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Curvature and relative Betti numbers.

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Many interesting relations between curvature and Betti numbers in a compact orientable Riemannian space are obtained by S. Bochner, A. Lichnerowicz and K. Yano (See [1]). We shall generalize these results to the case of the domain with boundary. Three quadratic forms related to the curvature tensors of the domain and of the boundary have an intimate connection with the absolute and relative Betti numbers of the doman. Our results are the consequences of the theorems concerning harmonic forms and Betti numbers given by Conner [3], Duff and Spencer [4].

Let $\mathfrak{M}$ be an n-dimensional orientable Riemannian manifold of C^∞ with positive definite metric $ds^2=g_{ij}dx^i dx^j$. Let $\mathfrak{D}$ be an open set in $\mathfrak{M}$ with $(n-1)$-dimensional boundary $\mathfrak{B}$ of C^∞ and $\tilde{\mathfrak{D}}$ be an open set containing $\mathfrak{D}$ and $\mathfrak{B}$. We assume that $\mathfrak{D}\cup\mathfrak{B}$ is compact. Local coordinates in $\tilde{\mathfrak{D}}$ and in $\mathfrak{B}$ are denoted by x^i $(i=1, 2,\cdots, n)$ and u^λ $(\lambda=2, 3,\cdots, n)$ respectively. Let N^i be the components of the unit contravariant outward normal vector to the boundary $\mathfrak{B}$ and we put $X_\lambda^i=\dfrac{\partial x^i}{\partial u^\lambda}$.

The local coordinates x^i in $\tilde{\mathfrak{D}}$ and u^λ in $\mathfrak{B}$ are both oriented in positive sense, so that we have

$$\delta_{i_1 i_2\cdots i_n}N^{i_1}X_{\lambda_2}^{i_2}\cdots X_{\lambda_n}^{i_n}\bar{\delta}^{\lambda_2\cdots\lambda_n}>0\,, \tag{1}$$

where $\delta_{i_1\cdots i_n}$ and $\bar{\delta}^{\lambda_2\cdots\lambda_n}$ are Kronecker deltas in $\tilde{\mathfrak{D}}$ and in $\mathfrak{B}$. Let D_i denote the covariant differential operators with respect to the metric form $g_{ij}dx^i dx^j$ in $\tilde{\mathfrak{D}}$ and $\varepsilon_{i_1\cdots i_n}=\sqrt{g}\,\delta_{i_1\cdots i_n}$.

We shall adopt the following notations in accordance with the previous paper [2]. Let $A_{(p)}$ and $B_{(q)}$ be anti-symmetric covariant tensors of order p and q defined in $\tilde{\mathfrak{D}}$ and of C^∞.

(2.1) $(*A)_{i_{p+1}\cdots i_n}=\frac{1}{p!}A_{i_1\cdots i_p}\varepsilon^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}$,

(2.2) $\circ A_{(p)}=(-1)^{(n-p)p}*A_{(p)}$,

(2.3) $(A\wedge B)_{i_1\cdots i_p j_1\cdots j_q}=\frac{1}{p!}\frac{1}{q!}A_{k_1\cdots k_p}B_{l_1\cdots l_q}\delta^{k_1\cdots k_p l_1\cdots l_q}_{i_1\cdots i_p j_1\cdots j_q}$,

(2.4) $(\triangle A)_{ii_1\cdots i_p}=\frac{1}{p!}\delta^{jj_1\cdots j_p}_{ii_1\cdots i_p}D_jA_{j_1\cdots j_p}$,

(2.5) $(\nabla A)_{i_2\cdots i_p}=D^{i_1}A_{i_1i_2\cdots i_p}$,

(2.6) $(\perp A)_{ii_1\cdots i_p}=\frac{1}{p!}\delta^{jj_1\cdots j_p}_{ii_1\cdots i_p}N_jA_{j_1\cdots j_p}$,

(2.7) $(\top A)_{i_2\cdots i_p}=N^{i_1}A_{i_1i_2\cdots i_p}$,

(2.8) $\square A=(\triangle\nabla+\nabla\triangle)A$.

$\perp A$ and $\top A$ are defined only on the boundary $\mathfrak{B}$. Associating the anti-symmetric covariant tensors $A_{(p)}$ and $B_{(q)}$ with the forms $A_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$ and $B_{i_1\cdots i_q}dx^{i_1}\cdots dx^{i_q}$, the notations $\triangle$, ∇ and $\square$ are usually denoted by d, $-\delta$ and $-\triangle$ respectively.

Schema: $\triangle\to d$, $\nabla\to -\delta$, $\square\to -\triangle$.

As it is easily seen, we have

(3.1) $\circ * A=*\circ A=A$,

(3.2) $*\triangle A_{(p)}=(-1)^p\nabla * A_{(p)}$, $\quad\circ\nabla A_{(p)}=(-1)^{p'}\triangle\circ A_{(p)}$,

(3.3) $*\perp A_{(p)}=(-1)^p\top * A_{(p)}$, $\quad\circ\top A_{(p)}=(-1)^{p'}\perp\circ A_{(p)}$,

(3.4) $(\perp\top+\top\perp)A=A$,

(3.5) $\perp\top\perp A=\top\perp\top A=A$,

(3.6) $\perp\perp A=\top\top A=0$,

where $p'=n-p$.

If we put $\bar{g}_{\lambda\mu}=g_{ij}X^i_\lambda X^j_\mu$, $\bar{g}_{\lambda\mu}du^\lambda du^\mu$ is the induced metric in $\mathfrak{B}$. We have at once

(4) $$X^i_\lambda X^j_\mu \bar{g}^{\lambda\mu}=g^{ij}-N^iN^j, \qquad N_iX^i_\lambda=0.$$

Operators $\bar{\triangle}$, $\bar{\nabla}$, $\bar{*}$ and $\bar{\circ}$ in $\mathfrak{B}$ are defined in the same way as in $\mathfrak{D}$.

Let I and J denote the following operators:

(5.1) $$I:\ A_{i_1\cdots i_p}\to A_{i_1\cdots i_p}X^{i_1}_{\lambda_1}\cdots X^{i_p}_{\lambda_p},$$

(5.2) $$J:\ A_{i_1\cdots i_p}\to A_{i_1 i_2\cdots i_p}N^{i_1}X^{i_2}_{\lambda_2}\cdots X^{i_p}_{\lambda_p},$$
$$\to 0 \qquad \text{by definition if } p=0.$$

Note that IA and JA are anti-symmetric covariant tensors in $\mathfrak{B}$. We get the following relations from (1), (2), (3), (4), (5) and similar formulas for $\bar{\circ}$, $\bar{*}$, $\bar{\triangle}$ and $\bar{\triangledown}$.

(6.1) $N_{j_1}\varepsilon^{j_1 j_2\cdots j_p}{}_{j_{p+1}\cdots j_n}X^{j_{p+1}}_{\lambda_{p+1}}\cdots X^{j_n}_{\lambda_n}=X^{j_2}_{\lambda_2}X^{j_3}_{\lambda_3}\cdots X^{j_p}_{\lambda_p}\varepsilon^{\lambda_2\cdots\lambda_p}{}_{\lambda_{p+1}\cdots\lambda_n}$, by (1) and (4);

(6.2) $I\perp A=0,\ J\top A=0,$ by (5.1), (5.2), (2.6), (2.7) and (4);

(6.3) $JA=I\top A,\ IA=J\perp A,$ by (5.1), (5.2), (2.6), (2.7) and (4);

(6.4) $JA=0$ if $\top A=0,$ $IA=0$ if $\perp A=0$ by (6.3);

(6.5) $J\circ A=\bar{\circ}IA,\ \bar{*}JA=I*A,$ by (2.1), (2.2), (6.1), (5.1), (5.2) and (3.1);

(6.6) $\bar{\triangle}IA=I\triangle A,\ \bar{\triangledown}JA=-J\triangledown A,$ by (2.4), (2.5), (5.1), (5.2) and (3.2).

Let $C_{i\lambda}$, for example, be any tensor with mixed indices i and λ defined on $\mathfrak{B}$. We shall define its covariant derivative along $\mathfrak{B}$ as follows:

$$C_{i\lambda;\,\mu}=\frac{\partial C_{i\lambda}}{\partial u^\mu}-\{{}_i{}^p{}_s\}X^s{}_\mu C_{p\lambda}-\overline{\{{}_\lambda{}^\sigma{}_\mu\}}C_{i\sigma},$$

where $\{{}_i{}^p{}_s\}$ and $\overline{\{{}_\lambda{}^\sigma{}_\mu\}}$ are Christoffel symbols with respect to g_{ij} and $\bar{g}_{\lambda\mu}$ respectively.

It is known that

(7.1) $$X^i{}_{\lambda;\,\mu}=-F_{\lambda\mu}N^i,\quad N^i{}_{;\lambda}=F_\lambda{}^\mu X^i{}_\mu,$$

(7.2) $$F_{\lambda\mu}=F_{\mu\lambda}.$$

(Sometimes $F_{\lambda\mu}$ defined as above is denoted by $-F_{\lambda\mu}$). $F_{\lambda\mu}du^\lambda du^\mu$ is the second fundamental form of $\mathfrak{B}$. If the boundary $\mathfrak{B}$ is convex outwards, the form is positive definite.

Let P, Q and K denote the following operators:

(8.1) $$P:\ A_{\lambda_1\cdots\lambda_p}\to F^\sigma_{\lambda_1}A_{\sigma\lambda_2\cdots\lambda_p}+\cdots+F^\sigma_{\lambda_p}A_{\lambda_1\cdots\lambda_{p-1}\sigma},$$
$$\to 0 \qquad \text{by definition if } p=0,$$

(8.2) $$Q:\ A_{\lambda_1\cdots\lambda_p}\to F^\sigma_\sigma A_{\lambda_1\cdots\lambda_p}-(PA)_{\lambda_1\cdots\lambda_p},$$

(8.3) $$K:\ A_{i_1\cdots i_p}\to A_{i_1\cdots i_p;\,s}N^s$$

where $A_{\lambda_1\cdots\lambda_p}$ and $A_{i_1\cdots i_p}$ are tensors in $\mathfrak{B}$ and in $\tilde{\mathfrak{D}}$ respectively.

We get at once, from (2.1), (2.2) and (8), the relations

(9.1) $$\bar{*}PA=Q\bar{*}A, \qquad \bar{\circ}PA=Q\bar{\circ}A,$$

(9.2) $$*KA=K*A, \qquad \circ KA=K\circ A.$$

If a tensor $T^{i_1\cdots i_p,j_1\cdots j_p}$ is anti-symmetric with respect to the indices $i_1,\cdots,i_p$ and satisfies $T^{i_1\cdots i_p,j_1\cdots j_p}=T^{j_1\cdots j_p,i_1\cdots i_p}$, we shall call it double anti-symmetric tensor. Put

$$(A,T,B)=\frac{1}{p!}\frac{1}{p!}T^{i_1\cdots i_p,j_1\cdots j_p}A_{i_1\cdots i_p}B_{j_1\cdots j_p}=\frac{1}{p!}\frac{1}{p!}T_{i_1\cdots i_p}^{j_1\cdots j_p}A^{i_1\cdots i_p}B_{j_1\cdots j_p}.$$

If

$$T^{i_1\cdots i_p,j_1\cdots j_p}=\begin{vmatrix} g^{i_1j_1}\cdots g^{i_1j_p} \\ \cdots\cdots\cdots\cdots \\ g^{i_pj_1}\cdots g^{i_pj_p}\end{vmatrix},$$

we shall put $(A,T,B)=(A,B)$, which is equal to $A^{i_1\cdots i_p}B_{i_1\cdots i_p}$. The quadratic form (A,T,A) of any anti-symmetric tensor $A_{(p)}$ of order p is denoted by $[T_{(p)}]$ for simplicity. Similar notations can be defined in $\mathfrak{B}$.

Let P and Q be two double anti-symmetric tensors in $\mathfrak{B}$ with components

(10.1) $$P_{\lambda_1\cdots\lambda_p}^{\mu_1\cdots\mu_p}=F_{\lambda_1}^{\sigma}\delta_{\sigma\lambda_2\cdots\lambda_p}^{\mu_1\mu_2\cdots\mu_p}+\cdots+F_{\lambda_p}^{\sigma}\delta_{\lambda_1\cdots\lambda_{p-1}\sigma}^{\mu_1\cdots\mu_{p-1}\mu_p},$$

$$=0 \qquad \text{by definition if } p=0,$$

(10.2) $$Q_{\lambda_1\cdots\lambda_p}^{\mu_1\cdots\mu_p}=F_{\sigma}^{\sigma}\delta_{\lambda_1\cdots\lambda_p}^{\mu_1\cdots\mu_p}-P_{\lambda_1\cdots\lambda_p}^{\mu_1\cdots\mu_p}.$$

We get easily from (8) and (10)

(11.1) $$(IA,P,IB)=(IA,PIB),$$

(11.2) $$(JA,Q,JB)=(JA,QJB).$$

If $A_{(p)}$ is harmonic, that is, $\Box A=(\triangle\nabla+\nabla\triangle)A=0$, it is known that

(12.1) $$A_{i_1\cdots i_p;s}{}^{s}=\frac{1}{p!}T_{i_1\cdots i_p}^{j_1\cdots j_p}A_{j_1\cdots j_p},$$

(12.2) $$T_{i_1\cdots i_p}^{j_1\cdots j_p}=\delta_{k_1\cdots k_p}^{j_1\cdots j_p}\left[\frac{1}{(p-1)!}\delta_{i_1i_2\cdots i_p}^{t_1k_2\cdots k_p}R^{k_1}{}_{t_1}+\frac{1}{2}\frac{1}{(p-2)!}\delta_{i_1i_2i_3\cdots i_p}^{t_1t_2k_3\cdots k_p}R^{k_1k_2}{}_{t_1t_2}\right],$$

$=0$ by definition if $p=0$,

$=R^{j_1}_{i_1}$ by definition if $p=1$,

where $R^k{}_t=R^{sk}{}_{ts}$ and R_{ijkl} are the components of the curvature tensor of $\tilde{\mathfrak{D}}$ [1].

It follows from (12.1) that, for harmonic tensor,

$$(13)\qquad \oint_{\mathfrak{B}} A^{i_1\cdots i_p} A_{i_1\cdots i_p;s} N^s d\sigma = \iint_{\mathfrak{D}} A^{i_1\cdots i_p;s} A_{i_1\cdots i_p;s} dv + \iint_{\mathfrak{D}} A^{i_1\cdots i_p} A_{i_1\cdots i_p;s}{}^s dv$$

$$= \iint_{\mathfrak{D}} A^{i_1\cdots i_p;s} A_{i_1\cdots i_p;s} dv + \iint_{\mathfrak{D}} (A, T, A) dv$$

holds, where $dv>0$ is the n-dimensional volume element of $\mathfrak{D}$ and $d\sigma>0$ is the $(n-1)$-dimensional surface element of $\mathfrak{B}$.

On the other hand, we get from (4) and (5)

$$(14.1)\qquad (IA, IA)=(A, B)-(\top A, \top B),$$

$$(14.2)\qquad (JA, JB)=(A, B)-(\perp A, \perp B).$$

Hence

$$(14.3)\qquad (IA, IB)=(A, B) \text{ if } \top A=0 \text{ or } \top B=0,$$

$$(14.4)\qquad (JA, JB)=(A, B) \text{ if } \perp A=0 \text{ or } \perp B=0.$$

Calculating $\overline{\nabla} IA$ and $\overline{\triangle} JA$ straightforward, we get

$$(15.1)\qquad JKA=I\nabla A-\overline{\nabla} IA-QJA,$$

$$(15.2)\qquad IKA=J\triangle A+\overline{\triangle} JA-PIA,$$

by (2.4), (2.5), (4), (8), (5) and (7.1).

Under the conditions $\square A=0$ in $\mathfrak{D}$ and $\perp A=\perp\nabla A=0$ in $\mathfrak{B}$, multiplying JA on both sides of (15.1), and making use of (6.4), (8.3), (11.2) and (14.4), we get

$$(A, KA)=A^{i_1\cdots i_p} A_{i_1\cdots i_p;s} N^s$$

$$=-(JA, QJA)=-(JA, Q, JA).$$

And similarly, under the condition $\square A=0$, $\top A=\top\triangle A=0$, we have

$$(A, KA)=-(IA, P, IA).$$

It follows from (13) that

$$\iint_{\mathfrak{D}} A^{i_1\cdots i_p;s}A_{i_1\cdots i_p;s}dv+\oint_{\mathfrak{B}}(JA, Q, JA)d\sigma+\iint_{\mathfrak{D}}(A, T, A)dv=0\,,$$

$$\iint_{\mathfrak{D}} A^{i_1\cdots i_p;s}A_{i_1\cdots i_p;s}dv+\oint_{\mathfrak{B}}(IA, P, IA)d\sigma+\iint_{\mathfrak{D}}(A, T, A)dv=0\,,$$

or denoting $A_{i_1\cdots i_p;s}$ simply by DA, we have

$$\text{(16.1)}\qquad \iint_{\mathfrak{D}}(DA, DA)dv+\oint_{\mathfrak{B}}(JA, Q, JA)dv+\iint_{\mathfrak{D}}(A, T, A)dv=0$$
$$\text{if } \Box A=0,\ \perp A=\perp\nabla A=0\,,$$

$$\text{(16.2)}\qquad \iint_{\mathfrak{D}}(DA, DA)dv+\oint_{\mathfrak{B}}(IA, P, IA)d\sigma+\iint_{\mathfrak{D}}(A, T, A)dv=0$$
$$\text{if } \Box A=0,\ \top A=\top\triangle A=0\,.$$

Since the first term of (16) is non-negative, we get

$$\text{(17.1)}\qquad \oint_{\mathfrak{B}}(JA, Q, JA)d\sigma+\iint_{\mathfrak{D}}(A, T, A)dv\leqq 0,\ \text{if } \Box A=0,\ \perp A=\perp\nabla A=0\,,$$

$$\text{(17.2)}\qquad \oint_{\mathfrak{B}}(IA, P, IA)d\sigma+\iint_{\mathfrak{D}}(A, T, A)dv\leqq 0,\ \text{if } \Box A=0,\ \top A=\top\triangle A=0\,.$$

If the quadratic forms $[Q_{(p-1)}]=(B, Q, B)$ of anti-symmetric tensor $B_{(p-1)}$ in $\mathfrak{B}$ and $[T_{(p)}]=(C, T, C)$ of anti-symmetric tensor $C_{(p)}$ in $\mathfrak{D}$ are positive definite respectively at every point of $\mathfrak{B}$ and $\mathfrak{D}$, $A_{(p)}$ in (17.1) must be zero. Hence the equation $\Box A=0$ in $\mathfrak{D}$ with boundary condition $\perp A=\perp\nabla A=0$ in $\mathfrak{B}$ has only the trivial solution, that is, the relative Betti number of $\mathfrak{D}$ is zero. Similar property holds for the equation $\Box A=0$ in $\mathfrak{D}$ with boundary condition $\top A=\top\triangle A=0$ in $\mathfrak{B}$ [2], [3], [4].

Restating the conditions at the beginning of this paper, we get the following theorem and corollaries under the conditions that *the Riemannian manifold* $\mathfrak{M}$ *with positive definite metric is orientable, the closure of the open set* $\mathfrak{D}$ *is compact and contained in an open set* $\tilde{\mathfrak{D}}$, *and all structures considered are of* C^∞.

THEOREM. *If the quadratic forms* $[T_{(p)}]$ *and* $[Q_{(p-1)}]$ *are positive definite respectively at every point of the open set* $\mathfrak{D}$ *and the boundary* $\mathfrak{B}$, *the p-th relative Betti number of* $\mathfrak{D}$ *is zero for* $n-1\geqq p\geqq 1$.

If the quadratic forms $[T_{(p)}]$ *and* $[P_{(p)}]$ *are positive definite respectively at every point of* $\mathfrak{D}$ *and* $\mathfrak{B}$, *the p-th absolute Betti number of* $\mathfrak{D}$ *is zero for* $n-1\geqq p\geqq 1$.

If $\mathfrak{B}$ is convex outwards, the second fundamental form $F_{\lambda\mu}du^{\lambda}du^{\mu}$ is positive definite and accordingly $[Q_{(p-1)}]$ and $[P_{(p)}]$ are all positive definite for $n-1\geqq p\geqq 1$. Hence we have

COROLLARY 1. *If $\mathfrak{B}$ is convex outwards and the quadratic form $[T_{(p)}]$ is positive definite, the p-th absolute and relative Betti numbers of $\mathfrak{D}$ are zero for $n-1\geqq p\geqq 1$.*

We get also the following corollaries.

COROLLARY 2. *$\mathfrak{M}$ is assumed to be Euclidean.*

If the quadratic form $[Q_{(p-1)}]$ is positive definite, the p-th relative Betti number of $\mathfrak{D}$ is zero for $n-1\geqq p\geqq 1$.

If the quadratic form $[P_{(p)}]$ is positive definite, the p-th absolute Betti number of $\mathfrak{D}$ is zero for $n-1\geqq p\geqq 1$.

COROLLARY 3. *If $\mathfrak{M}$ is Euclidean and $\mathfrak{B}$ is convex outwards, all the p-th absolute and relative Betti numbers of $\mathfrak{D}$ are zero for $n-1\geqq p\geqq 1$.*

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