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# Cohomology mod $p$ of the $p$-fold symmetric products of spheres.

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## 1. Introduction

In this note we shall study the cohomology modulo $p$ of the $p$-fold symmetric product $\mathfrak{S}_p(S^n)$ of an $n$-sphere $S^n$ $(n \geqq 1)$, where $p$ is an odd prime. Let $Z_p$ denote the group of integers modulo $p$. As usual, denote by $\mathcal{P}^s$ the reduced $p$-th power, by $\Delta_p$ the Bockstein homomorphism and by $\smile$ the cup product. Then our main result is stated as follows:

THEOREM 1. *We have $H^n(\mathfrak{S}_p(S^n); Z_p) \approx Z_p$. Denote by h a generator of $H^n(\mathfrak{S}_p(S^n); Z_p)$. Then a set of generators for the cohomology group $H^*(\mathfrak{S}_p(S^n); Z_p)$ can be formed with all elements of the following four types:*

(I) $\mathbf{1}$,[1] (II) $\mathcal{P}^s(h)$ $(0 \leqq s \leqq n/2)$, (III) $\Delta_p\mathcal{P}^s(h)$ $(1 \leqq s < n/2)$,

(IV) $h^q$ $(2 \leqq q < p)$, *where* $h^q = h \smile h \smile \cdots \smile h$ ($q$ *factors*).

*Define $B(n, p)$ as a set of all elements of the above types* (I)$\sim$(IV) *or* (I)$\sim$(III) *according as n is even or odd. Then $B(n, p)$ is linearly independent.*

We shall also calculate the cup products and the reduced powers in $H^*(\mathfrak{S}_p(S^n); Z_p)$ (§ 5).

Our proof depends on the results about the cohomology of the $p$-fold cyclic product $\mathfrak{Z}_p(S^n)$ of $S^n$, which I have obtained in the paper [4],[2] together with the technique which was used by Steenrod to prove Theorem 4.8 in his paper [7].

Throughout this paper, the coefficient group is always $Z_p$, and hence we shall hereafter omit to mention it.

## 2. Symmetric, cyclic, cartesian products

Denote by $\mathfrak{S}_p$ the symmetric group of the letters $1, 2, \cdots, p$. Let $t \in \mathfrak{S}_p$ be an element defined by $t(j) = j+1 \bmod p$ $(j = 1, 2, \cdots, p)$, and

1) $\mathbf{1}$ denotes the unit cohomology class.

2) Numbers in square brackets refer to the bibliography at the end of the paper.

denote by $\mathfrak{Z}_p$ the subgroup generated by $t$.

Given a Hausdorff space $K$, each element $\alpha \in \mathfrak{S}_p$ yields a transformation $\bar{\alpha}$ in the $p$-fold *cartesian product* $\mathfrak{X}_p(K)$ of $K$:

$$\bar{\alpha}(x_1 \times x_2 \times \cdots \times x_p) = x_{\alpha(1)} \times x_{\alpha(2)} \times \cdots \times x_{\alpha(p)} \quad (x_i \in K) .$$

Thus $\mathfrak{Z}_p$ and $\mathfrak{S}_p$ may be regarded as transformation groups acting on $\mathfrak{X}_p(K)$. By definition, the $p$-fold *cyclic product* $\mathfrak{Z}_p(K)$ (or *symmetric product* $\mathfrak{S}_p(K)$) of $K$ is the orbit space[3] $O(\mathfrak{X}_p(K), \mathfrak{Z}_p)$ (or $O(\mathfrak{X}_p(K), \mathfrak{S}_p)$) over $\mathfrak{X}_p(K)$ relative to $\mathfrak{Z}_p$ (or $\mathfrak{S}_p$).

We assume in the following that $K$ is a finite simplicial complex. Then, as was shown in [3, 4], $\mathfrak{X}_p(K)$ may be regarded as a finite simplicial complex having the following property: every transformation $\bar{\alpha}$ is simplicial, and if a simplex of $\mathfrak{X}_p(K)$ is mapped onto itself by $\bar{\alpha}$ then it remains point-wise fixed under $\bar{\alpha}$. Therefore $\mathfrak{Z}_p$ (or $\mathfrak{S}_p$) operates naturally on the alternating cochain complex $\{C^r(\mathfrak{X}_p(K)), \delta\}$. The *special cohomology group* ${}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K))$ (or ${}^{\mathfrak{S}_p^{-1}}H(\mathfrak{X}_p(K))$) is defined as the cohomology group of the cochain complex $\{C^r(\mathfrak{X}_p(K))^{\mathfrak{Z}_p}, \delta\}$ (or $\{C^r(\mathfrak{X}_p(K))^{\mathfrak{S}_p}, \delta\}$).[4] Then, according to Liao [2], the following isomorphisms hold:

$$\text{(2.1)} \qquad \begin{aligned} \mathrm{I}_1^* &: \; H^r(\mathfrak{Z}_p(K)) \approx {}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K)) , \\ \mathrm{I}_2^* &: \; H^r(\mathfrak{S}_p(K)) \approx {}^{\mathfrak{S}_p^{-1}}H^r(\mathfrak{X}_p(K)) , \end{aligned}$$

where $\mathrm{I}_1^*$ and $\mathrm{I}_2^*$ are induced by the projections

$$\pi_1 : \mathfrak{X}_p(K) \to \mathfrak{Z}_p(K), \quad \pi_2 : \mathfrak{X}_p(K) \to \mathfrak{S}_p(K)$$

respectively.

Let us now consider the *inclusion homomorphisms*

$$\begin{aligned} i_1 = i(0, \mathfrak{Z}_p) &: \; C^r(\mathfrak{X}_p(K))^{\mathfrak{Z}_p} \subset C^r(\mathfrak{X}_p(K)) , \\ i_2 = i(0, \mathfrak{S}_p) &: \; C^r(\mathfrak{X}_p(K))^{\mathfrak{S}_p} \subset C^r(\mathfrak{X}_p(K)) , \\ i_3 = i(\mathfrak{Z}_p, \mathfrak{S}_p) &: \; C^r(\mathfrak{X}_p(K))^{\mathfrak{S}_p} \subset C^r(\mathfrak{X}_p(K))^{\mathfrak{Z}_p} , \end{aligned}$$

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3) Let $Y$ be a Hausdorff space on which a group $\Gamma$ acts. Then the orbit space $O(Y, \Gamma)$ over $Y$ relative to $\Gamma$ is defined as a space obtained from $Y$ by identifying each point $y \in Y$ with its image $\gamma(y)$ $(\gamma \in \Gamma)$.

4) Let $\Gamma$ be a group operating on an additive group $A$. Then we denote by $A^\Gamma$ the subgroup of $A$ which consists of all $a \in A$ for which $\gamma(a) = a$ for all $\gamma \in \Gamma$.

and the *transfer homomorphisms*

$$T_1=T(\mathfrak{Z}_p, 0) \ : \ C^r(\mathfrak{X}_p(K)) \ \to \ C^r(\mathfrak{X}_p(K))^{\mathfrak{Z}_p},$$
$$T_2=T(\mathfrak{S}_p, 0) \ : \ C^r(\mathfrak{X}_p(K)) \ \to \ C^r(\mathfrak{X}_p(K))^{\mathfrak{S}_p},$$
$$T_3=T(\mathfrak{S}_p, \mathfrak{Z}_p): \ C^r(\mathfrak{X}_p(K))^{\mathfrak{Z}_p} \ \to \ C^r(\mathfrak{X}_p(K))^{\mathfrak{S}_p}\text{[5]}.$$

Denote by $\alpha^{\#}: C^r(\mathfrak{X}_p(K))\to C^r(\mathfrak{X}_p(K))$ the cochain map induced by $\bar{\alpha}$. Then it is easily seen that

$$(2.2)\qquad \begin{aligned} & i_1T_1=\sum_{i=0}^{p-1} t^{i\#}, \qquad i_2T_2=\sum_{\alpha\in\mathfrak{S}_p}\alpha^{\#}, \\ & T_3T_1=T_2, \qquad i_1i_3=i_2, \\ & T_3i_3=(p-1)\,!=-1. \end{aligned}$$

The cochain maps $i_1$, $i_2$ and $i_3$ induce respectively the homomorphisms $i_1^*: {}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K))\to H^r(\mathfrak{X}_p(K))$, $i_2^*: {}^{\mathfrak{S}_p^{-1}}H^r(\mathfrak{X}_p(K))\to H^r(\mathfrak{X}_p(K))$ and $i_3^*: {}^{\mathfrak{S}_p^{-1}}H^r(\mathfrak{X}_p(K))\to {}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K))$. The following relations are obvious.

$$(2.3)\qquad i_1^*\mathrm{I}_1^*=\pi_1^*, \quad i_2^*\mathrm{I}_2^*=\pi_2^*, \quad \mathrm{I}_1^*\pi_3^*=i_3^*\mathrm{I}_2^*,$$

where

$$\pi_3 \ : \ \mathfrak{Z}_p(K)\to\mathfrak{S}_p(K)$$

is the projection, and $\pi_i^*$ $(i=1, 2, 3)$ denotes the homomorphism induced by $\pi_i$.

Using the homomorphisms $T_1^*: H^r(\mathfrak{X}_p(K))\to {}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K))$, $T_2^*: H^r(\mathfrak{X}_p(K))\to {}^{\mathfrak{S}_p^{-1}}H^r(\mathfrak{X}_p(K))$ and $T_3^*: {}^{\mathfrak{Z}_p^{-1}}H^r(\mathfrak{X}_p(K))\to {}^{\mathfrak{S}_p^{-1}}H^r(\mathfrak{X}_p(K))$ induced by $T_1$, $T_2$ and $T_3$ respectively, define homomorphisms $\phi_i{}^*$ $(i=1, 2, 3)$ as follows:

$$\phi_1^*=\mathrm{I}_1^{*-1}T_1^* \quad : \ H^r(\mathfrak{X}_p(K)) \to H^r(\mathfrak{Z}_p(K)),$$
$$\phi_2^*=\mathrm{I}_2^{*-1}T_2^* \quad : \ H^r(\mathfrak{X}_p(K)) \to H^r(\mathfrak{S}_p(K)),$$
$$\phi_3^*=\mathrm{I}_2^{*-1}T_3^*\mathrm{I}_1^* \quad : \ H^r(\mathfrak{Z}_p(K)) \to H^r(\mathfrak{S}_p(K)).$$

Then it follows from (2.2) and (2.3) that

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5) As for the transfer homomorphism, see p. 254 of the book [2] of Cartan-Eilenberg. 0 denotes the zero group.

$$
(2.4)\qquad
\begin{aligned}
&\pi_1^*\phi_1^* = \sum_{i=0}^{p-1} t^{i*}, \qquad \pi_2^*\phi_2^* = \sum_{\alpha\in\mathfrak{S}_p} \alpha^*, \\
&\phi_3^*\phi_1^* = \phi_2^*, \qquad \pi_1^*\pi_3^* = \pi_2^*, \qquad \phi_3^*\pi_3^* = -1,
\end{aligned}
$$

where $\alpha^*: H^r(\mathfrak{X}_p(K)) \to H^r(\mathfrak{X}_p(K))$ denotes the homomorphism induced by $\bar{\alpha}$. The last relation leads to

PROPOSITION 1. *$\pi_3^*$ is isomorphic into and $\phi_3^*$ is onto.*

For each $k$ $(1 \leqq k < p)$, define $\theta_k \in \mathfrak{S}_p$ by $\theta_k(j) = kj \bmod p$ $(j = 1, 2, \cdots, p)$. Then we have

$$
(2.5)\qquad \bar{\theta}_k \bar{t}^i = \bar{t}^{ik} \bar{\theta}_k .
$$

Hence $\bar{\theta}_k$ defines a map

$$
\tilde{\theta}_k \ : \ \mathfrak{Z}_p(K) \to \mathfrak{Z}_p(K)
$$

such that $\pi_1\bar{\theta}_k = \tilde{\theta}_k\pi_1$. Since $\pi_3\tilde{\theta}_k = \pi_3$, the commutativity holds in the diagram

$$
\begin{array}{ccc}
H^r(\mathfrak{Z}_p(K)) & \xrightarrow{\ \tilde{\theta}_k^*\ } & H^r(\mathfrak{Z}_p(K)) \\
 & {\scriptstyle \pi_3^*}\nwarrow \qquad \nearrow{\scriptstyle \pi_3^*} & \\
 & H^r(\mathfrak{S}_p(K)) &
\end{array}
$$

Namely we have $(1-\tilde{\theta}_k^*)\ \pi_3^* = 0$, and this proves the following[6]:

LEMMA 1. *The image of $\pi_3^*$ is contained in the kernel of $(1-\tilde{\theta}_k^*)$ for every $k$ $(1 \leqq k < p)$.*

Write

$$
\sigma = \sum_{i=0}^{p-1} t^{i\#} \qquad \tau = 1 - t^{\#}
$$

as in [3], and consider the cochain map $\theta_k^{\#}$ induced by $\bar{\theta}_k$. Then it follows directly from (2.5) that we have

$$
(2.6)\qquad
\begin{aligned}
&\theta_k^{\#}\sigma = \sigma\theta_k^{\#}, \quad \tau\theta_k^{\#} = \theta_k^{\#}(1 + t^{\#} + \cdots + t^{k-1\#})\tau, \\
&\theta_k^{\#}\tau = \tau(1 + t^{\#} + \cdots + t^{i-1\#})\theta_k^{\#} \qquad (ik = 1 \bmod p,\ 1 \leqq i < p).
\end{aligned}
$$

6) Compare with the proof of Theorem 4.8 in [7].

### 3. The homomorphisms $E_q$ and $\hat{\theta}_k^*$

Let $\mathfrak{D}_p(K)=\{x\times x\times\cdots\times x \mid x\in K\}$ denote the diagonal of $\mathfrak{X}_p(K)$, and $\mathfrak{d}_p(K)$ the image of $\mathfrak{D}_p(K)$ by the projection $\pi_1$. To study $H^*(\mathfrak{Z}_p(K))$, I have defined in [4] the homomorphisms

$$E_q \ : \ H^r(K)\to H^{r+q}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$$

for all $q, r>0$. The construction of $E_q$ is based on the following diagram:

$$E_{2\alpha+1}:\ H^r(K)\underset{\approx}{\overset{d_1^*}{\longleftarrow}} H^r(\mathfrak{d}_p(K))\overset{\delta^*}{\longrightarrow} H^{r+1}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$$

$$\overset{\mu^\alpha}{\longrightarrow} H^{r+2\alpha+1}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K)) \qquad \text{for } q=2\alpha+1,$$

$$E_{2\alpha+2}:\ H^r(K)\underset{\approx}{\overset{d_1^*}{\longleftarrow}} H^r(\mathfrak{d}_p(K))\overset{\delta^*}{\longrightarrow} H^{r+1}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$$

$$\overset{\nu}{\longrightarrow} H^{r+2}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\overset{\mu^\alpha}{\longrightarrow} H^{r+2\alpha+2}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$$

$$\text{for } q=2\alpha+2.$$

We now explain the meaning of the homomorphisms which have not yet been defined.

$d_1^*$ is the homomorphism induced by the composite $d_1$ of the diagonal map $d_0$ and the projection $\pi_1$ (i. e. $d_1(x)=\pi_1(x\times x\times\cdots\times x)$).

$\delta^*$ is the coboundary homomorphism in the cohomology exact sequence of $(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$.

For $\rho=\sigma$ and $\tau$, define ${}^\rho H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))$ as the cohomology group of the cochain complex $\{{}_\rho C^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K)),\ \delta\}$. It is easily seen [4] that

$$\mathrm{I}^*:\ H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\approx {}^\sigma H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K)),$$

where $\mathrm{I}^*$ is induced by the projection $\pi_1$. Write $\bar\rho=\tau$ or $\sigma$ according as $\rho=\sigma$ or $\tau$, and consider homomorphisms

$$r_\rho\ :\ {}^\rho H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))\to {}^{\bar\rho}H^{r+1}(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K)),$$

$$\psi_\sigma:\ {}^\tau H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))\to {}^\sigma H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))$$

defined by

$$r_\rho(\{\rho u\})=\{\delta u\},\qquad \psi_\sigma(\{\tau u\})=\{\sigma u\},$$

where $u\in C^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))$ and $\{v\}$ denotes the appropriate cohomology class containing the cocycle $v$. Then we write

$$\mu=\mathrm{I}^{*-1}\gamma_\tau\gamma_\sigma\mathrm{I}^*:\ H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\to H^{r+2}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K)),$$
$$\nu=\mathrm{I}^{*-1}\psi_\sigma\gamma_\sigma\mathrm{I}^*:\ H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\to H^{r+1}(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K)),$$
$$\mu^\alpha=\mu\mu\cdots\mu\quad(\alpha \text{ factors}).$$

Since $\tilde{\theta}_k(\mathfrak{d}_p(K))\subset\mathfrak{d}_p(K)$, the map $\tilde{\theta}_k$ induces the homomorphism

$$\tilde{\theta}_k^*:\ H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\to H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K)).$$

PROPOSITION 2. $\tilde{\theta}_k^*E_{2\alpha+1}=k^\alpha E_{2\alpha+1}$, $\tilde{\theta}_k^*E_{2\alpha+2}=k^{\alpha+1}E_{2\alpha+2}$

It is easily seen that this Proposition is a consequence of the following:

LEMMA 2. $\tilde{\theta}_k^*\mu=k\mu\tilde{\theta}_k^*$, $\tilde{\theta}_k^*\nu=k\nu\tilde{\theta}_k^*$, $\tilde{\theta}_k^*\delta^*=\delta^*$.

PROOF. Since $\tilde{\theta}_k$ is the identity on $\mathfrak{d}_p(K)$, the last relation is obvious by the well-known property of $\delta^*$. The first two are proved as follows:

It follows from (2.6) that $\theta_k{}^\#$ maps $\rho C^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))$ in itself. Therefore $\theta_k{}^\#$ induces the homomorphism

$$\theta_{k,\rho}:\ {}^\rho H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K))\to{}^\rho H^r(\mathfrak{X}_p(K),\ \mathfrak{D}_p(K)).$$

Similarly $t^\#$ induces the homomorphism $t_\rho$. (Note that $t_\sigma=$the identity.) It follows easily from (2.6) and the definitions of $\gamma_\rho, \psi_\sigma$ that

$$\theta_{k,\sigma}\gamma_\tau=\gamma_\tau\theta_{k,\tau}(1+t_\tau+\cdots+t_\tau^{k-1}),\quad \theta_{k,\tau}\gamma_\sigma=\gamma_\sigma\theta_{k,\sigma},$$
$$\theta_{k,\sigma}\psi_\sigma=\psi_\sigma\theta_{k,\tau}(1+t_\tau+\cdots+t_\tau^{k-1}),\quad t_\tau\gamma_\sigma=\gamma_\sigma.$$

On the other hand, it is obvious that

$$\theta_{k,\sigma}\mathrm{I}^*=\mathrm{I}^*\tilde{\theta}_k^*.$$

Therefore we have

$$\tilde{\theta}_k^*\mu=\tilde{\theta}_k^*\mathrm{I}^{*-1}\gamma_\tau\gamma_\sigma\mathrm{I}^*=\mathrm{I}^{*-1}\theta_{k,\sigma}\gamma_\tau\gamma_\sigma\mathrm{I}^*=\mathrm{I}^{*-1}\gamma_\tau\theta_{k,\tau}(1+t_\tau+\cdots+t_\tau^{k-1})\gamma_\sigma\mathrm{I}^*$$
$$=k\mathrm{I}^{*-1}\gamma_\tau\theta_{k,\tau}\gamma_\sigma\mathrm{I}^*=k\mathrm{I}^{*-1}\gamma_\tau\gamma_\sigma\theta_{k,\sigma}\mathrm{I}^*=k\mathrm{I}^{*-1}\gamma_\tau\gamma_\sigma\mathrm{I}^*\tilde{\theta}_k^*=k\mu\tilde{\theta}_k^*,$$

and similarly $\tilde{\theta}_k^*\nu=k\nu\tilde{\theta}_k^*$. q. e. d.

Let $j^*:\ H^r(\mathfrak{Z}_p(K),\ \mathfrak{d}_p(K))\to H^r(\mathfrak{Z}_p(K))$ denote the injection homomorphism. Then we have

PROPOSITION 3. *Let* $q\neq 2s(p-1)+\varepsilon$ ($\varepsilon=0,1$ ; $s=0,\ \pm1,\ \pm2,\cdots$).

*Then if* $j^*E_q(c)$ $(c\in H^r(K))$ *is in the image of* $\pi_3^*: H^*(\mathfrak{S}_p(K))\rightarrow H^*(\mathfrak{Z}_p(K))$, *then* $j^*E_q(c)=0$.

PROOF. Let $j^*E_q(c)\in\pi_3^*H^*(\mathfrak{S}_q(K))$. Then it follows from Lemma 1 and Proposition 2 that

$$0=(1-\tilde{\theta}_k^*)j^*E_q(c)=j^*E_q(c)-j^*\tilde{\theta}_k^*E_q(c)$$
$$=j^*E_q(c)-k^\alpha j^*E_q(c)=(1-k^\alpha)j^*E_q(c)\,,$$

where $q=2\alpha+\varepsilon$ and $1\leqq k<p$.

Assume now that $j^*E_q(c)\neq 0$. Then $k^\alpha=1 \bmod p$. Take as $k$ especially a primitive root of the prime $p$. Then we see that $\alpha$ is a multiple of $p-1$. Thus $q=2s(p-1)+\varepsilon$, which contradicts the assumption of the Proposition. Therefore $j^*E_q(c)=0$. q. e. d.

## 4. Proof of Theorem 1.

From now on we consider only the case $K=S^n$, and give a proof of Theorem 1. First we recall some results of [4] about $H^*(\mathfrak{Z}_p(S^n))$.

Let $e_n$ denote a generator of $H^n(S^n)$. Write

$$a_r=j^*E_{r-n}(e_n)\in H^r(\mathfrak{Z}_p(S^n)) \qquad (n+2\leqq r\leqq pn)\,.$$

For a set $\{m_1, m_2,\cdots, m_q\}$ of $q$ $(1\leqq q<p)$ different integers such that $1\leqq m_i\leqq p$ $(i=1,2,\cdots,q)$, define

$$g_{nq}(m_1, m_2,\cdots, m_q)\in H^{nq}(\mathfrak{Z}_p(S^n))$$

by

$$\phi_1^*(c_1\times c_2\times\cdots\times c_p) \quad (c_1\times c_2\times\cdots\times c_p\in H^{nq}(\mathfrak{X}_p(S^n)))\,,$$

(4.1) $$c_i=e_n \qquad \text{if } i=m_1, m_2,\cdots, m_q\,,$$
$$=1 \qquad \text{otherwise.}$$

The following has been proved in [4]. (See Theorems **(3.1)**, **(13.2)** and **(11.1)**.)

PROPOSITION 4. *The above* $a_r$ *and* $g_{nq}$ $(m_1, m_2,\cdots, m_q)$ *are not zero, and a set of generators for* $H^*(\mathfrak{Z}_p(S^n))$ *can be formed with all such elements, together with the unit cohomology class* 1; $H^n(\mathfrak{Z}_p(S^n))\approx Z_p$ *is generated by* $g_n(1)=\cdots=g_n(p)$ $(=g)$; *The kernel of* $\pi_1^*$ *is generated by all* $a_r$ $(n+2\leqq r<pn)$; *and we have*

(4.2) $$\mathcal{P}^s(g)=(-1)^{s+1}a_{n+2s(p-1)}\,, \quad \Delta_p a_{n+2\alpha}=a_{n+2\alpha+1}\,.$$

Since $\pi_2^* = \pi_1^* \pi_3^*$, the following is clear from (4.2) and Proposition 3.

LEMMA 3. *If $\pi_2^*(b)=0$ $(b \in H^*(\mathfrak{S}_p(S^n)))$, then $\pi_3^*(b)$ belongs to a subgroup $F$ of $H^*(\mathfrak{Z}_p(S^n))$ generated by a set of all elements of type $a_{n+2s(p-1)+\varepsilon}$ $(\varepsilon=0,1)$.*

Now we proceed to prove Theorem 1.

It is known [3] that $H^r(\mathfrak{S}_p(K))$ has a subgroup isomorphic with $H^r(K)$. Hence it follows that $H^n(\mathfrak{S}_p(S^n)) \neq 0$. On the other hand, $\phi_3^*$ is onto by Proposition 1. Therefore it follows from Proposition 4 that $\phi_3^*: H^n(\mathfrak{Z}_p(S^n)) \approx H^n(\mathfrak{S}_p(S^n))$. Namely we have

$$H^n(\mathfrak{S}_p(S^n)) \approx Z_p \,, \text{[7]} \tag{4.3}$$

which is the first part of Theorem 1.

Write

$$h = \phi_3^*(g) \neq 0\,, \tag{4.4}$$

then $h$ is a generator of $H^n(\mathfrak{S}_p(S^n))$ and

$$g = -\pi_3^*(h) \tag{4.5}$$

is obvious by (2.4).

Using the naturality of $\mathcal{P}^s$ and $\Delta_p$, it follows from (4.5) and (4.2) that

$$\begin{aligned} &\pi_3^* \mathcal{P}^s(h) = (-1)^s a_{n+2s(p-1)} \neq 0\,, \\ &\pi_3^* \Delta_p \mathcal{P}^s(h) = (-1)^s a_{n+2s(p-1)+1} \neq 0 \qquad (s>0)\,. \end{aligned} \tag{4.6}$$

By (2.4) and (4.5), we have

$$\begin{aligned} \pi_2^*(h) &= \pi_1^* \pi_3^*(h) = -\pi_1^*(g) = -\pi_1^* \phi_1^*(e_n \times 1 \times \cdots \times 1) \\ &= -\sigma^*(e_n \times 1 \times \cdots \times 1) \qquad (\sigma^* = \sum_{i=0}^{p-1} t^{i*})\,, \end{aligned}$$

and, by the product formulas of $\mathcal{P}^s$ and $\smile$ [7], we obtain readily

$$\mathcal{P}^s(e_n \times 1 \times \cdots \times 1) = 0 \qquad \text{for} \qquad s>0\,;$$

$$(\sigma^*(e_n \times 1 \times \cdots \times 1))^q = q!\ (\textstyle\sum e_n \times \cdots \times e_n \times 1 \times \cdots \times 1) \text{ for even } n\,,$$

where the summation is extended over all elements $c_1 \times c_2 \times \cdots \times c_p$ such that, for a subset $\{i_1 < i_2 < \cdots < i_q\}$ of $\{1, 2, \cdots, p\}$, $c_j = e_n$ if $j = i_1$,

7) I have proved in [5] a generalization of this fact: If $K$ is $(r-1)$-connected, then the *integral* cohomology group $H^q(\mathfrak{S}_p(K); Z)$ is isomorphic with $H^q(K; Z)$ for $q \leqq r+1$.

$i_2, \cdots, i_q$ and $=1$ otherwise. Consequently, using the naturality of $\mathcal{P}^s$, $\Delta_p$ and $\smile$, it follows that

$$\pi_2^*\mathcal{P}^s(h)=0, \quad \pi_2^*\Delta_p\mathcal{P}^s(h)=0 \qquad (s>0);$$

(4.7)

$$\pi_2^*(h^q)=(-1)^q q!\,(\sum e_n\times\cdots\times e_n\times 1\times\cdots\times 1) \text{ for even } n.$$

Now it is easily seen from (4.6) and (4.7) that the set $B(n, p)$ is linearly independent. Thus we have proved the third part of Theorem 1.

It follows from Proposition 3 that the image group $\pi_3^*\phi_3^*H^*(\mathfrak{Z}_p(S^n))$ is contained in a subgroup generated by all elements of the following types:

$$1,\ a_{n+2s(p-1)}\ (1\leqq s\leqq n/2),\quad a_{n+2s(p-1)+1}\ (1\leqq s<n/2)$$
$$g_{nq}(m_1, m_2, \cdots, m_q) \qquad (1\leqq q<p).$$

Therefore we obtain by Proposition 1 and (2.4) that the image group $H^*(\mathfrak{S}_p(S^n))=\phi_3^*H^*(\mathfrak{Z}_p(S^n))=\phi_3^*\pi_3^*\phi_3^*H^*(\mathfrak{Z}_p(S^n))$ is contained in a subgroup generated by all elements of the following types:

$$1,\quad \phi_3^*a_{n+2s(p-1)}\ (1\leqq s\leqq n/2),\quad \phi_3^*a_{n+2s(p-1)+1}\ (1\leqq s<n/2),$$
$$\phi_3^*g_{nq}(m_1, m_2, \cdots, m_q)\ (1\leqq q<p).$$

However it follows from (4.2), (4.5), (2.4) and (4.4) that

$$\phi_3^*a_{n+2s(p-1)}=(-1)^{s+1}\phi_3^*\mathcal{P}^s(g)=(-1)^s\phi_3^*\mathcal{P}^s\pi_3^*(h)$$
$$=(-1)^s\phi_3^*\pi_3^*\mathcal{P}^s(h)=(-1)^{s+1}\mathcal{P}^s(h),$$

(4.8)

$$\phi_3^*a_{n+2s(p-1)+1}=(-1)^{s+1}\phi_3^*\Delta_p\mathcal{P}^s(g)=(-1)^s\phi_3^*\Delta_p\mathcal{P}^s\pi_3^*(h)$$
$$=(-1)^s\phi_3^*\pi_3^*\Delta_p\mathcal{P}^s(h)=(-1)^{s+1}\Delta_p\mathcal{P}^s(h),$$
$$\phi_3^*g_n(m)=\phi_3^*g=h=\mathcal{P}^0(h).$$

Therefore, to complete the proof of the second part of Theorem 1, it is sufficient to show that $\phi_3^*g_{nq}(m_1, m_2, \cdots, m_q)$ $(2\leqq q<p)$ is a linear combination of the elements described in Theorem 1.

Let $g_{nq}(m_1, m_2, \cdots, m_q)=\phi_1^*(c_1\times c_2\times\cdots\times c_p)$ as in (4.1). Then it follows from (2.4) that

$$\pi_2^*\phi_3^*g_{nq}(m_1, m_2, \cdots, m_q)$$
$$=\pi_2^*\phi_2^*(c_1\times c_2\times\cdots\times c_p)$$

$$= \sum_{\alpha \in \mathfrak{S}_p} \alpha^*(c_1 \times c_2 \times \cdots \times c_p)$$

$$= q!\,(p-q)!\,(\sum e_n \times \cdots \times e_n \times 1 \times \cdots \times 1) \quad \text{for even } n,$$

$$= 0 \quad \text{for odd } n.$$

Hence, together with (4.7), we obtain

$$\pi_2^* \phi_3^* g_{nq}(m_1, m_2, \cdots, m_q)$$

$$= (-1)^q (p-q)!\, \pi_2^*(h^q) \qquad \text{for even } n,$$

$$= 0 \qquad \text{for odd } n.$$

Since $h^q = 0$ if $n$ is odd, it follows that

$$\pi_2^*(\phi_3^* g_{nq}(m_1, m_2, \cdots, m_q) - (-1)^q (p-q)!\, h^q) = 0$$

for any $n$. This and Lemma 3 show that $\pi_3^*(\phi_3^* g_{nq}(m_1, m_2, \cdots, m_q) - (-1)^q (p-q)!\, h^q)$ is contained in the subgroup $F$. Therefore, by (2.4) and (4.8), we see that $\phi_3^* g_{nq}(m_1, m_2, \cdots, m_q) - (-1)^q (p-q)!\, h^q$ is contained in the subgroup $\phi_3^* F$ of $H^*(\mathfrak{S}_p(S^n))$ generated by all elements of types $\mathcal{P}^s(h)$, $\Delta_p \mathcal{P}^s(h)$. This completes the proof of the second part.

## 5. Supplement

We have

THEOREM 2. *For any $s, t > 0$, the following hold:*

$$h \smile \mathcal{P}^t(h) = 0, \quad h \smile \Delta_p \mathcal{P}^t(h) = 0,$$

$$\mathcal{P}^s(h) \smile \mathcal{P}^t(h) = 0, \quad \mathcal{P}^s(h) \smile \Delta_p \mathcal{P}^t(h) = 0,$$

$$\Delta_p \mathcal{P}^s(h) \smile \Delta_p \mathcal{P}^t(h) = 0,$$

$$\mathcal{P}^s \mathcal{P}^t(h) = (-1)^s {}_{t(p-1)-1}C_s \mathcal{P}^{s+t}(h),$$

$$\mathcal{P}^s \Delta_p \mathcal{P}^t(h) = (-1)^s {}_{t(p-1)}C_s \Delta_p \mathcal{P}^{s+t}(h).$$

PROOF. These are obtained by the naturality of $\smile$, $\Delta_p$ and $\mathcal{P}^s$ from the corresponding results in $H^*(\mathfrak{Z}_p(S^n))$ which I have proved in [4]. (See Theorems (11.8) and (13.2).)

Since the proof is similar, we shall only give an account of that of the last relation. It follows from (4.6) and Theorem (13.2) in [4] that

$$\pi_3^* \mathcal{P}^s \Delta_p \mathcal{P}^t(h) = (-1)^t \mathcal{P}^s a_{n+2t(p-1)+1}$$

$$=(-1)^t{}_{t(p-1)}C_s a_{n+2(s+t)(p-1)+1}$$

$$=(-1)^s \pi_3^*{}_{t(p-1)}C_s \Delta_p \mathcal{P}^{s+t}(h).$$

Hence we obtain by Proposition 1 that $\mathcal{P}^s \Delta_p \mathcal{P}^t(h) = (-1)^s{}_{t(p-1)}C_s \Delta_p \mathcal{P}^{s+t}(h)$.

By the product formula, we have

COROLLARY $\mathcal{P}^s(h^q)=0 \qquad (s>0, q\geqq 2)$.

Compare the cohomology mod $p$ of $\mathfrak{S}_p(S^n)$ with that of the Eilenberg-Maclane space $K(Z, n)$, and use the $\mathcal{C}$-theory due to J-P. Serre. (See [1] and [6].) Then we obtain directly

THEOREM 3. *If $n$ is sufficiently large, the $p$-component of the homotopy group $\pi_i(\mathfrak{S}_p(S^n))$ is zero for $i<n+2p^2-3$, and is isomorphic with $Z_p$ for $i=n+2p^2-3$.*

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