then for every $f(z)\in\mathfrak{M}$, D_f must always contain some circle $|w|=r$ with $a<r<b$. q. e. d.

6. Class of bounded analytic functions.

Let $w=f(z)\in\mathfrak{R}$ and be bounded in the annulus D: $1<|z|<R$, i. e., $1<|f(z)|<M$ in D. In this section we consider the class of such functions $f(z)$ and deduce some theorems on the range of values D_f. We obtain immediately from Theorem 1 the following

Theorem 6. *Suppose that $w=f(z)\in\mathfrak{R}$ and is bounded in D: $1<|z|<R$, i. e., $1<|f(z)|<M$ in D. Then we have*

$$R\leqq M. \tag{37}$$

The equality sign in (37) *holds if and only if $w=f(z)\equiv e^{i\theta}z$* ($\theta$: real).

Proof. We consider the function $w_1=1/f(1/z)$, single-valued and analytic in the annulus $1/R<|z|<1$ $(Q=1/R)$, and apply Theorem 1 to this function. Let $\tilde{E}$ be the complement of the range of values taken by the function. Then we have

$$\tau(\tilde{E})\leqq Q=\frac{1}{R}. \tag{38}$$

Since the closed circular disk $|w_1|\leqq 1/M$ is contained in $\tilde{E}$, its hyperbolic transfinite diameter $(=1/M)$ is not greater than that of $\tilde{E}$, i. e.,

$$\frac{1}{M}\leqq\tau(\tilde{E}). \tag{39}$$

From (38) and (39), we obtain

$$\frac{1}{M} \leqq \frac{1}{R}, \quad R \leqq M.$$

It is obvious that the equality sign in (37) holds if and only if $w=f(z)$ is univalent in D and D_f coincides with the ring-domain $1<|w|<M$, and hence $w=f(z)\equiv e^{i\theta}z$ (θ: real). q. e. d.

Here we notice that this theorem is also obtainable from the principle of harmonic measure or Schiffer's theorem (see Remark 2 of Sec. 2).

Let the annulus $D: 1<|z|<R$ be univalently mapped by some bounded function $w=f(z)\in\mathfrak{R}$ onto the annulus $1<|w|<M$ slit along a rectilinear segment $<\gamma, M>$. Then there holds $M\geqq R$ by Theorem 6. It is well-known that the value of γ uniquely depends upon only $M(\geqq R)$ and yet is a strictly decreasing function of M such that

$$\lim_{M\to R}\gamma=R \quad \text{and} \quad \lim_{M\to\infty}\gamma=P \qquad (R=\Phi(P)).$$

Hereafter, for preciseness, we denote the value of γ by $\gamma(M)$. Then we prove the following

THEOREM 7. *Suppose that $w=f(z)\in\mathfrak{R}$ and is bounded in $D: 1<|z|<R$, i.e., $1<|f(z)|<M$ in D, and let d be the shortest distance from the origin $w=0$ to the outer boundary component of D_f. Then we have*

$$d\geqq\gamma(M). \tag{40}$$

The equality sign holds if and only if $w=e^{i\theta}f(z)$ is the above described function which univalently maps D onto the annulus $1<|w|<M$ slit along the segment $<\gamma(M), M>$, where θ is a real number.

PROOF. Without any loss of generality, we may assume that a point $w=d$ (>0) on the outer boundary component of D_f has the shortest distance from the origin $w=0$. Let $\zeta=F(w; M)$ be the Grötzsch's extremal function which univalently maps the annulus $1<|w|<M$ onto the domain $|\zeta|>1$ slit along the rectilinear segment $<P', +\infty>$ $(M=\Phi(P'))$. Then there obviously holds

$$P\leqq P' \ (R=\Phi(P),\ M=\Phi(P')), \tag{41}$$

since $\Phi(P)$ is the strictly increasing function of P and $R\leqq M$ from (37).

Consider the composite function $\zeta=F(f(z); M)\in\mathfrak{R}$ and apply Theorem 4 to this function. Then we obtain

$$P\leqq F(d; M). \tag{42}$$

On the other hand, since the annulus $1<|w|<M$ slit along $<\gamma(M), M>$ and the domain $|w|>1$ slit along $<P, +\infty>$ are conformally equivalent to the annulus $D: 1<|z|<R$, there holds

$$F(\gamma(M); M)=P. \tag{43}$$

From (42) and (43), we obtain

(44) $$F(\gamma(M)\,;\,M)\leqq F(d\,;\,M)\,,$$

and therefore

$$\gamma(M)\leqq d\,,$$

since $F(z\,;\,M)$ is a strictly increasing function for real $z\in<1, M>$.

The equality of (40): $\gamma(M)=d$ entails the equality of (42): $P=F(d\,;\,M)$. Hence, by Theorem 4, we have

$$F(f(z)\,;\,M)\equiv F(z\,;\,R)\,,$$

in other words, $w=f(z)$ univalently maps the annulus D onto the annulus $1<|w|<M$ slit along the segment $<\gamma(M), M>$. It is obvious from (41).

Conversely, the equality sign holds in (40) for such functions. q. e. d.

REMARK. Under the additional assumption that $f(z)$ is univalent in D, the above theorem was proved by Grötzsch [4] and Komatu [8].

7. Circular disks omitted by functions $\in\mathfrak{R}$.

A few years ago, Goodman [3] proved the following theorem:

Let $w=f(z)=\sum_{n=1}^{\infty}a_n z^n$, $(|a_1|=1)$, *be regular and univalent in* E: $|z|<1$. *Let* c *be fixed and suppose that for* z *in* E, $f(z)$ *omits all* ξ *for which*

$$|\xi-c|\leqq R_0\,.$$

Then

$$R_0\leqq|c|\,\frac{4|c|-1}{4|c|+1}\,.$$

The result is best possible.

In this section we attempt to extend the result to the case of class $\mathfrak{R}$ of analytic functions in D: $1<|z|<R$ and prove the following

THEOREM 8. *Let* $w=f(z)\in\mathfrak{R}$ *and let* c $(|c|>P, R=\Phi(P))$ *be any fixed point not belonging to the simply-connected domain which is bounded by the outer boundary of* D_f *and contains* D_f. *Suppose that in the annulus* D, $w=f(z)$ *omits all* ξ *for which*

(45) $$|\xi-c|<R_0\,.$$

Then we have

(46) $$R_0\leqq M\Big(\frac{|c|}{\gamma(M)}-1\Big)\,,$$

M being uniquely determined by the relation

(47) $$|c|=\frac{\gamma(M)(M^2-1)}{M^2-\gamma(M)^2}$$

for each value of $|c|$. *The result is best possible.*

PROOF. Without loss of generality we may assume that $c>0$. For the proof, we now introduce the following linear transformation [7]:

$$\zeta=\frac{tM}{\bar{R}_0}\frac{w-c+s}{w-c+t}, \tag{48}$$

s and t being the roots of the equations

$$s\cdot t=\bar{R}_0,\quad (c-s)(c-t)=1,\quad (s>t), \tag{48'}$$

and two constants $M, \bar{R}_0$ are connected with each other by a relation

$$M=\frac{\bar{R}_0(c-t)}{t}. \tag{48''}$$

By this transformation, the whole w-plane with two circular holes: $|w|\leqq 1$ and $|w-c|\leqq \bar{R}_0$, is univalently mapped onto the annulus $1<|\zeta|<M$, in such a way that the circles $|\zeta|=1$ and $|\zeta|=M$ correspond to the circles $|w|=1$ and $|w-c|=\bar{R}_0$, respectively. Two points $w=c+\bar{R}_0$ and $w=\infty$ are carried into $\zeta=M$ and $\zeta=tM/\bar{R}_0$. Determining M so as to satisfy the relation

$$\frac{tM}{\bar{R}_0}=\gamma(M), \tag{48'''}$$

and using (48′) and (48′′), we obtain

$$\bar{R}_0=M\Big(\frac{c}{\gamma(M)}-1\Big), \tag{49}$$

and

$$c=\frac{\gamma(M)(M^2-1)}{M^2-\gamma(M)^2}. \tag{50}$$

If c is regarded as a function of $M(\geqq R)$, it becomes a strictly decreasing function such that $\lim_{M\to R} c=+\infty$ and $\lim_{M\to\infty} c=P$. Therefore, for any fixed $c(>P)$, M and so $\bar{R}_0$ can be uniquely determined from (50) and (49), respectively. The details of calculations will be omitted here, since these seem to be essentially analogous to those appearing in the preceding paper by the author [9]. Thus the annulus D: $1<|z|<R$ becomes conformally equivalent to the annulus $1<|\zeta|<M$ slit along $<tM/\bar{R}_0, M>$ and also to the ring-domain bounded by $|w|=1$, $|w-c|=\bar{R}_0$ and the slit $<c+\bar{R}_0, +\infty>$.

After the above preparations we prove the theorem. Suppose now that $R_0>\bar{R}_0$ for some function $w=f(z)\in\mathfrak{R}$ and for some point c, and consider the image domain D_ζ in the ζ-plane of D by the composite function of $w=f(z)$ and (48). It belongs to the class $\mathfrak{R}$ and is bounded above by M. Then we have for *every* point ζ on the outer boundary of D_ζ

$$|\zeta|>\gamma(M) \tag{51}$$

from Theorem 7. It should be noticed that the equality sign never holds here, since the assumption $R_0>\bar{R}_0$ entails the result that D_ζ is contained in some smaller annulus: $1<|\zeta|<M'$ $(<M)$.

On the other hand, since $w=f(z)$ is single-valued and analytic in D, the point $\zeta=\gamma(M)$ in the ζ-plane never belongs to the simply-connected domain bounded by the outer boundary of D_ζ, containing D_ζ. Hence there exists *some* point ζ' on the outer boundary such that

$$|\zeta'|\leqq\gamma(M).$$

The result is contradictory to (51). Thus we have $R_0\leqq\bar{R}_0$.

Using again Theorem 7, we can prove that the equality sign of (46) holds if and only if $w=f(z)$ univalently maps D onto the ring-domain bounded by $|w|=1$, $|w-c|=\bar{R}_0$ and the slit $<c+\bar{R}_0, +\infty>$, except a rotation about the origin $w=0$. But the details will be omitted here. q. e. d.

Remark. Under the additional assumption that $f(z)$ is univalent in D, the author [9] proved the above theorem.

8. Class of analytic functions with bounded real parts.

In this section we consider a subclass of $\mathfrak{R}$, i. e., a class $\mathfrak{R}_c$ of functions $f(z)$ such that $f(z)\in\mathfrak{R}$ and satisfy the condition

$$\mathrm{Re}\{f(z)\}>-c\,, \tag{52}$$

in D, where c is a fixed positive constant. Thus we obtain the following

Theorem 9. *Suppose that $w=f(z)\in\mathfrak{R}_c$. Let τ be any point on the outer boundary component of the range of values D_f taken by $f(z)$ in D. Then we have*

$$|\tau|\leqq\frac{1+M\gamma(M)}{M+\gamma(M)}\,, \tag{53}$$

M being uniquely determined so as to satisfy the condition

$$M+\frac{1}{M}=2c \qquad (M\geqq R)\,, \tag{54}$$

for any fixed $c\geqq(R+R^{-1})/2$. The result is best possible.

Proof. We introduce the following transformation [7]:

$$\zeta=-tM^2\frac{w+2c-s}{w+2c-t} \qquad (t<s)\,, \tag{55}$$

s and t being the roots of the equations

$$s\cdot t=1,\quad s+t=2c \qquad (t<s)\,, \tag{56}$$

and M being connected with s and t by the relation

$$\frac{1}{M^2}=\frac{t}{2c-t}\left(=\frac{t}{s}\right). \tag{57}$$

By this transformation, the w-plane with two circular holes $|w+2c|\leqq 1$ and $|w|\leqq 1$ is univalently mapped onto an annulus $1<|\zeta|<M^2$ in such a way that the circles $|\zeta|=1$ and $|\zeta|=M^2$ correspond to the circles $|w|=1$ and $|w+2c|=1$ respectively. Therefore, by the reflection principle, the ring-domain bounded by the straight line $\mathrm{Re}\{w\}=-c$ and the circle $|w|=1$ is univalently mapped onto the annulus $1<|\zeta|<M$ in such a way that the circle $|w|=M$ corresponds to the line $\mathrm{Re}\{w\}=-c$. From (56) and (57), (55) may be rewritten in the from

$$\zeta=-M\frac{w+2c-M}{w+2c-\dfrac{1}{M}}, \tag{58}$$

M being uniquely determined by the relation

$$M+\frac{1}{M}=2c. \tag{59}$$

The details of calculations will be omitted here, since these seem to be essentially analogous to those appearing in the preceding paper by the author [9].

In the ζ-plane, we consider the range of values D_ζ taken in D by the composite function of $w=f(z)$ and (58). By Theorem 7, for any point ζ' on the outer boundary of the range D_ζ, there holds

$$|\zeta'|\geqq\gamma(M). \tag{60}$$

Using (58)

$$w=-\frac{\zeta+\dfrac{1}{M}}{1+\dfrac{\zeta}{M}}. \tag{61}$$

For any point τ on the outer boundary of D_f, from (60) and (61), we obtain

$$|\tau|=\left|\frac{\zeta'+\dfrac{1}{M}}{1+\dfrac{\zeta'}{M}}\right|\geqq\frac{|\zeta'|+\dfrac{1}{M}}{1+\dfrac{|\zeta'|}{M}}\geqq\frac{\gamma(M)+\dfrac{1}{M}}{1+\dfrac{\gamma(M)}{M}}.$$

Thus we have the inequality (53). Using again Theorem 7 we can prove that the equality sign of (53) holds if and only if $w=f(z)$ univalently maps D onto the ring-domain bounded by the circle $|w|=1$, the straight line $\mathrm{Re}\{w\}=-c$ and the slit along the real axis from $-c$ to $-\left\{\gamma(M)+\dfrac{1}{M}\right\}\Big/\left\{1+\dfrac{\gamma(M)}{M}\right\}$.

q. e. d.

REMARK 1. Applying Theorem 6 to the composite function of $w=f(z)$ and (58), and considering (59), it is easily proved that for the fixed constant c we necessarily have $c\geqq(R+R^{-1})/2$.

Remark 2. Under the additional assumption that $f(z)$ is univalent in D, the author [9] proved the above theorem.

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