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Affine connections in a quaternion manifold and transformations preserving the structure.

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Let M be an almost complex manifold with an affine connection with respect to which the almost complex structure is covariant constant and with an irreducible homogeneous holonomy group. We have proved, in a previous paper [6], that if the largest connected group $A_0(M)$ of affine transformations in M does not preserve the structure, the manifold M must admit a quaternion structure and there exists a homomorphism of $A_0(M)$ into the special orthogonal group $SO(3)$, where by a quaternion structure we mean a pair $(\phi_i{}^h, \psi_i{}^h)$ of two almost complex structures such that $\phi_i{}^a\psi_a{}^h+\psi_i{}^a\phi_a{}^h=0$.

On the other hand, we have studied in another paper [7] affine connections on almost complex, quaternion and Hermitian manifolds. In particular we have shown that, on a quaternion manifold, (1) it is possible to introduce an affine connection, called a (ϕ, ψ)-connection, with respect to which $\phi_i{}^h$, $\psi_i{}^h$ are both covariant constant, that (2) (ϕ, ψ)-connections are uniquely determined by their torsion tensors; and that (3) in the set of all (ϕ, ψ)-connections there exist three special connections whose torsion tensors are constructed by the almost complex structures and their Nijenhuis tensors.

Now, since transformations preserving the almost complex structure are not necessarily affine (consider, for example, the projective transformations in a complex projective space) it might be interesting to study transformations preserving the quaternion structure. Are such transformations affine transformations with respect to some (ϕ, ψ)-connection? This question will be answered in the affirmative by showing that they are affine transformations just with respect to the three special connections mentioned above.

1. Transformations

In this section we discuss influences of general transformations upon geometric objects in a manifold. Conditions for a transformation to be affine will be given.

We consider an n-dimensional differentiable manifold[1] M of class C^∞ covered by coordinate neighborhoods and a differentiable homeomorphism or a transformation f of M onto itself. If $f(p)=q$, $(p, q\in M)$, we take coordinate neighborhood U of p and V of q endowed with respective local coordinates x^h and $'x^h$, such that $f(U)=V$, then the homeomorphism f is represented by differentiable functions

$$(1.1)\qquad 'x^h=f^h(x^1, x^2,\cdots, x^n) \quad \text{with} \quad \left|\frac{\partial f^h}{\partial x^i}\right| \neq 0 .$$

Let Ω be a geometric object. We regard **(1.1)** as a transformation of coordinates: $(x^h)\to('x^h)$, and calculate the components in V of Ω, then we get a new geometric object $f\Omega$ in the manifold. We call $f\Omega$ thus obtained the *transform of Ω by f.* If $\Omega=f\Omega$, we say that Ω *is invariant under f*, or that *f preserves Ω.*

We shall give some examples.

(1) Let us consider a general tensor field T, say, of the type (1,1) whose components in U are T_i^h. Then the transform fT of T by f has components fT_i^h in V given by

$$(1.2)\qquad fT_i^h=\frac{\partial' x^h}{\partial x^a}\frac{\partial x^b}{\partial' x^i}\,T_b^a .$$

The differential of f introduced by Chevalley [1] is nothing but this correspondence $T\to fT$ described above with local coordinates.

(2) The partial derivatives $\partial_j T_i^h$ in each coordinate neighborhood U give a new geometric object. The transform of this new object by f is, by definition, given, in each coordinate neighborhood $V=f(U)$, by the partial derivatives $\partial_j(fT_i^h)$ of the transform fT of T. The following theorem is easy to prove.

THEOREM **1.1** *If a tensor field T is invariant under a transformation f, so is the field of its partial derivatives.*

(3) Let us consider an affine connection Γ whose components in U are $\Gamma_{j\ i}^{\ h}$. Then the transform $f\Gamma$ of Γ by f is, by definition, an object whose components in V are given by

1) In this paper by "differentiable" we mean "differentiable of class C^∞" and we shall also restrict ourselves to connected manifolds satisfying the second axiom of countability. In such a manifold one can always introduce an affine connection.

$$(1.3)\qquad f\Gamma_{j}{}^{h}{}_{i}=\frac{\partial' x^h}{\partial x^a}\left(\frac{\partial x^c}{\partial' x^j}\frac{\partial x^b}{\partial' x^i}\,\Gamma_{c}{}^{a}{}_{b}+\frac{\partial^2 x^a}{\partial' x^j \partial' x^i}\right).$$

Now if we denote by ∇ the operation of the covariant differentiation with respect to the affine connection Γ and by ∇T the tensor field whose components in U are $\nabla_j T_i^h$, then the transform of Γ is characterized as the uniquely determined affine connection $'\Gamma$ such that, for any tensor field T[2],

$$(1.4)\qquad f(\nabla T)={}'\nabla(fT),$$

$'\nabla$ denoting the operation of the covariant differentiation with respect to $'\Gamma$. If $\Gamma=f\Gamma$, f is called an *affine transformation* (with respect to the given affine connection Γ) [5,9]. From this definition follows:

THEOREM 1.2. *In order that a transformation f be affine with respect to an affine connection Γ, it is necessary and sufficient that f and the operation of the covariant differentiation with respect to Γ be commutative:*

$$f(\nabla T)=\nabla(fT)\quad \textit{for any tensor field } T^{2)}.$$

Moreover we have

THEOREM 1.3. *If T is a tensor field invariant by f and is covariant constant with respect to an affine connection Γ, then T is also covariant constant with respect to the transform $f\Gamma$ of Γ by f.*

In fact, we have

$$'\nabla T={}'\nabla(fT)=f(\nabla T)=0.$$

Now we consider the torsion tensor of the transform of an affine connection Γ:

$$'S_{ji}{}^{h}=\frac{1}{2}\,(f\Gamma_{j}{}^{h}{}_{i}-f\Gamma_{i}{}^{h}{}_{j})\quad \text{in } V.$$

Substituting (1.3) into the above we find easily

$$'S_{ji}{}^{h}=fS_{ji}{}^{h},\quad (S_{ji}{}^{h}=\Gamma_{[j}{}^{h}{}_{i]}).$$

So we have

THEOREM 1.4. *If S is the torsion tensor of an affine connection Γ, the torsion tensor of $f\Gamma$ is fS.*

From Theorem 1.3 and Theorem 1.4 we have

THEOREM 1.5. *Let $\overset{1}{T}, \overset{2}{T}, \cdots, \overset{p}{T}$ be tensor fields such that two affine*

2) This holds if (1.4) does for any vector field.

connections, with respect to which $\overset{1}{T},\cdots,\overset{p}{T}$ *are all covariant constant, are identical if and only if their respective torsion tensors coincide. Then in order that a transformation preserving* $\overset{1}{T},\cdots,\overset{p}{T}$ *be affine with respect to an affine connection with respect to which* $\overset{1}{T},\cdots,\overset{p}{T}$ *are covariant constant, it is necessary and sufficient that the transformation preserve the torsion tensor of the connection. In particular, if we can introduce a symmetric affine connection with respect to which* $\overset{1}{T},\cdots,\overset{p}{T}$ *are covariant constant, a transformation preserving* $\overset{1}{T},\cdots,\overset{p}{T}$ *is always an affine transformation with respect to the symmetric affine connection.*

As an immediate consequence of this theorem, every transformation preserving the Riemannian metric is an affine transformation with respect to the Riemannian connection, since the metric connection without torsion is uniquely determined. It is, however, to be noted that a metric-preserving transformation is not necessarily affine with respect to a general metric connection. Since a metric connection is uniquely determined by its torsion tensor, Theorem **1.5** is also valid in this case.

2. Affine connections in quaternion manifolds [7].

We first summarize the results of [7, Chap. 3].

We consider a $4m$-dimensional manifold admitting a quaternion structure $(\phi_i{}^h, \psi_i{}^h)$[3] defined by two almost complex structures of class C^∞ such that

$$\phi_i{}^a\phi_a{}^h=\psi_i{}^a\psi_a{}^h=-\delta_i^h\,, \qquad \phi_i{}^a\psi_a{}^h+\psi_i{}^a\phi_a{}^h=0\,, \tag{2.1}$$

where $h, i, j,\cdots$ run over the range $1,\cdots, 4m$, $(n=4m)$. If we put $\kappa_i{}^h=\phi_i{}^a\psi_a{}^h$, then $\kappa_i{}^h$ is also an almost complex structure and $\phi_i{}^h$, $\psi_i{}^h$, $\kappa_i{}^h$ are anti-commutative with one another and

$$\phi_i{}^a\psi_a{}^h=-\psi_i{}^a\phi_a{}^h=\kappa_i{}^h,\ \ \psi_i{}^a\kappa_a{}^h=-\kappa_i{}^a\psi_a{}^h=\phi_i{}^h,\ \ \kappa_i{}^a\phi_a{}^h=-\phi_i{}^a\kappa_a{}^h=\psi_i{}^h\,, \tag{2.2}$$

from which it is easily seen that if any statement is proved about $\phi_i{}^h$, $\psi_i{}^h$, $\kappa_i{}^h$ then an analogous statement is established by exactly the same reasoning by replacing everywhere $\phi_i{}^h$, $\psi_i{}^h$, $\kappa_i{}^h$ by $\psi_i{}^h$, $\kappa_i{}^h$, $\phi_i{}^h$, or

3) Since a geometric object is defined by its components in each local coordinate neighborhood, we denote it by its components in a certain coordinate neighborhood. We follow the notations in Schouten [8] as a rule.

κ_i^h, ϕ_i^h, ψ_i^h respectively.

In terms of these almost complex structures, we define linear operators acting on tensor fields.

DEFINITION. Let P_{ji}^h be a tensor field. We define the operators $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ by

$$\Phi_1 P_{ji}^h = \frac{1}{2}(P_{ji}^h - \phi_i^b P_{jb}^a \phi_a^h), \quad \Phi_2 P_{ji}^h = \frac{1}{2}(P_{ji}^h + \phi_i^b P_{jb}^a \phi_a^h),$$

$$\Phi_3 P_{ji}^h = \frac{1}{2}(P_{ji}^h - \phi_j^c \phi_i^b P_{cb}^h), \quad \Phi_4 P_{ji}^h = \frac{1}{2}(P_{ji}^h + \phi_j^c \phi_i^b P_{cb}^h).$$

We define further Ψ_s and K_s $(1 \leqq s \leqq 4)$ in the same way by replacing ϕ_i^h by ψ_i^h and κ_i^h respectively.

These operations satisfy various relations, but we mention here only those used later.

(2.3) *Φ_s $(1 \leqq s \leqq 4)$ commute with one another and* $\Phi_1\Phi_2 = 0$, $\Phi_3\Phi_4 = 0$.

(2.4) *$\Phi_1, \Phi_2, \Psi_1, \Psi_2, K_1, K_2$ commute with one another and $\Phi_3, \Phi_4, \Psi_3, \Psi_4, K_3, K_4$ do also.*

(2.5) $$(\Psi_3 + K_3)\Phi_3 = \Phi_3$$

(2.6) *Let $N_{ji}^h(\phi)$ be the Nijenhuis tensor formed with ϕ_i^h:*

$$N_{ji}^h(\phi) = \frac{1}{2}(\phi_{[j}{}^a \partial_{|a|} \phi_{i]}{}^h - \phi_{[j}{}^a \partial_{i]} \phi_a^h);$$

then we have $$\Phi_2 N_{ji}^h(\phi) = \Phi_3 N_{ji}^h(\phi) = N_{ji}^h(\phi)$$

If $\Phi_1 P_{ji}^h = 0$ ($\Phi_2 P_{ji}^h = 0$) we say that P_{ji}^h is *hybrid* (*pure*) *in i and h* with respect to ϕ_i^h and if $\Phi_3 P_{ji}^h = 0$ ($\Phi_4 P_{ji}^h = 0$) we say that P_{ji}^h is *hybrid* (*pure*) *in j and i* with respect to ϕ_i^h.

Now let $\Gamma_j{}^h{}_i$ be an affine connection in a quaternion manifold. If ϕ_i^h and ψ_i^h are both covariant constant:

(2.7) $$\nabla_j \phi_i^h = \nabla_j \psi_i^h = 0 \text{ (and consequently } \nabla_j \kappa_i^h = 0),$$

where ∇ denotes the operation of the covariant differentiation, then Γ_{ji}^h is called a *(ϕ, ψ)-connection.* We know that it is always possible to introduce a (ϕ, ψ)-connection in a quaternion manifold and that the torsion tensor S_{ji}^h of a (ϕ, ψ)-connection $\Gamma_j{}^h{}_i$ satisfies

(2.8) $$2\Phi_2\Phi_3 S_{ji}^h = N_{ji}^h(\phi), \quad 2\Psi_2\Psi_3 S_{ji}^h = N_{ji}^h(\psi), \quad 2K_2K_3 S_{ji}^h = N_{ji}^h(\kappa),$$

where $N_{ji}{}^h(\phi)$, $N_{ji}{}^h(\psi)$, $N_{ji}{}^h(\kappa)$ are the Nijenhuis tensors formed with $\phi_i{}^h$, $\psi_i{}^h$, $\kappa_i{}^h$ respectively. Conversely

THEOREM A. *Let $S_{ji}{}^h$ be a tensor field in a quaternion manifold. If $S_{ji}{}^h$ is skew-symmetric in its lower indices and satisfies* (2.8), *then there exists one and only one (ϕ, ψ)-connection whose torsion tensor is just $S_{ji}{}^h$.*

Among (ϕ, ψ)-connections, we can find connections of some specified character:

THEOREM B. *It is possible to find, in a quaternion manifold, three (ϕ, ψ)-connections $\overset{1}{\Gamma}{}_j{}^h{}_i$, $\overset{2}{\Gamma}{}_j{}^h{}_i$, $\overset{3}{\Gamma}{}_j{}^h{}_i$ whose respective torsion tensors $\overset{1}{S}{}_{ji}{}^h$, $\overset{2}{S}{}_{ji}{}^h$, $\overset{3}{S}{}_{ji}{}^h$ are*

$$\overset{1}{S}{}_{ji}{}^h = \Phi_3(N_{ji}{}^h(\psi) + N_{ji}{}^h(\kappa)), \quad \overset{2}{S}{}_{ji}{}^h = \Psi_3(N_{ji}{}^h(\kappa) + N_{ji}{}^h(\phi)),$$

$$\overset{3}{S}{}_{ji}{}^h = K_3(N_{ji}{}^h(\phi) + N_{ji}{}^h(\psi)).$$

Now starting from these connections we are going to construct a new (ϕ, ψ)-connection. The $\overset{0}{\Gamma}{}_j{}^h{}_i$ defined by

$$\overset{0}{\Gamma}{}_j{}^h{}_i = \frac{1}{3}(\overset{1}{\Gamma}{}_j{}^h{}_i + \overset{2}{\Gamma}{}_j{}^h{}_i + \overset{3}{\Gamma}{}_j{}^h{}_i),$$

is an affine connection and satisfies

$$\overset{0}{\nabla}_j \phi_i{}^h = \frac{1}{3}(\overset{1}{\nabla}_j \phi_i{}^h + \overset{2}{\nabla}_j \phi_i{}^h + \overset{3}{\nabla}_j \phi_i{}^h) = 0,$$

$$\overset{0}{\nabla}_j \psi_i{}^h = \frac{1}{3}(\overset{1}{\nabla}_j \psi_i{}^h + \overset{2}{\nabla}_j \psi_i{}^h + \overset{3}{\nabla}_j \psi_i{}^h) = 0,$$

and consequently it is a (ϕ, ψ)-connection. The torsion tensor $\overset{0}{S}{}_{ji}{}^h$ of $\overset{0}{\Gamma}{}_{ji}{}^h$ is

$$\begin{aligned}\overset{0}{S}{}_{ji}{}^h &= \frac{1}{3}(\overset{1}{S}{}_{ji}{}^h + \overset{2}{S}{}_{ji}{}^h + \overset{3}{S}{}_{ji}{}^h) \\ &= \frac{1}{3}((\Psi_3 + K_3)N_{ji}{}^h(\phi) + (K_3 + \Phi_3)N_{ji}{}^h(\psi) + (\Phi_3 + \Psi_3)N_{ji}{}^h(\kappa)).\end{aligned}$$

From (2.4), (2.5) and (2.6) we see

$$\begin{aligned}(\Psi_3 + K_3)N_{ji}{}^h(\phi) &= (\Psi_3 + K_3)\Phi_3 N_{ji}{}^h(\phi) \\ &= \Phi_3 N_{ji}{}^h(\phi) = N_{ji}{}^h(\phi),\end{aligned}$$

from which and formulas obtained from this by cyclic permutations of $(\phi_i^h, \psi_i^h, \kappa_i^h)$, we find

$$\overset{0}{S}_{ji}{}^h = \frac{1}{3}(N_{ji}{}^h(\phi) + N_{ji}{}^h(\psi) + N_{ji}{}^h(\kappa)) \tag{2.9}$$

THEOREM 2.1. *It is possible to introduce, in a quaternion manifold, a* (ϕ, ψ)*-connection* $\overset{0}{\Gamma}_j{}^h{}_i$ *whose torsion tensor* $\overset{0}{S}_{ji}{}^h$ *is given by* (2.9).

The torsion tensors of these four (ϕ, ψ)-connections are expressed in terms of the structure and the Nijenhuis tensors formed with three almost complex structures. In case any two of the three Nijenhuis tensors vanish identically, the third vanishes automatically, and the torsion tensors of these connections vanish too. In such a case, these four (ϕ, ψ)-connections are all symmetric and coincide with one another. Conversely if there exists a symmetric (ϕ, ψ)-connection, then it is unique and three Nijenhuis tensors all vanish identically. For these reasons we call them the *canonical connections* in a quaternion manifold.

3. Transformations preserving the structure

In this section, we shall prove, applying the results of §1 to a quaternion manifold, that transformations preserving the structure are necessarily affine with respect to the four canonical connections (Theorem 3.3). The group $Q(M)$ of all transformations preserving the structure will be proved to be a Lie group with respect to the natural topology (Theorem 3.4), and then the maximum dimensions of $Q(M)$ and the space admitting the group $Q(M)$ of the maximum dimensions will be discussed.

We consider, in a quaternion manifold, a differentiable transformation f preserving the quaternion structure:

$$f\phi_i^h = \phi_i^h,\ f\psi_i^h = \psi_i^h \text{ (and consequently } f\kappa_i^h = \kappa_i^h).$$

We consider an arbitrary (ϕ, ψ)-connection $\Gamma_j{}^h{}_i$ with torsion tensor $S_{ji}{}^h$. Since ϕ_i^h and ψ_i^h are both invariant by f and are both covariant constant, they are covariant constant also with respect to $f\Gamma_j{}^h{}_i$, by virture of Theorem 1.3, so that $f\Gamma_j{}^h{}_i$ is also a (ϕ, ψ)-connection. Following Theorem 1.4, the torsion tensor of $f\Gamma_j{}^h{}_i$ is $fS_{ji}{}^h$. Thus

THEOREM 3.1. *Let* $\Gamma_j{}^h{}_i$ *be a* (ϕ, ψ)*-connection with torsion tensor* $S_{ji}{}^h$ *and* f *be a transformation which preserves the quaternion structure. Then*

$f\Gamma_j{}^h{}_i$ is a (ϕ, ψ)-connection with torsion tensor $fS_{ji}{}^h$.

Moreover if f is an affine transformation: $f\Gamma_j{}^h{}_i=\Gamma_j{}^h{}_i$, then we have $fS_{ji}{}^h=S_{ji}{}^h$. If, conversely, f preserves the torsion tensor: $fS_{ji}{}^h=S_{ji}{}^h$, then the (ϕ, ψ)-connection $f\Gamma_j{}^h{}_i$ must have the torsion tensor $S_{ji}{}^h$, so we have $f\Gamma_j{}^h{}_i=\Gamma_j{}^h{}_i$ and consequently f is an affine transformation. We have thus proved

THEOREM 3.2. *Let $\Gamma_j{}^h{}_i$ be a (ϕ, ψ)-connection with torsion tensor $S_{ji}{}^h$ and f be a transformation which preserves the quaternion structure. Then in order that f be an affine transformation with respect to $\Gamma_j{}^h{}_i$, it is necessary and sufficient that f preserve the torsion tensor $S_{ji}{}^h$.*

This is nothing but a special case of Theorem **1.5.**

Since the invariance of $\phi_i{}^h$ by f implies the invariance of the field of its partial derivatives, it follows that a transformation preserving $\phi_i{}^h$ also preserves the Nijenhuis tensor $N_{ji}{}^h(\phi)$ formed with $\phi_i{}^h$. Therefore a transformation leaving the quaternion structure invariant preserves the torsion tensors of the four canonical connections. Thus by Theorem **3.2** we have

THEOREM 3.3. *A transformation preserving the quaternion structure is always an affine transformation with respect to any one of the four canonical connections.*

Let us take one of the four canonical connections, say, $\overset{0}{\Gamma}{}_j{}^h{}_i$ in a quaternion manifold M. We denote by $A(M)$ the group of all affine transformations with respect to $\overset{0}{\Gamma}{}_j{}^h{}_i$ and by $Q(M)$ that of all transformations preserving the quaternion structure. $A(M)$ is a Lie group with respect to the natural topology [3]. Theorem **3.3** implies that $Q(M)$ is a subgroup of $A(M)$ and is obviously closed, so that $Q(M)$ is a Lie group.

THEOREM 3.4. *$Q(M)$ is a closed subgroup of $A(M)$ and thus a Lie group.*

Now we denote by $Q_p(M)$ and $A_p(M)$ the isotropy subgroups at a point p of $Q(M)$ and $A(M)$ respectively. Then $Q(M)$ is contained in $A(M)$. To each transformation f of $A_p(M)$ corresponds a linear transformation $\tilde{f}$ in the tangent space T_p at p. This correspondence $\rho: f\rightarrow\tilde{f}$ gives an isomorphism of $A_p(M)$ onto a subgroup $\tilde{A}_p$ of the general linear group $L(n)$ in T_p, called the *linear isotropy group of $A(M)$ at p.* ρ gives also an isomorphism of $Q_p(M)$ onto a subgroup $\tilde{Q}_p$ of $\tilde{A}_p$, called the *linear isotropy group of $Q(M)$ at p.*

Now we take a complex coordinate system in the complexification

T_p^C of T_p and we transform ϕ_i^h and ψ_i^h into their normal forms [6]:

$$(\phi_i^h)\to(I_i^h)=\begin{pmatrix} i\delta_\lambda^\kappa & 0 \\ 0 & -i\delta_{\bar\lambda}^{\bar\kappa}\end{pmatrix},\quad (\psi_i^h)\to(J_i^h)=\begin{pmatrix} 0 & 0 & 0 & -\delta_s^r \\ 0 & 0 & \delta_s^r & 0 \\ 0 & -\delta_s^r & 0 & 0 \\ \delta_s^r & 0 & 0 & 0\end{pmatrix},$$

where $\kappa, \lambda=1, \cdots, 2m$, $\bar\kappa=\kappa+2m$, $r, s=1,\cdots, m$. Then all the transformations leaving I_i^h, J_i^h and T_p invariant constitute the quaternion linear group $QL(m, C)$, so that Q_p is a subgroup of the real representation $QL(m, R)$ of $QL(m, C)$. Thus we have

THEOREM 3.5. *The isotropy subgroup of $Q(M)$ at any point of the manifold is isomorphic with its linear isotropy group which is a subgroup of $QL(m, R)$.*

Now we consider an n-dimensional quaternion manifold $(n=4m)$ which admits a group, with the maximum dimensions, of transformations preserving the quaternion structure.

Since $QL(m, C)$ consists of non-singular matrices of the form

$$\begin{pmatrix} A & -B & 0 & 0 \\ \bar B & \bar A & 0 & 0 \\ 0 & 0 & \bar A & -\bar B \\ 0 & 0 & B & A\end{pmatrix},$$

A, B being both complex matrices of degree m and $\bar A$ denoting the complex conjugate of A, we have

$$\dim QL(m, R)=\dim QL(m, C)=4m^2,$$

so that at any point p,

$$\dim Q_p(M)=\dim \tilde Q_p \leqq \dim QL(m, R)=4m^2.$$

Since $Q(M)$ takes, in our case, the maximum dimensions, we suppose that at some point p, $\tilde Q_p$ takes the dimension $4m^2$ and then $\tilde Q_p= QL(m, R)$. Now since we have in general

$$\dim Q(M)\leqq \dim Q_p(M)+\dim M,$$

we assume further that

$$\dim Q(M) = \dim Q_p(M) + \dim M = 4m(m+1)\,.$$

For any point q of M, we have

$$\dim Q_q(M) \geqq \dim Q(M) - \dim M = 4m^2\,.$$

But we must have $\dim Q_q(M) \leqq 4m^2$, and therefore we see that, at any point q of M, $Q_q(M) = QL(m, R)$. Since $QL(m, R)$ contains a transformation represented by the matrix $(-\delta_i^h)$, we can find in $Q_p(M)$ a transformation σ_p which gives a symmetry at p in some neighborhood of p. It follows that $Q(M)$ acts on M transitively and the canonical connection is complete [5].

Furthermore $\tilde{Q}_p$ contains, for example, a transformation represented by the matrix $(2\delta_i^h)$, so the torsion and curvature tensors of a canonical connection should vanish identically [4], because $Q(M)$ is a group of affine transformations with respect to the canonical connection and leaves the torsion and curvature tensors invariant. Thus the canonical connection is locally flat and the four canonical connections coincide, hence the quaternion structure is integrable [7].

On the other hand, we know [2, p. 50] that a complete, simply-connected, homogeneous affinely connected manifold is uniquely determined by its local properties. Hence if M is simply-connected, M is equivalent to the whole coordinate space with the vanishing connection $\Gamma_j{}^h{}_i = 0$.

Gathering the results we obtain

THEOREM 3.6. *Let M be an n-dimensional simply-connected manifold ($n = 4m$) with a quaternion structure. Suppose that M is endowed with one of the four canonical connections and admits a group $Q(M)$ of transformations preserving the quaternion structure with the maximum dimensions. Then this maximum dimension is $n(n/4+1)$ and the quaternion structure is integrable. Moreover, the manifold with this canonical connection is equivalent to the coordinate space with the vanishing connection $\Gamma_j{}^h{}_i = 0$ and the group $Q(M)$ is given by linear equations:*

$$y^h = x^i A_i{}^h + A^h \quad \textit{with} \quad (A_i{}^h) \in QL(m, R)\,.$$

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