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On Umezawa's criteria for univalence II.

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1. In this note we discuss two of Umezawa's criteria for the univalence of a mapping by an analytic function [4]. We use one of his results, in a slightly generalized form, to extend results due to Rogozhin [3] and the present author [2], and we give a simple proof of the other result.

2. Umezawa proved the following

THEOREM 1 ([4], p. 212). *Let $f(z)$ be analytic in the closure of the finite domain $\mathfrak{D}$ bounded by the simple closed analytic curve Γ, and let $f'(z)\neq 0$ for z on Γ. If the relation*

$$\int_C d(\arg df(z)) > -\pi \tag{1}$$

holds for all arcs C on Γ, and if the relation

$$\int_\Gamma d(\arg df(z)) = 2\pi \tag{2}$$

holds, then $f(z)$ is univalent in $\mathfrak{D}$.

We have given another proof of the preceding result, one in which we show that $f(z)$ maps $\mathfrak{D}$ onto a close-to-convex domain [1]. An examination of our proof shows that $f(z)$ also maps Γ onto a simple closed analytic curve. Indeed, an even closer examination of our (or Umezawa's) proof of Theorem 1, shows that we can establish the following slightly more general result.

THEOREM 2. *Let Γ be a simple closed piece-wise analytic curve with a finite number of corners, and let Γ have well-defined one-sided tangent vectors at those corners. If $f(z)$ is analytic in the closure of the finite domain $\mathfrak{D}$ bounded by Γ, if $f'(z)\neq 0$ there, if there is a positive ε such that*

$$\int_C d(\arg df(z)) \geqq -\pi+\varepsilon$$

holds for all arcs C on Γ, and if (2) holds, then $f(z)$ is univalent in $\mathfrak{D}$, and the image of $\mathfrak{D}$ is a close-to-convex domain.

PROOF. We shall merely sketch the proof. If we write

$$d(\arg df(z)) = d(\arg f'(z)) + d(\arg dz)\,, \tag{3}$$

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then we see that our hypothesis gives us information on the rotation of the tangent vector to the image of Γ under the mapping defined by $w=f(z)$. We now use this last fact, the continuity of the function $\arg f'(z)$ and the geometric character of the domain $\mathfrak{D}$ in order to construct a nested sequence of domains $D_1 \subset D_2 \cdots$, $\bigcup_n D_n = \mathfrak{D}$ such that each D_n is bounded by a simple closed analytic curve Γ_n for which (1) and (2) hold. We now appeal to Theorem 1 to conclude that $f(z)$ is univalent in each D_n, and close-to-convex there. The rest of the result now follows.

We now use the preceding result to prove the following theorem which is a generalization of a result announced by us, at the International Congress held in Amsterdam 1954, for what we called "almost convex" domains [2]; it is essentially a theorem due to Rogozhin [3].

THEOREM 3. *Let φ be fixed, $0 \leqq \varphi < \pi$. Let $\mathfrak{D}$ be a domain in which it is possible to join each pair of distinct points z_1, z_2, by a pair of straight line segments z_1z_3, z_3z_2, lying in $\mathfrak{D}$, such that $\left|\arg \frac{z_3-z_1}{z_2-z_3}\right| \leqq \varphi$. If $f(z)$ is analytic in $\mathfrak{D}$, and if the relation*

$$\left|\int_C d(\arg f'(z))\right| < \pi - \varphi \tag{4}$$

holds for all arcs C in $\mathfrak{D}$, then $f(z)$ is univalent in $\mathfrak{D}$.

PROOF. Let z_1, z_2 be distinct points in $\mathfrak{D}$, and let $z_1z_3z_2$ denote the broken line in $\mathfrak{D}$ joining the points. We construct a narrow band about the broken line. The band is constructed out of six line segments; two are parallel to z_1z_3, two are parallel to z_3z_2, one is perpendicular to z_1z_3, and the sixth is perpendicular to z_3z_2. This band bounds a domain $\mathfrak{D}'$ whose closure is in $\mathfrak{D}$, and whose interior contains the broken line $z_1z_3z_2$. We denote the oriented boundary curve of $\mathfrak{D}'$ by Γ'.

It is geometrically evident that the relation

$$\int_C d(\arg dz) \geqq -\varphi \tag{5}$$

holds for each arc C of Γ'; here we have made use of the hypothesis that $\left|\arg \frac{z_3-z_1}{z_2-z_3}\right| \leqq \varphi$ as well as of the geometry of the figure. From (3), (4) and (5) we obtain

$$\int_C d(\arg df(z)) = \int_C d(\arg f'(z)) + \int_C d(\arg dz) > -\pi$$

for each arc C on Γ'. From (4) we obtain

$$\int_{\Gamma'} d(\arg f'(z)) = 0$$

which combines with (3) to yield

$$\int_{\Gamma'} d(\arg df(z))=2\pi\,,$$

We also note that (4) guarrantees that $f'(z)\neq 0$ in $\mathcal{D}'$. Hence we can appeal to Theorem 2 to conclude that $f(z)$ is univalent in $\mathcal{D}'$. Therefore $f(z_1)\neq f(z_2)$ for $z_1\neq z_2$. The theorem now follows.

The domain $\mathcal{D}'$ in the preceding proof is typical of what we [2] called "almost convex" domains. Domains of that type can be used to prove the following theorem due to Rogozhin.

THEOREM 4 ([3]). *Let k be fixed, $0\leqq k<1$. If $f(z)$ is analytic for $|z|>k$, and if the inequality*

$$\left|\int_C d(\arg f'(z))\right|<\pi-4\arctan k \tag{6}$$

holds for all arcs C lying outside the circle $|z|=k$, then $f(z)$ is univalent for $|z|>1$.

PROOF. With Rogozhin, we consider the circular arc γ through $z=\pm 1$ which is tangent to the circle $|z|=k$ and we note that its radian measure is $4\arctan k$. Let z_1, z_2 be distinct points outside the circle $|z|=1$. It will be clear from our construction that we need only consider the extreme case when z_1, z_2 lie on a line through $z=0$ and on opposite sides of the origin. Without any loss in generality, we take z_1, z_2 to lie on the real axis; one point is on the positive real axis, the other is on the negative real axis. We now join z_1 and z_2 by means of two line segments tangent to γ, such that z_3 lies just above γ. It is clear that

$$\left|\arg\frac{z_3-z_1}{z_2-z_3}\right|<4\arctan k\,. \tag{7}$$

Now we can construct a band domain as in the proof of Theorem 3 and then make use of (6) and Theorem 2 to conclude that $f(z_1)\neq f(z_2)$ when $z_1\neq z_2$. It is also geometrically clear that each pair of points outside the circle $|z|=1$ can also be joined by a broken line outside the circle $|z|=k$ (though not necessarily by lines tangent to γ) such that (7) holds. Again we can use (6), (7), and the Theorem 2 to conclude that different points have different images under $f(z)$. This completes the proof.

It is worth noting that the arcs C in the preceding theorem, as well as the broken line, were permitted to leave the presumed domain of univalence. We also note that we were able to obtain more information than Rogozhin did with his proof. For example, it is clear from our proof that the function $f(z)$ is also univalent in any half-plane not containing the circle $|z|=k$. This last result can also be extended a little.

We now come to another criterion for the univalence of functions defined in certain domains considered by Rogozhin.

Definition. A domain $\mathcal{D}$ is said to be of type R if and only if each pair of distinct points z_1, z_2 can be joined by a circular arc $\gamma(z_1, z_2)$ lying in $\mathcal{D}$. Let $\theta(z_1, z_2)$ denote the greatest lower bound of the radian measures of all arcs $\gamma(z_1, z_2)$ joining z_1, z_2 and let θ denote the least upper bound of all values $\theta(z_1, z_2)$ as the points z_1, z_2 range over $\mathcal{D}$. Similarly, $l(z_1, z_2)$ denotes the greatest lower bound of the lengths of all $\gamma(z_1, z_2)$ joining z_1, z_2 and l denotes the least upper bound of all values $l(z_1, z_2)$ as z_1, z_2 range over $\mathcal{D}$. We call θ and l the parameters of $\mathcal{D}$.

Theorem 5 ([3]). *Let $\mathcal{D}$ be a domain of type R, with parameters θ and l. If $f(z)$ is analytic in $\mathcal{D}$, and if the inequality*

$$\left|\frac{f''(z)}{f'(z)}\right| < \frac{\pi-\theta}{l} \tag{8}$$

holds throughout $\mathcal{D}$, then $f(z)$ is univalent in $\mathcal{D}$.

Proof. We content ourselves with the following sketch of the proof, but only for the case $0 \leqq \theta < \pi$. If z_1, z_2 are distinct points in $\mathcal{D}$, then it is clear that they can be joined by a circular arc $\widehat{z_1 z_2}$ lying in $\mathcal{D}$, such that the radian measure of the arc is less than θ and such that the length of the arc $\widehat{z_1 z_2}$ is less than l. We can construct a band about $\widehat{z_1 z_2}$ made up of two circular arcs close to, and parallel to, $\widehat{z_1 z_2}$, and of two small line segments perpendicular to the extension of $\widehat{z_1 z_2}$ through z_1 and z_2. We use (3) and the geometry of the figure to show that the hypotheses of Theorem 2 are satisfied, just as in the proof of Theorems 3 and 4 above, in order to be able to conclude that $f(z_1) \neq f(z_2)$. This completes the sketch of the proof.

3. We now consider one of Umezawa's criteria for the univalence of functions defined in a doubly connected domain.

Theorem 6 ([4], p. 217). *Let $\mathcal{D}$ be the domain bounded by the simple closed analytic curve Γ and the point z_0 inside Γ. If $f(z)$ is analytic in the closure of $\mathcal{D}$, except for a simple pole at z_0, if $f'(z) \neq 0$ for z on Γ, and if the relations*

$$d(\arg df(z)) < 0\,, \qquad z \text{ on } \Gamma\,, \tag{9}$$

and

$$\int_{\Gamma} d(\arg df(z)) = -2\pi \tag{10}$$

hold, then $f(z)$ is univalent in $\mathcal{D}$.

Proof. Our proof seems a little simpler than Umezawa's geometric proof. Since $f(z)$ has a simple pole, a simple application of the classic integral of the logarithmic derivative shows that $f'(z) \neq 0$ in $\mathcal{D}$; of course we have made use of (10). If we refer back to (3), then we see that the inequality (9) states that the tangent vector to the image of Γ always rotates in the same direction, i. e., it is a curve with negative curvature. Because of (10), we find that the tangent vector to the image of Γ must make only one complete revolution. Hence that image curve is a simple closed analytic convex curve. The univalence now follows from a classic argument. This completes the proof.

A proof of the preceding result could also be given along the lines of our proof of Umezawa's Theorem 1 [1]; this would entail the use of a conformal map of the inside of Γ onto a unit disc such that z_0 is mapped onto the center of the disc.

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Bibliography

[1] M. O. Reade, On Umezawa's criteria for univalence, J. Math. Soc. Japan, **9** (1957), 234-238.

[2] M. O. Reade, Some remarks on schlicht functions, Proc. Int. Cong. Math., Amsterdam, 1954.

[3] V. S. Rogozhin, Two sufficient conditions for the univalence of mappings, Rostov. Gos. Univ. Uc. Zap. Fiz.-Mat. Fak., **32** (1955), 135-137.

[4] T. Umezawa, On the theory of univalent functions, Tôhoku Math. J., **7** (1955), 212-228.