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A unique continuation theorem for solutions of a parabolic differential equation.

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Introduction.

It was shown by N. Aronszajn [1], [2] that, if $u(x)$ satisfies a second order linear elliptic differential equation $Au(x)=0$ on a domain D and has a zero point of infinite order in D, then it vanishes identically in D. Recently one of the authors has proved a similar result for a parabolic equation $\partial u(t,x)/\partial t=Au(t,x)$ $(0<t<\infty, x\in D)$ for the case when D is bounded. The purpose of this paper is to extend this result to the case when D is not necessarily bounded.

§ 1. Assumptions and the main theorems.

Let D be a (not-necessarily bounded) domain in a euclidean m-space whose boundary $B=\bar{D}-D$ consists of at most countably many C^3-hypersurfaces of $m-1$ dimension. Consider an elliptic differential operator A defined by

$$\text{(A)}\qquad Au=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{a(x)}\,a^{ij}\frac{\partial}{\partial x^j}u\right)+c(x)u \qquad \text{for } x\in D$$

with a boundary condition

$$\text{(B)}\qquad \alpha(\xi)u+\{1-\alpha(\xi)\}\partial u/\partial n_\xi=0 \qquad \text{for } \xi\in B.$$

Here $\|a^{ij}(x)\|$ denotes a strictly positive-definite symmetric matrix for any $x\in\bar{D}$, $0\leqq\alpha(\xi)\leqq 1$ on B, $\partial^2 a^{ij}(x)/\partial x^k\partial x^l$ $(i,j,k,l=1,\cdots,m)$ and $\partial^2\alpha(\xi)/\partial\xi^p\partial\xi^q$ $(p,q=1,\cdots,m-1)$ are Lipschitz continuous in $x\in\bar{D}$ and in $\xi\in B$ respectively, where local coordinates on B are denoted by $<\xi^1,\cdots,\xi^{m-1}>$.

Moreover $c(x)$ is assumed to be Lipschitz continuous in $x\in\bar{D}$, and satisfies

$$\text{(C)}\qquad -\infty<c(x)\leqq C<\infty$$

for some constant C. Here the differentiability of functions on $\bar{D}$ at any point $\xi\in B$ and normal derivatives $\partial u/\partial n_\xi$ $(\xi\in B)$ with respect to the metric tensor $a^{ij}(x)$ should be understood as those defined in one of Itô's papers [6].

Under these assumption shown above, it was shown in [6] that there exists a so-called fundamental solution $U(t,y,x)=U(t,x,y)\geqq 0$ of a parabolic equation

(1.1) $$\partial u(t, x)/\partial t = Au(t, x) \qquad (t>0, x\in D)$$

associated with the boundary condition (B). Namely, for any $f\in L^p(D)$ with $p\geqq 1$ (with respect to the measure $dx=\sqrt{a(x)}\,dx^1\cdots dx^m$), the function

(1.2) $$u(t, x)=[T_t f](x)=\int_D f(y)U(t, y, x)dy$$

belongs to $L^p(D)$ and is a solution of (1.1) satisfying both the initial condition

(1.3) $$\lim_{t\downarrow 0}\|T_t f-f\|_p=0, \quad (\|\ \|_p \text{ denotes the norm in } L^p(D))$$

and the boundary condition (B). The main result in the present paper is the following

THEOREM 1. *If* i) *$u(t, x)$ is defined by* (1.2) *with $f\in L^2(D)$ and* ii) *there exist $t_0>0$ and an open set $D_0\subset D$ such that $u(t_0, x)=0$ for any $x\in D_0$, then $u(t, x)=0$ for any $t>0$ and any $x\in \bar{D}$, and consequently $f(x)=0$ almost everywhere in D.*

The proof will be given in § 3.

The uniqueness of the solution of the equation (1.1) with the initial condition (1.3) and with the boundary condition (B), does not necessarily hold (see [6], Appendix I). If we *assume* that

(1.4) $\left\{\begin{array}{l}\textit{the equation (1.1) has a unique solution } u(t,x)\in L^2(D) \textit{ satisfying both} \\ \textit{the initial condition (1.3) with } p=2 \textit{ and the boundary condition (B),}\end{array}\right.$

then the solution can be expressed by (1.2), and hence it follows from Theorem 1 that

THEOREM 2. *Let $u(t, x)$ be a solution of a parabolic equation* (1.1) *satisfying* (B). *If $u(t, x)$ belongs to $L^2(D)$ for any $t>0$, and if the assumption* ii) *in Theorem* 1 *holds, then $u(t, x)=0$ for any $<t, x>, t>0$.*

The uniqueness (1.4) holds for any $p\geqq 1$ if, for example,

(1.5) $$a^{ij}(x) \textit{ are bounded and } \|a^{ij}(x)\| \textit{ is uniformly elliptic in } \bar{D}.$$

Hence our result covers the case when $\bar{D}$ is compact. Without the assumption (1.4) Theorem 2 does not hold (see § 4).

If (1.5) is satisfied, then $U(t, y, x)$ is bounded in $<x, y>\in D\times D$ for any $t>0$ and we can prove that, for any $f\in L^p(D)$ with $1\leqq p\leqq 2$, $u(t, \cdot)$ defined by (1.2) belongs to $L^2(D)$ for $t>0$. Hence Theorem 2 is valid for $u(t, x)\in L^p(D)$ if $1\leqq p\leqq 2$, and therefore we have: *If $\mu(X)$ is an additive set function of bounded variation on D, and if the function $u(t, x)=\int_D U(t, y, x)d\mu(y)$ satisfies the assumption* ii) *in Theorem* 1, *then $u(t, x)=0$ for any $<t, x>$, and furthermore $\mu(X)=0$ for any Borel set $X\subset D$.*

§ 2. Some properties of solutions of a parabolic equation $\partial u/\partial t = Au$.

Consider the elliptic differential operator A with the boundary condition (B) defined in § 1, and assume that $C=0$ in (C). In one of Itô's papers [6], it is shown that*) : *There exist a sequence* $\{\phi_p(x; \lambda); p=1, 2, \cdots\}$, $(x \in \bar{D}, 0 \leqq \lambda < \infty)$ *of solutions of* $A\phi + \lambda\phi = 0$ *satisfying the boundary condition* (B) *and a sequence* $\{\rho_p; p=1, 2, \cdots\}$ *of Borel measures on* $[0, \infty)$ *with* $\rho_p([0, \infty))=1$ *for any* p *such that:*

a) *any* $f \in L^2(D)$ *is expressible in the form*

$$f(x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{p=1}^{N} \int_0^N \phi_p(x; \lambda) f_p(\lambda) d\rho_p(\lambda) \tag{2.1}$$

where

$$f_p(\lambda) = \underset{F:\,\text{compact}\,\uparrow \bar{D}}{\text{s-lim}} \int_F \overline{\phi_p(x; \lambda)} f(x) dx \tag{2.2}$$

(s-lim *means the strong limit in* $\sum_{p=1}^{\infty} \oplus L^2([0, \infty), \rho_p)$), *and*

$$\sum_{p=1}^{\infty} \int_0^{\infty} |f_p(\lambda)|^2 d\rho_p(\lambda) = \int_D |f(x)|^2 dx. \tag{2.3}$$

b) *the fundamental solution* $U(t, y, x)$ *of the equation* $\partial u/\partial t = Au$ *associated with the boundary condition* (B) *can be expressed as*

$$U(t, y, x) = \sum_{p=1}^{\infty} \int_0^{\infty} e^{-\lambda t}\, \overline{\phi_p(y; \lambda)}\, \phi_p(x; \lambda) d\rho_p(\lambda). \tag{2.4}$$

It can be seen from the argument in [6], Chapter III that both the summation and the integral in the right hand side of (2.4) converge uniformly in $\langle t, y, x \rangle$ on any compact subset of $(0, \infty) \times \bar{D} \times \bar{D}$, and

c) $\phi_p(x; \lambda)$, $p=1, 2, \cdots$, *are measurable in the variable* $\langle x, \lambda \rangle$.

A set function $\rho(\Lambda) = \sum_{p=1}^{\infty} \rho_p(\Lambda) 2^{-p}$ defines a Borel measure ρ on $[0, \infty)$ satisfying $\rho([0, \infty))=1$. Besides all ρ_p's are absolutely continuous with respect to ρ. Hence there exist non-negative functions $\omega_p(\lambda), p=1, 2, \cdots$ such that

$$d\rho_p(\lambda) = \omega_p(\lambda) d\rho(\lambda) \tag{2.5}$$

and

$$\int_0^{\infty} \omega_p(\lambda) d\rho(\lambda) = 1.$$

Now, let $u(t, x)$ be the function defined by (1.2) with $f(x) \in L^2(D)$. Then, by a), b) and (2.5), we have

*) Similar results were proved by F. E. Browder [3] [4], L. Gåring [5] and others. Here we quote the expression taken in [6] for the convenience of notations in the present paper.

(2.6) $$f(x)=\underset{N\to\infty}{\mathrm{l.i.m.}}\sum_{p=1}^{N}\int_0^N \phi_p(x;\lambda)f_p(\lambda)\omega_p(\lambda)d\rho(\lambda),$$

(2.7) $$\int_D |f(x)|^2dx=\sum_{p=1}^{\infty}\int_0^\infty |f_p(\lambda)|^2\omega_p(\lambda)d\rho(\lambda),$$

and

(2.8) $$U(t,x,y)=\sum_{p=1}^{\infty}\int_0^\infty e^{-\lambda t}\phi_p(y;\lambda)\overline{\phi_p(x;\lambda)}\,\omega_p(\lambda)d\rho(\lambda).$$

It follows from (2.6) and (2.8) that

(2.9) $$u(t,x)=\int_D f(y)U(t,y,x)dy$$
$$=\sum_{p=1}^{\infty}\int_0^\infty \phi_p(x;\lambda)e^{-\lambda t}f_p(\lambda)\omega_p(\lambda)d\rho(\lambda)$$

and therefore

(2.10) $$\underset{F:\,\mathrm{compact}\uparrow\bar{D}}{\text{s-lim}}\int_F \overline{\phi_p(x;\lambda)}\,u(t,x)dx=e^{-\lambda t}f_p(\lambda).$$

If we put

(2.11) $$v_p(x;\lambda)=\phi_p(x,\lambda)f_p(\lambda)\omega_p(\lambda),$$

we have

(2.12) $$\Big\{\sum_{p=1}^{\infty}\int_0^\infty |e^{-\lambda t}v_p(x;\lambda)|d\rho(\lambda)\Big\}^2$$
$$\leqq\Big\{\sum_{p=1}^{\infty}\int_0^\infty |f_p(\lambda)|^2\omega_p(\lambda)d\rho(\lambda)\Big\}\Big\{\sum_{p=1}^{\infty}\int_0^\infty e^{-2\lambda t}|\phi_p(x;\lambda)|^2\omega_p(\lambda)d\rho(\lambda)\Big\}$$
$$=U(2t,x,x)\int_D |f(x)|^2dx$$

by virtue of Schwarz's inequality and of (2.7) and (2.8).

LEMMA 1. *For every integer $n\geqq 0$, $A^nu(t,x)$ is real-analytic in $t>0$, and of class C^2 in x. Furthermore*

(2.13) $$\partial^n u(t,x)/\partial t^n=A^nu(t,x)=\sum_{p=1}^{\infty}\int_0^\infty e^{-\lambda t}(-\lambda)^n v_p(x;\lambda)d\rho(\lambda)$$

for $n\geqq 1$. (Notice that $a^{ij}(x)$'s and $c(x)$ are not assumed to be analytic.)

PROOF. It follows from (2.12) that

(2.14) $$\sum_{p=1}^{\infty}\int_0^\infty |e^{-\lambda t}\lambda^n v_p(x;\lambda)|d\rho(\lambda)\leqq w_n(t)\Big\{U(t,x,x)\int_D |f(x)|^2dx\Big\}^{1/2}$$

for any $n\geqq 0$ where $w_n(t)=\sup\{e^{-\lambda t/2}\lambda^n;\ \lambda\geqq 0\}$. Hence, for any fixed $n\geqq 0$, a sequences of functions $\{u_N^{(n)}(t,x);\ N=1,2,\cdots\}$ defined by

(2.15) $$u_N^{(n)}(t,x)=\sum_{p=1}^{N}\int_0^N e^{-\lambda t}(-\lambda)^n v_p(x;\lambda)d\rho(\lambda)$$

converges uniformly in any compact subset of $\{t;\,t>0\}\times\bar{D}$ to the function

$$u^{(n)}(t, x)=\sum_{p=1}^{\infty}\int_0^{\infty}e^{-\lambda t}(-\lambda)^n v_p(x;\lambda)d\rho(\lambda). \tag{2.16}$$

This does converge even if we consider t's in (2.14) and in (2.15) as complex variables, namely (2.16) converges uniformly in any compact subset of $\{t\cdot \Re t>0\}\times \bar{D}$. Since $u_N^{(n)}(t, x)$'s are analytic in $\{\Re t>0\}$ for any fixed x, so are $u^{(n)}(t, x)$'s. In particular these functions are real-analytic. Evidently

$$\partial^n u(t, x)/\partial t^n=\partial^n u^{(0)}(t, x)/\partial t^n=u^{(n)}(t, x) \qquad (n=1, 2,\cdots). \tag{2.17}$$

Let $\psi(x)$ be a function of class C^2 with a compact support $\subset D$. Then

$$\int_D u_N(t, x)A\psi(x)dx=\int_D Au_N(t, x)\psi(x)dx=\int_D u_N^{(1)}(t, x)\psi(x)dx.$$

By letting N go to infinity, we get

$$\int_D u(t, x)A\psi(x)dx=\int_D u^{(1)}(t, x)\psi(x)dx.$$

Consequently $u^{(1)}(t, x)$ is of class C^2 in $x\in D$, according to the [6], Theorem 5, [7], Chapitre V, Theoreme XII and we have

$$Au(t, x)=u^{(1)}(t, x). \tag{2.18}$$

Successive uses of similar arguments will prove that $u^{(n)}(t, x)$ is of class C^2 in $x\in D$ and

$$Au^{(n)}(t, x)=u^{(n+1)}(t, x). \tag{2.19}$$

Combining (2.16), (2.17), (2.18) and (2.19) we have (2.13). Lemma 1 is thus proved.

Lemma 2. *For ρ-almost every λ, the function*

$$v(x;\lambda)=\sum_{p=1}^{\infty}v_p(x;\lambda) \tag{2.20}$$

is of class C^2 in x, and satisfies

$$Av(x;\lambda)=-\lambda v(x;\lambda) \tag{2.21}$$

on D, and

$$u(t, x)=\int_0^{\infty}e^{-\lambda t}v(x;\lambda)d\rho(\lambda). \tag{2.22}$$

Remark. It is important that (2.22) holds for all $\langle t, x\rangle\in(0, \infty)\times\bar{D}$ by virtue of (2.12).

Proof. It follows from (c) that $v_p(x;\lambda)$ $(p=1, 2,\cdots)$ are measurable in $\langle x, \lambda\rangle$ and hence, by Fubini's theorem, (2.12) implies

$$\int_F\sum_{p=1}^{\infty}|v_p(x;\lambda)|dx<\infty \tag{2.23}$$

for any compact $F\subset\bar{D}$, except for $\lambda\in\Lambda_0$ of ρ measure 0. Hence $v(x;\lambda)$ defined by (2.20) is locally summable in x for $\lambda \notin \Lambda_0$. Since $V_N(x;\lambda)=\sum_{p=1}^{N} v_p(x;\lambda)$ satisfies

$$\int_D V_N(x;\lambda)A\psi(x)dx=\int_D AV_N(x;\lambda)\psi(x)dx=-\int_D \lambda V_N(x;\lambda)\psi(x)dx$$

for any function $\psi(x)$ of class C^2 with its compact support $\subset \bar{D}$, we obtain for $\lambda \notin \Lambda_0$

$$\int_D v(x;\lambda)A\psi(x)dx=-\int_D \lambda v(x;\lambda)\psi(x)dx$$

as N tends to infinity. This implies (2.21) and (2.22) follows from (2.12), (2.9) and (2.11), q. e. d.

§ 3. Proof of theorems.

Lemma 3. *If* $v(\lambda)\in L^2(\rho)$ *and* $\int_0^\infty e^{-\lambda t}v(\lambda)d\rho(\lambda)=0$ *for any* $t>0$, *then* $v(\lambda)=0$ *ρ-almost everywhere.*

Proof. Any continuous function $\psi(\lambda)$ on $[0,\infty)$ satisfying $\lim_{\lambda\to\infty}\psi(\lambda)=0$ can be approximated uniformly on $[0,\infty)$ by a linear combination of $e^{-p\lambda}$'s ($p=1,2,\cdots$); this fact may be proved by applying Weierstrass' polynomial approximation theorem to the function $h(\xi)=\psi(-\log\xi)$ for $0<\xi\leqq 1$ and $=0$ at $\xi=0$, which is continuous in $[0.1]$. Therefore the assumption of this lemma enables us to state that $\int_0^\infty \psi(\lambda)v(\lambda)d\rho(\lambda)=0$ for any continuous ψ with its compact support, and consequently for any $\psi\in L^2(\rho)$. Hence $v(\lambda)=0$ ρ-almost everywhere.

Proof of Theorem 1. For any constant c, $e^{-ct}U(t,y,x)$ is a fundamental solution of the equation $\partial u(t,x)/\partial t=(A-c)u(t,x)$ associated with the boundary condition (B). Therefore it is sufficient to prove Theorem 1 when $C=0$ in the condition (C) in § 1, and hence we are able to use results in § 2.

By Lemma 1, $u(t,x)$ is real-analytic in $t>0$. However from the assumption of Theorem 1,

$$\partial^n u(t_0,x)/\partial t^n=[A^n u](t_0,x)=0$$

for any $x\in D_0$, $n=1,2,\cdots$ Hence

$$u(t,x)=0 \tag{3.1}$$

for any $<t,x>\in(0,\infty)\times D_0$.

The function $V(x;\lambda)=e^{-\lambda t_0}v(x;\lambda)$ belongs to $L^2(\rho)$ for any $x\in D$ by virtue of (2.12) and (2.20). Moreover on account of (2.22) and (3.1),

$$\int_0^\infty e^{-\lambda t}V(x;\lambda)d\rho(\lambda)=0 \tag{3.2}$$

for any $<t, x> \in (0, \infty)\times D_0$.

Hence, by Lemma 3, there exist a countable set E dense in D_0 and a Borel set Λ_1 of ρ-measure 0 such that $v(x;\lambda)=e^{\lambda t_0}V(x;\lambda)=0$ for any $x\in E$ and for any $\lambda\notin\Lambda_1$. On the other hand, Lemma 2 shows that $v(x;\lambda)$ is of class C^2 and satisfies $(A+\lambda)v(x;\lambda)=0$ in D for ρ-almost every λ. Hence $v(x;\lambda)=0$ for all $x\in D_0$ for ρ-almost every λ. Therefore, by a theorem of Aronszajn [1], [2], $v(x;\lambda)=0$ for any $x\in D$ for ρ-almost every λ. This means $u(t, x)=0$ for any $<t, x>\in(0, \infty)\times D$ on account of (2.22) and consequently for any $<t, x>\in(0, \infty)\times\bar{D}$ because of the continuity of $u(t, x)$. Theorem 1 is thus proved.

Theorem 2 follows from (1.4) and Theorem 1, as was explained in § 1.

§ 4. A counter example and a conjecture.

Set $a^1(x)=e^{-2x}$ and $a(x)=a_1(x)=e^{2x}(x\in R^1)$, and consider the differential operator A:

$$Au=\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x}\left(\sqrt{a(x)}\,a^1(x)\frac{\partial u}{\partial x}\right)=e^{-2x}u_{xx}-e^{-2x}u_x\,.$$

Then, for any fixed $t_0>0$, the function

$$u(t, x)=\begin{cases} 0 & \text{for } t\leqq t_0 \\ \int_{t_0}^{t}(t-\tau)^{1/2}\exp[-e^{2x}/4(t-\tau)]d\tau & \text{for } t>t_0 \end{cases}$$

satisfies $u_t=Au$ in $(0, \infty)\times R^1$. However $u(t, x)=0$ for $t\leqq t_0$ and $u(t, x)>0$ for $t>t_0$.

This example shows that, even if a solution of $u_t=Au$ vanishes identically in x for some $t_0>0$, it may not necessarily vanish for $t>t_0$.

However the authors propose a conjecture: *If $u(t_0, x)$ vanishes on any open set, then $u(t, x)=0$ for any x when $t<t_0$.*

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References

[1] N. Aronszajn, Sur l'unicité du prolongement des solution des équation aux derivées partielles elliptiques du second order, C. R. Acad. Sci. Paris, **242** (1956), 723-725.

[2] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. Tech. Report 16, Univ. of Kansas (1956).

3] F. E. Browder, The eigenfunction expansion theorem for the general self-adjoint singular elliptic partial differential operator, I and II, Proc. Nat. Acad. Sci., **40** (1954), 454-459, 459-463.

[4] F. E. Browder, The eigenfunction expansions for formally self-adjoint partial differential operator, I and II, ibid., **42** (1956), 769-771, 870-872.

[5] L. Gâring, Eigenfunction expansion connected with elliptic differential operators, Talfte Skandinaviks Matematikerkongressen, Lund (1953).

[6] S. Itô, Fundamental solutions of parabolic differential equations and boundary value problems, to appear on Jap. J. Math.

[7] L. Schwartz, Théorie des distributions, I et II, Paris, 1950, 1951.

[8] H. Yamabe, " A unique continuation theorem of a diffusion equasion ", to appear on Ann. of Math.