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Note on Random Riemann Sum.

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§1. Let $f\in L_p(a, b)$ and consider the Riemann sum $S_n\equiv S_n(f; t_1,\cdots, t_n)$ for random division points t_i. It was the idea of P. Lévy [1] to define 'generalized integrals' as 'limits' of such Riemann sums. S. Takahashi [2] proved that S_n converges to $\int_0^1 f(t)dt$ with probability 1, for $p>1$ and in probability for $p=1$, if the t_i, $1\leqq i<\infty$, are independent with the uniform distribution on (0,1).

We shall now consider the case $p=1$, $a=0$, $b=\infty$, taking, for each $n\geqq 1$, division-points t_i^n with probability ndt in $(t, t+dt)$ and mutually independently in non-overlapping time-intervals. Rigorously speaking, we adopt the jumping times of the Poisson process with parameter n as the division-points. We then let n tend to ∞.

In §2 we shall prove that if $f\in L_1(0, \infty)$, S_n converges to the Lebesgue integral of f over $(0, \infty)$ in probability as n tends to infinity and in §3 that if $f\in L_1(0, \infty)\cap L_2(0, \infty)$, the subsequence S_{2^n} converges to the same value with probability one.

It should be noticed that our way of picking the division-points by means of Poisson process has made much easier the analytical treatment compared with the case treated by Takahashi [2].

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§2. Let (Ω, B, P) be the probability space on which we define Poisson processes $\{P_n(t, \omega), t\in(0, \infty)\}$, n being the mean value of $P_n(1, \omega)$; it makes no difference here whether these processes are independent or dependent. Let $t_i^n(\omega)$ be the i-th jumping point of the Poisson process $P_n(t, \omega)$.

In this note we shall often use the following well-known fact.

LEMMA. $\{t_{i+1}^n(\omega)-t_i^n(\omega)\}$, $i=0, 1, 2,\cdots$ $(t_0^n(\omega)\equiv 0)$ *are independent random variables with the following probability law*:

$$P\{t_{i+1}^n(\omega)-t_i^n(\omega)<t\}=1-e^{-nt} \quad (t\geqq 0), \tag{2.1}$$

and so we have

$$(2.2)\qquad P\{t_{i+j}^n(\omega)-t_j^n(\omega)\in(t,t+dt)\}=\frac{(nt)^{i-1}e^{-nt}}{(i-1)!}n\,dt$$

Now we shall prove the following

THEOREM 1. *If* $f\in L_1(0,\infty)$, *then* $S_n(\omega)=\sum_{i=1}^{\infty}[f(t_i^n(\omega))\cdot(t_{i+1}^n(\omega)-t_i^n(\omega))]$ *converges to* $\int_0^\infty f(t)dt$ *in probability as n tends to infinity.*

PROOF. Keeping the above Lemma in mind, we have

$$E|S_n(\omega)-\int_0^\infty f(t)dt|\leqq E|\sum_{i=1}^{\infty}\int_{t_i^n(\omega)}^{t_{i+1}^n(\omega)}(f(t_i^n(\omega))-f(t))dt|+E|\int_0^{t_1^n(\omega)}f(t)dt|$$

$$\equiv A_n+B_n.$$

$$A_n\leqq\sum_{i=1}^{\infty}E\left\{\int_{t_i^n(\omega)}^{t_{i+1}^n(\omega)}|f(t_i^n(\omega))-f(t)|\,dt\right\}$$

$$=\sum_{i=1}^{\infty}\int_{s=0}^{\infty}e^{-ns}\frac{(ns)^{i-1}}{(i-1)!}nds\int_{h=0}^{\infty}e^{-nh}ndh\int_{t=s}^{s+h}|f(s)-f(t)|\,dt$$

$$=\int_{s=0}^{\infty}n\,ds\int_{h=0}^{\infty}e^{-nh}ndh\int_{t=0}^{h}|f(t+s)-f(s)|\,dt$$

$$=\int_{s=0}^{\infty}dt\int_{t=0}^{\infty}|f(t+s)-f(s)|\,dt\int_{h=t}^{\infty}n^2e^{-nh}dh$$

$$=\int_{s=0}^{\infty}ds\int_{t=0}^{\infty}|f(t+s)-f(s)|\,ne^{-nt}dt$$

$$=\int_0^\infty\|f_t-f\|ne^{-nt}dt=\int_0^\infty\|f_{t/n}-f\|e^{-t}dt\to0\quad(n\to\infty)$$

where $f_t(s)\equiv f(t+s)$ and $\|\ \|$ means the norm in $L_1(0,\infty)$.

$$B_n\leqq\int_0^\infty e^{-nh}n\,dh\int_0^h|f(t)|\,dt=\int_0^\infty e^{-h}dh\int_0^{h/n}|f(t)|\,dt\to0\quad(n\to\infty)$$

Hence we have $p\text{-}\lim_{n\to\infty}S_n(\omega)=\int_0^\infty f(t)dt.$

§3. THEOREM 2. *For* $f\in L_1(0,\infty)\cap L_2(0,\infty)$, $S_{2^n}(\omega)$ *converges to* $\int_0^\infty f(t)dt$ *with probability one as n tends to infinity.*

PROOF. Let us put $k=\int_0^\infty f^2(t)dt$, $k'=\int_0^\infty|f(t)|\,dt$ and $I=\int_0^\infty f(t)dt$. Making use of the above Lemma, we shall estimate $ES_n^2(\omega)$.

$$ES_n^2(\omega)=E\left(\sum_{i=1}^{\infty}(t_{i+1}^n(\omega)-t_i^n(\omega))^2f^2(t_i^n(\omega))\right)$$

$$+2E\left(\sum_{i=1}^{\infty}\sum_{j=i+2}^{\infty}(t_{j+1}^n(\omega)-t_j^n(\omega))(t_{i+1}^n(\omega)-t_i^n(\omega))f(t_j^n(\omega))f(t_i^n(\omega))\right)$$

$$+2E\left(\sum_{i=1}^{\infty}(t_{i+2}^n(\omega)-t_{i+1}^n(\omega))(t_{i+1}^n(\omega)-t_i^n(\omega))f(t_{i+1}^n(\omega))f(t_i^n(\omega))\right)$$

$$\equiv A_n+B_n+C_n\,.$$

$$A_n=\sum_{i=1}^{\infty}E(t_{i+1}^n(\omega)-t_i^n(\omega))^2f^2(t_i^n(\omega))=\sum_{i=1}^{\infty}\left[\int_0^{\infty}f^2(t)e^{-nt}\frac{(nt)^{i-1}}{(i-1)!}n\,dt\cdot\right.$$

$$\left.\int_0^{\infty}h^2e^{-nh}n\,dh\right]=2k/n\,.$$

$$B_n=2\sum_{i=1}^{\infty}\sum_{j=i+2}^{\infty}\left[\int_0^{\infty}f(t)e^{-nt}\frac{(nt)^{i-1}}{(i-1)}n\,dt\int^{\infty}he^{-nh}n\,dh\cdot\right.$$

$$\left.\int_{t+h}^{\infty}f(s)e^{-n(s-t-h)}\frac{\{n(s-t-h)\}^{j-i-2}}{(j-i-2)!}n\,ds\int_0^{\infty}ue^{-nu}n\,du\right].$$

Since we can exchange the integral and summation by virtue of $f\in L_1(0,\infty)$, we have

$$B_n=2\int_0^{\infty}f(t)n\,dt\int_0^{\infty}h\,e^{-nh}n\,dh\int_{t+h}^{\infty}f(s)ds$$

$$=2\int_0^{\infty}f(t)dt\int_t^{\infty}f(s)ds-2\int_0^{\infty}f(t)n\,dt\int_0^{\infty}he^{-nh}n\,dh\int_t^{t+h}f(s)ds=B_n'-B_n''\,.$$

$$B_n'=\left(\int_0^{\infty}f(t)dt\right)^2=I^2\,.$$

By Schwarz' inequality we have $|\int_t^{t+h}f(s)ds|\leqq(kh)^{1/2}$, so that we get

$$|B_n''|=2|\int_0^{\infty}f(t)n\,dt\int_0^{\infty}he^{-nh}n\,dh\int_t^{t+h}f(s)ds|\leqq 2k^{1/2}k'\int_0^{\infty}h^{3/2}n^2e^{-nh}dh=O(n^{-1/2})\,.$$

Therefore we have

$$B_n=I^2+O(n^{-1/2})\,.$$

$$C_n=2\sum_{i=1}^{\infty}\int_0^{\infty}h'e^{-nh'}n\,dh'\int_0^{\infty}f(t)e^{-nt}\frac{(nt)^{i-1}}{(i-1)!}n\,dt\int_0^{\infty}hf(t+h)e^{-nh}n\,dh$$

$$=\frac{2}{n}\int_0^{\infty}f(t)n\,dt\int_0^{\infty}hf(t+h)e^{-nh}n\,dh$$

$$=\frac{2}{n}\int_0^\infty f(t)dt\int_0^\infty hf(t+h/n)e^{-h}dh$$

$$=\frac{2}{n}\int_0^\infty\int_0^\infty f(t)f(t+h/n)dt\, h\, e^{-h}dh$$

Using Schwarz' inequality, we have

$$|C_n|\leqq\frac{2}{n}\int_0^\infty\left(\int_0^\infty f^2(t)dt\right)^{1/2}\left(\int_0^\infty f^2(t+h/n)dt\right)^{1/2}he^{-h}dh\leqq 2k/n\,.$$

Summing up the above estimation, we obtain

$$ES_n^2(\omega)=I^2+O(n^{-1/2})\,.$$

On the other hand

$$ES_n(\omega)=\sum_{i=1}^\infty\int_0^\infty f(t)e^{-nt}\frac{(nt)^{i-1}}{(i-1)\,!}\,n\,dt\int_0^\infty he^{-nh}n\,dh$$

$$=n\int_0^\infty f(t)dt\int_0^\infty he^{-nh}n\,dh$$

$$=\int_0^\infty f(t)dt=I\,.$$

Thus we have

$$E(S_n(\omega)-I)^2=ES_n^2(\omega)-I^2=O(n^{-1/2})\,,$$

so that

$$E\left\{\sum_{n=1}^\infty(S_{2^n}(\omega)-I)^2\right\}<\infty\,,$$

which implies

$$\sum_{n=1}^\infty(S_{2^n}(\omega)-I)^2<\infty$$

and so $S_{2^n}(\omega)\rightarrow I$ with probability one.

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References

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