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On the change of rings in the homological algebra.

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The object of this note consists in making a response partly to Massey's problem 22 ([4]), generalizing the problem from the case of the homology of groups to the case of the homological algebra. Thus we shall study the change of rings $\varphi: \Lambda \rightarrow \Gamma$ in the homological algebra by the use of algebraic mapping cylinder as was indicated in Massey's paper. In our description, we shall make free use of the notations of the book of H. Cartan and S. Eilenberg ([1]).

In § 1, we define the algebraic mapping cylinder by analogy of the topological mapping cylinder (cf. p. 73, [1]). In § 2, we introduce new functors Tor^{φ} and Ext_{φ}, which have similar properties as the absolute Tor- and Ext-functors, as will be shown in § 3.

The functors Tor^{φ} and Ext_{φ} yields a "relative" theory for the change of rings $\varphi: \Lambda \rightarrow \Gamma$. Another "relative" theory of the homological algebra was discussed in [3]. According to what is announced in the same paper, our problems seem to be also investigated by M. Auslander. But I have not yet access to his results. In § 4, we shall consider in particular the relative cohomology group of dimension 2, $H^2(\mathfrak{G}, \mathfrak{K}: M)$ (M being a $\mathfrak{G}$-module) of a group $\mathfrak{G}$ and its subgroup $\mathfrak{K}$, and bring it in relation with the classes of group extensions of $\mathfrak{G}$, which are trivial over $\mathfrak{K}$.

The author has in view to analyse further the relations between the relative homology of a pair $(\mathfrak{G}, \mathfrak{K})$ and the homology of factor group $\mathfrak{G}/\mathfrak{K}$ in case $\mathfrak{K}$ is normal. We can readily see that if $H^r(\mathfrak{K}: M)=0$, for $0<r<n$, then $H^n(\mathfrak{G}, \mathfrak{K}: M) \approx H^n(\mathfrak{G}/\mathfrak{K}: M^{\mathfrak{K}})$ by our reduction theorem 3.4* in § 3 and by the exact sequence of Hochschild and Serre ([2]). These topics will be treated in a forthcoming paper.

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§ 1. Algebraic mapping cylinder.

Let $A=\sum_{-\infty<n<+\infty} A_n$ and $C=\sum_{-\infty<n<+\infty} C_n$ be chain complexes. A *map* $\varphi: A\rightarrow C$ is a homomorphism satisfying $\partial\varphi=\varphi\partial$, where ∂ denotes the boundary operators in A and C respectively. For a given map $\varphi: A\rightarrow C$, the *algebraic mapping cylinder* $M(\varphi)=\sum_n M_n$ of φ, and the *φ-relative chain complex* $N(\varphi)=\sum_n N_n$ are defined as follows:

$$\begin{cases} M_n=A_n\dot{+}A_{n-1}\dot{+}C_n, \quad \text{(direct sum)}, \\ \partial_n(a, b, c)=(\partial a-b, -\partial b, \partial c+\varphi b), (a, b, c)\in M_n, \text{ and } \partial\cdot\partial=0. \end{cases}$$

$$\begin{cases} N_n=A_{n-1}\dot{+}C_n, \quad \text{(direct sum)}, \\ \partial_n(b, c)=(-\partial b, \partial c+\varphi b), (b, c)\in N_n, \text{ and } \partial\cdot\partial=0. \end{cases}$$

LEMMA 1.1. (i) *There exist maps $\mu: C\rightarrow M(\varphi)$ and $\nu: M(\varphi)\rightarrow C$ such that the composite maps $\mu\nu$ and $\nu\mu$ are homotopic with the identity map of $M(\varphi)$ and C respectively.* (ii) *By setting $\alpha(a)=(a, 0, 0)$ and $\beta(a, b, c)=(b, c)$, we obtain an exact sequence $0\rightarrow A\xrightarrow{\alpha} M(\varphi)\xrightarrow{\beta} N(\varphi)\rightarrow 0$, where α and β are maps.* (iii) $\nu\alpha=\varphi$. (iv) *Thus we have an exact sequence*

$$\cdots\longrightarrow H_n(A)\xrightarrow{\varphi_*} H_n(C)\longrightarrow H_n(N(\varphi))\longrightarrow H_{n-1}(A)\longrightarrow\cdots,$$

with the induced homomorphism φ_*.

PROOF. We define μ and ν by setting $\mu(c)=(0, 0, c)$, $c\in C$, and $\nu(a, b, c)=c+\varphi a$. We define a homomorphism $D: M(\varphi)\rightarrow M(\varphi)$ of degree $+1$ by setting $D_n(a, b, c)=(0, a, 0)$, $(a, b, c)\in M_n$. Then D is the homotopy connecting $\mu\nu$ with the identity map of $M(\varphi)$. (ii), (iii) and (iv) will be readily seen.

REMARK. (1) The boundary homomorphism ∂_* of the homology sequence of $0\rightarrow A\rightarrow M(\varphi)\rightarrow N(\varphi)\rightarrow 0$ coincides with the homomorphism induced by the map $(b, c)\rightarrow -b: N(\varphi)\rightarrow A$ of degree -1. (2) The boundary operator ∂ in $M(\varphi)$ is determined by the conditions: (i) $\alpha\partial=\partial\alpha$, (ii) $\beta\partial=\partial\beta$, (iii) $\mu\partial=\partial\mu$, (iv) $\partial D+D\partial=\mu\nu-1$. In fact, if we set formally $\partial(a, b, c)=(f_1a+g_1b+h_1c, f_2a+g_2b+h_2c, f_3a+g_3b+h_3c)$, then by (i) we have $f_1=\partial$; by (ii): $f_2=f_3=0$; by (iii): $h_3=\partial$ and $h_1=h_2=0$; and by (iv): $g_1=-1$, $g_2=-\partial$ and $g_3=\varphi$.

LEMMA 1.2. *Let A, A', C and C' be chain complexes and let $\varphi: A\rightarrow C$,*

$\varphi' : A' \to C'$, $s : A \to A'$ and $t : C \to C'$ be respective maps. For the existence of the maps $u : M(\varphi) \to M(\varphi')$ and $v : N(\varphi) \to N(\varphi')$ such that the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & M(\varphi) & \xrightarrow{\beta} & N(\varphi) & \longrightarrow & 0 \\ & & \downarrow{s} & & \downarrow{u} & & \downarrow{v} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\alpha']{} & M(\varphi') & \xrightarrow[\beta']{} & N(\varphi') & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{\mu} & M(\varphi) & \xrightarrow{w} & \mathrm{Coker}\,\mu & \longrightarrow & 0 \\ & & \downarrow{t} & & \downarrow{u} & & \downarrow{\bar{u}} & & \\ 0 & \longrightarrow & C' & \xrightarrow[\mu']{} & M(\varphi') & \xrightarrow[w']{} & \mathrm{Coker}\,\mu' & \longrightarrow & 0 \end{array}$$

are commutative, (where Coker μ *and* Coker μ' *do not depend on C, so that we can define $\bar{u}$ by $\bar{u}(a, b) = (sa, sb)$, $(a, b) \in$* Coker μ*), it is necessary and sufficient that the composed maps $t\varphi$ and $\varphi' s$ are homotopic.*

PROOF. If we set formally $u(a, b, c) = (f_1 a + g_1 b + h_1 c, f_2 a + g_2 b + h_2 c, f_3 a + g_3 b + h_3 c)$, then we have $f_2 = f_3 = 0$ and $f_1 = s$ by $u\alpha = \alpha' s$; $h_1 = h_2 = 0$ and $h_3 = t$ by $u\mu = u' t$; $g_1 = 0$, $g_2 = s$ by $\bar{u} w = w' u$; and finally by $\partial u = u \partial$, we have

$$\partial g_3 b + g_3 \partial b = t\varphi b - \varphi' s b\,.$$

Thus $t\varphi$ and $\varphi' s$ are homotopic. Conversely, if $t\varphi$ and $\varphi' s$ are homotopic with the homotopy g_3, then we have

$$u(a, b, c) = (sa, sb, tc + g_3 b) \text{ and } v(b, c) = (sb, tc + g_3 b)$$

as the maps desired.

LEMMA 1.3. *In the situation of the Lemma* 1.2, *let the maps $s' : A \to A'$, $t' : C \to C'$ be given. If s and t be homotopic with s' and t' respectively, then $t'\varphi$, u and v are homotopic with $\varphi' s'$, u' and v' respectively, where u' and v' are the maps induced by (s', t').*

PROOF. Let g_3, Δ and Δ' be homotopies of $t\varphi \simeq \varphi' s$, $s \simeq s'$, and $t \simeq t'$ respectively. Then $\bar{g}_3 = -\Delta'\varphi + g_3 + \varphi'\Delta$ is the homotopy of $t'\varphi \simeq \varphi' s'$, and $D : M(\varphi) \to M(\varphi')$ defined by $D(a, b, c) = (\Delta a, \Delta b, \Delta' c)$ is the homotopy of $u \simeq u'$. Q. E. D.

For the cochain complexes $A = \sum_n A^n$ and $C = \sum_n C^n$ and a *map* $\varphi : C \to A$, $(\varphi\delta = \delta\varphi)$, the *algebraic mapping cylinder* $M^*(\varphi) = \sum_n M^n$ and

φ-relative cochain complex $N^*(\varphi)=\sum_n N^n$ are defined as follows:

$$\begin{cases} M^n = A^n \dotplus A^{n-1} \dotplus C^n, \text{ (direct sum)}, \\ \delta^n(a,b,c) = (\delta a, \ -a-\delta b+\varphi c, \delta c), \ (a,b,c)\in M^n, \text{ and } \delta\delta=0. \end{cases}$$

$$\begin{cases} N^n = A^{n-1} \dotplus C^n, \text{ (direct sum)}, \\ \delta^n(b,c) = (-\delta b+\varphi c, \delta c), \ (b,c)\in N^n, \text{ and } \delta\delta=0. \end{cases}$$

In a similar manner as in the case of the chain complexes, we can set up lemmas analogous to the Lemmas **1.1, 1.2** and **1.3**. We remark only that there exists an exact sequence

$$\cdots\to H^n(N^*(\varphi))\to H^n(C)\xrightarrow{\varphi^*} H^n(A)\to H^{n+1}(N^*(\varphi))\to\cdots,$$

with the induced homomorphism φ^*.[1]

§ 2. Definitions of Tor$^\varphi$ and Ext$_\varphi$.

In the sequel, we assume that the rings Λ and Γ and the ring homomorphism $\varphi:\Lambda\to\Gamma$ are given. We will call thereby $\varphi:\Lambda\to\Gamma$ a *change of rings*. For the terminology and the notations on the change of rings in the homological algebra, we refer to the book [1], Chap. II, § 5, § 6, Chap. V and Chap. VI, § 4.

DEFINITION of Tor$^\varphi$.

Let X, Y be Γ-projective resolutions of A and C respectively in the situation $(A_\Gamma, {}_\Gamma C)$. By $\varphi:\Lambda\to\Gamma$, the situation $(A_\Gamma, {}_\Gamma C)$ is converted to $(A_\Lambda, {}_\Lambda C)$. Let X', Y' be Λ-projective resolutions of A and C in this situations. By virtue of the Λ-projectivity of X' and Λ-acyclicity of X, there exists a Λ-map $F: X'\to X$ over the identity map of A. By the same reason, there exists a Λ-map $G: Y'\to Y$ over the identity map of C. We define a map

$$\Phi^{(1)}: X'\otimes_\Lambda Y'\to X\otimes_\Gamma Y$$

by $\Phi^{(1)}(x'\otimes_\Lambda y')=Fx'\otimes_\Gamma Gy'$. Let $M(\Phi^{(1)})$ and $N(\Phi^{(1)})$ be the algebraic mapping cylinder of $\Phi^{(1)}$ and the $\Phi^{(1)}$-relative chain complex respectively.

1) This exact sequence is mentioned in Exercise 3, [1] p. 73. It is to be noticed however that we have to change the sign of φ, ψ in (1) 1.c., and modify accordingly the meaning of their notations.

We define $\mathbf{Tor}_n^{\varphi}(A, C)$ by

$$\mathrm{Tor}_n^{\varphi}(A, C) = H_n(N(\Phi^{(1)}))\,.$$

From the exact sequence

$$0 \to X' \otimes_\Lambda Y' \to M(\Phi^{(1)}) \to N(\Phi^{(1)}) \to 0\,,$$

we obtain the exact homology sequence:

$$\cdots \mathrm{Tor}_n^{\Lambda}(A, C) \xrightarrow{\varphi_n} \mathrm{Tor}_n^{\Gamma}(A, C) \to \mathrm{Tor}_n^{\varphi}(A, C) \to \mathrm{Tor}_{n-1}^{\Lambda}(A, C) \to \cdots$$

where φ_n is the homomorphism induced by $\Phi^{(1)}$.

THEOREM 2.1. (Uniqueness). (i) *Let $X, \overline{X}$ and $X', \overline{X}'$ be respectively the Γ- and the Λ-projective resolutions of A. Let $Y, \overline{Y}$ and $Y', \overline{Y}'$ be respectively the Γ- and the Λ-projective resolutions of C. Consider the Λ-maps $F: X' \to X$, $\overline{F}: \overline{X}' \to \overline{X}$, $G: Y' \to Y$ and $\overline{G}: \overline{Y}' \to \overline{Y}$. We define $\Phi^{(1)}: X' \otimes_\Lambda Y' \to X \otimes_\Gamma Y$ and $\overline{\Phi}^{(1)}: \overline{X}' \otimes_\Lambda \overline{Y}' \to \overline{X} \otimes_\Gamma \overline{Y}$ as above. Then we have*

$$H(N(\Phi^{(1)})) \approx H(N(\overline{\Phi}^{(1)}))\,.$$

(ii) *We define the maps*

$$\Phi^{(2)}: X' \otimes_\Lambda C = X'_{(\varphi)} \otimes_\Gamma C \to X \otimes_\Gamma C\,,$$

$$\Phi^{(3)}: A \otimes_\Lambda Y' = A \otimes_\Gamma ({}_{(\varphi)}Y') \to A \otimes_\Gamma Y\,,$$

$$\Phi^{(4)}: X' \otimes_\Lambda Y \to X \otimes_\Gamma Y\,,$$

and

$$\Phi^{(5)}: X \otimes_\Lambda Y' \to X \otimes_\Gamma Y$$

by

$$\Phi^{(2)}(x' \otimes_\Lambda c) = F(x') \otimes_\Gamma c\,,$$

$$\Phi^{(3)}(a \otimes_\Lambda y') = a \otimes_\Gamma G(y')\,,$$

$$\Phi^{(4)}(x' \otimes_\Lambda y) = F(x') \otimes_\Gamma y\,,$$

and

$$\Phi^{(5)}(x \otimes_\Lambda y') = x \otimes_\Gamma G(y')$$

respectively. Then we have

$$H(N(\Phi^{(1)})) \approx H(N(\Phi^{(i)}))\,,\ i = 2, \cdots, 5\,.$$

(iii) *For the given change of rings $\varphi: \Lambda \to \Gamma$, $\mathrm{Tor}_n^{\varphi}(A, C)$ is unique as a functor on the category $\mathfrak{A}$ of right Γ-modules and the category $\mathfrak{C}$*

of left Γ-modules.

PROOF. It is well-known that all the maps $\Phi^{(i)}$, $(i=1,2,\cdots,5)$, induce one and the same homomorphism φ_n: $\mathrm{Tor}_n^A(A,C)\to\mathrm{Tor}_n^\Gamma(A,C)$. Since the diagram

$$(*)\qquad \begin{array}{ccc} X'\otimes_A Y' & \longrightarrow & \overline{X}'\otimes_A \overline{Y}' \\ \downarrow & & \downarrow \\ X\otimes_\Gamma Y & \longrightarrow & \overline{X}\otimes_\Gamma \overline{Y} \end{array}$$

is homotopically commutative and the diagrams

$$\begin{array}{ccccc} A\otimes_\Gamma({}_{(\varphi)}Y')=A\otimes_A Y' & \longleftarrow & X'\otimes_A Y' & \longrightarrow & X'\otimes_A C=X'_{(\varphi)}\otimes_\Gamma C \\ \Phi^{(3)}\downarrow & & \Phi^{(1)}\downarrow & & \Phi^{(2)}\downarrow \\ A\otimes_\Gamma Y & \longleftarrow & X\otimes_\Gamma Y & \longrightarrow & X\otimes_\Gamma C \end{array}$$

and

$$\begin{tikzcd} X\otimes_A Y' \arrow[r] \arrow[dr, "\Phi^{(5)}"'] & X'\otimes_A Y' \arrow[r] \arrow[d, "\Phi^{(1)}"] & X'\otimes_A Y \arrow[dl, "\Phi^{(4)}"] \\ & X\otimes_\Gamma Y & \end{tikzcd}$$

are commutative, we can apply Lemma **1.2** and the Five Lemma. Hence (i) and (ii) are proved.

To prove (iii), it remains to show that for any Γ-homomorphisms $f: A\to\overline{A}$ and $g: C\to\overline{C}$, there exists a unique homomorphism $\mathrm{Tor}_n^\varphi(f,g)$: $\mathrm{Tor}_n^\varphi(A,C)\to\mathrm{Tor}_n^\varphi(\overline{A},\overline{C})$. This can easily be proved by Lemmas **1.2** and **1.3** obtaining the homotopically commutative diagram similar to the above one (*).

DEFINITION of Ext^φ.

Taking a Γ-projective resolution X of A, a Λ-projective resolution X' of A, a Γ-injective resolution Y of C and a Λ-injective resolution Y' of C, we consider the Λ-maps $F: X'\to X$, $G: Y\to Y'$ and the map $\Phi_{(1)}$: $\mathrm{Hom}_\Gamma(X,Y)\to\mathrm{Hom}_\Lambda(X',Y')$, where $\Phi_{(1)}$ is defined by $\Phi_{(1)}(\alpha)=G\alpha F$, $\alpha\in\mathrm{Hom}_\Gamma(X,Y)$.

We define $\mathbf{Ext}_n^\varphi$ by the cohomology modules of the $\Phi_{(1)}$-relative cochain complex $N^*(\Phi_{(1)})$:

$$\mathrm{Ext}_\varphi^n(A,C)=H^n(N^*(\Phi_{(1)})),\ ({}_\Gamma A,{}_\Gamma C)\,.$$

From the exact sequence

$$0\to N^*(\Phi_{(1)})\to M^*(\Phi_{(1)})\to\mathrm{Hom}_\Lambda(X',Y')\to 0$$

we obtain the exact cohomology sequence

$$\cdots\to \mathrm{Ext}_{\varphi}^{n}(A, C)\to \mathrm{Ext}_{\Gamma}^{n}(A, C)\xrightarrow{\varphi^{n}}\mathrm{Ext}_{\Lambda}^{n}(A, C)\to \mathrm{Ext}_{\varphi}^{n+1}(A, C)\to\cdots$$

where φ^{n} is the homomorphism induced by $\Phi_{(1)}$.

THEOREM 2.1*. (i) *The definition of* $\mathrm{Ext}_{\varphi}^{n}(A, C)$ *is determined independently of the choice of resolutions of* A *and* C. (ii) *We define the maps*

$$\Phi_{(2)}: \mathrm{Hom}_{\Gamma}(X, C)\to \mathrm{Hom}_{\Gamma}({}_{(\varphi)}X', C)=\mathrm{Hom}_{\Lambda}(X', C)\,,$$

$$\Phi_{(3)}: \mathrm{Hom}_{\Gamma}(A, Y)\to \mathrm{Hom}_{\Gamma}(A, {}^{(\varphi)}Y')=\mathrm{Hom}_{\Lambda}(A, Y')\,,$$

$$\Phi_{(4)}: \mathrm{Hom}_{\Gamma}(X, Y)\to \mathrm{Hom}_{\Lambda}(X', Y)$$

and

$$\Phi_{(5)}: \mathrm{Hom}_{\Gamma}(X, Y)\to \mathrm{Hom}_{\Lambda}(X, Y')$$

respectively by

$$\Phi_{(2)}\alpha=\alpha\circ\overline{F},\ \alpha\in \mathrm{Hom}_{\Gamma}(X, C),\ \overline{F}(\gamma\otimes_{\Lambda}x')=\gamma F(x'),\ \gamma\in\Gamma\,,$$

$$\Phi_{(3)}\alpha=\overline{G}\circ\beta,\ \beta\in \mathrm{Hom}_{\Gamma}(A, Y),\ \overline{G}(y)(\gamma)=G(\gamma y),\ \gamma\in\Gamma\,,$$

$$\Phi_{(4)}\alpha=\alpha\circ F,\ \alpha\in \mathrm{Hom}_{\Gamma}(X, Y)$$

and

$$\Phi_{(5)}\alpha=G\circ\alpha,\ \alpha\in \mathrm{Hom}_{\Gamma}(X, Y)\,.$$

Then we have

$$H(N^{*}(\Phi_{(1)}))\approx H(N^{*}(\Phi_{(i)})),\ i=2,\cdots,5\,.$$

§ 3. Properties of Tor^{φ} and Ext_{φ}.

Projective φ-relative resolutions.

In the sequel, we shall consider the right Γ-module A. Similar arguments will hold for the left Γ-module A.

Let X and X' be Γ- and Λ-projective resolutions of A respectively. For a Λ-map $F: X'\to X$ over the identity map of A, we define the Γ-map $\overline{F}: X'\otimes_{\Lambda}\Gamma\to X$ by setting $\overline{F}(x'\otimes\gamma)=F(x')\gamma$, $x'\in X'$, $\gamma\in\Gamma$. The algebraic mapping cylinder $M^{\Lambda}(\overline{F})$ of $\overline{F}$ and the $\overline{F}$-relative chain complex $N^{\Lambda}(\overline{F})$ are defined as follows:

$$\begin{cases} M_0(\bar{F}) = X_0' \otimes_\Lambda \Gamma \dotplus X_0, \text{ (direct sum)}, \\ \partial_0(x_0' \otimes \xi, x_0) = 0, \quad (x_0' \otimes \xi, x_0) \in M_0(\bar{F}), \end{cases}$$

$$\begin{cases} M_n(\bar{F}) = X_n' \otimes_\Lambda \Gamma \dotplus X_{n-1}' \otimes_\Lambda \Gamma \dotplus X_n, \text{ (direct sum)}, \ n \geqq 1, \\ \partial_n(x_n' \otimes \xi, x_{n-1}' \otimes \eta, x_n) = ((\partial x_n') \otimes \xi - x_{n-1}' \otimes \eta, -(\partial x_{n-1}') \otimes \eta, \\ \partial x_n + \bar{F}(x_{n-1}' \otimes \eta)), \ (x_n' \otimes \xi, x_{n-1}' \otimes \eta, x_n) \in M_n(\bar{F}), \ n \geqq 2, \\ \partial_1(x_1' \otimes \xi, x_0' \otimes \eta, x_1) = ((\partial x_1') \otimes \xi - x_0' \otimes \eta, \partial x_1 + \bar{F}(x_0' \otimes \eta)), \end{cases}$$

and

$$N_0(\bar{F}) = X_0, \ \partial_0 x_0 = 0,$$

$$N_n(\bar{F}) = X_{n-1}' \otimes_\Lambda \Gamma \dotplus X_n, \text{ (direct sum)}, \ n \geqq 1,$$

$$\partial_n(x_{n-1}' \otimes \eta, x_n) = (-(\partial x_{n-1}') \otimes \eta, \partial x_n + \bar{F}(x_{n-1}' \otimes \eta)), \ n \geqq 2,$$

$$\partial_1(x_0' \otimes \eta, x_1) = \partial x_1 + \bar{F}(x_0' \otimes \eta).$$

LEMMA 3.1. *The algebraic mapping cylinder $M^\Lambda(\bar{F})$ of the map $\bar{F}: X' \otimes_\Lambda \Gamma \to X$ is a Γ-projective resolution of A, if we define the augmentation $\bar{\varepsilon}: M^\Lambda(\bar{F}) \to A$ by setting*

$$\bar{\varepsilon}(x_0' \otimes \xi, x_0) = \varepsilon'(x_0')\xi + \varepsilon(x_0),$$

where ε and ε' are the augmentations of X and X' respectively.

PROOF. $\bar{\varepsilon}$ is the augmentation, since

$$\begin{aligned} \bar{\varepsilon}\partial_1(x_1' \otimes \xi, x_0' \otimes \eta, x_1) &= -\varepsilon'(x_0')\eta + \varepsilon(\bar{F}(x_0' \otimes \eta)) \\ &= -\varepsilon'(x_0')\eta + \varepsilon'(x_0')\eta = 0. \end{aligned}$$

We have also $\operatorname{Im} \partial_1 \supset \operatorname{Ker} \bar{\varepsilon}$. In fact, if $(x_0' \otimes \xi, x_0) \in \operatorname{Ker} \bar{\varepsilon}$, then $F(x_0')\xi + x_0 \in \operatorname{Ker} \varepsilon$, since $\bar{\varepsilon}(x_0' \otimes \xi, x_0) = \varepsilon'(x_0')\xi + \varepsilon(x_0) = \varepsilon(F(x_0')\xi) + \varepsilon(x_0)$. By the acyclicity of X, there exists $x_1 \in X_1$ such that $\partial_1 x_1 = F(x_0')\xi + x_0$. Since $\partial_1(0, -x_0' \otimes \xi, x_1) = (x_0' \otimes \xi, x_0)$, we have $(x_0' \otimes \xi, x_0) \in \operatorname{Im} \partial_1$.

$\operatorname{Im} \partial_{n+1} = \operatorname{Ker} \partial_n$, $n \geqq 1$, is evident from the fact that $H_n(M^\Lambda(\bar{F})) \approx H_n(X)$, $n \geqq 0$. Finally, $M^\Lambda(\bar{F})$ is projective by the Proposition II. 5.3 of [1]. Q. E. D.

LEMMA 3.2. (i) *$N^\Lambda(\bar{F}) \otimes_\Gamma C$ is the relative chain complex of $\Phi^{(2)}$: $(X' \otimes_\Lambda \Gamma) \otimes_\Gamma C \to X \otimes_\Gamma C$, and we have $H_n(N^\Lambda(\bar{F}) \otimes_\Gamma C) = \operatorname{Tor}_n^\varphi(A, C)$.* (ii) *If Y be a Γ-projective resolution of C, then $N^\Lambda(\bar{F}) \otimes_\Gamma Y$ is the relative chain* *complex of* $\Phi^{(4)}$: $(X'\otimes_A\Gamma)\otimes_\Gamma Y\to X\otimes_\Gamma Y$, *and we have* $H_n(N^A(\bar{F})\otimes_\Gamma Y)=\mathrm{Tor}_n^\varphi(A, C)$.

PROOF. (i) is evident. (ii):

$$\begin{aligned}&\sum_{i+j=n-1}((X_i'\otimes_A\Gamma)\otimes_\Gamma Y_j)+\sum_{i+j=n}X_i\otimes_\Gamma Y_j\\ &=X_0\otimes_\Gamma Y_n+(X_0'\otimes_A\Gamma+X_1)\otimes_\Gamma Y_{n-1}+\cdots+(X_{n-1}'\otimes_A\Gamma+X_n)\otimes_\Gamma Y_0\\ &=\sum_{i+j=n}N_i(\bar{F})\otimes_\Gamma Y_j. \qquad\qquad \text{Q. E. D.}\end{aligned}$$

We shall refere to $N^A(\bar{F})$ as a *projective* φ*-relative resolution* for the sake of brevity.

THEOREM 3.3. *For any exact sequence* $0\to C'\to C\to C''\to 0$, *which is composed of* Γ*-modules and* Γ*-homomorphisms, we have the exact sequence:*

$$\cdots\to\mathrm{Tor}_n^\varphi(A, C')\to\mathrm{Tor}_n^\varphi(A, C)\to\mathrm{Tor}_n^\varphi(A, C'')\to\mathrm{Tor}_{n-1}^\varphi(A, C')\to\cdots.$$

PROOF. There exist Γ-projective resolutions Y, Y' and Y'' of C, C' and C'' such that the diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&Y'&\longrightarrow&Y&\longrightarrow&Y''&\longrightarrow&0\\ &&\downarrow&&\downarrow&&\downarrow&&\\ 0&\longrightarrow&C'&\longrightarrow&C&\longrightarrow&C''&\longrightarrow&0\end{array}$$

is commutative and each row is exact. Since $N^A(\bar{F})$ is Γ-projective by the Proposition II. 5.3. of [1], the sequence

$$0\to N^A(\bar{F})\otimes_\Gamma Y'\to N^A(\bar{F})\otimes_\Gamma Y\to N^A(\bar{F})\otimes_\Gamma Y''\to 0$$

is exact. From this exact sequence we have the exact homology sequence of the theorem.

REMARK. (i) Since $g: A\otimes_A\Gamma\to A$ is always an epimorphism, we have $\mathrm{Tor}_0{}^\varphi(A, C)=0$ for any $(A_\Gamma, {}_\Gamma C)$. (ii) By (ii) of the Lemma 3.2, some of the formal properties of $\mathrm{Tor}_n^\varphi(A, C)$ can be derived from the properties of $\mathrm{Tor}_n^\Gamma(A, C)$, which are obtained by the properties of Γ-projective resolution of one of the variables. Theorem 3.3 is an example of this.

THEOREM 3.4. (Reduction theorem). *If* Γ *be* Λ*-projective as a left* Λ*-module, then*

$$\mathrm{Tor}_n^\varphi(A, C)\approx\mathrm{Tor}_{n-1}^\Gamma(K_A, C),\ n\geqq 1,$$

where $K_A=\mathrm{Ker}(A\otimes_A\Gamma\to A)$.

PROOF. By the definition

$$0 \to X' \otimes_{\Lambda} \Gamma \to M(\overline{F}) \to N(\overline{F}) \to 0$$

is exact, and its homology sequence gives the isomorphisms

$$H_1(N(\overline{F})) \approx \operatorname{Ker}(A \otimes_{\Lambda} \Gamma \to A),$$

and

$$H_n(N(\overline{F})) \approx \operatorname{Tor}^{\Lambda}_{n-1}(A, \Gamma), \ n \geqq 2.$$

If Γ be Λ-projective, then

$$H_n(N(\overline{F})) = \begin{cases} K_A, & n=1 \\ 0, & n \neq 1. \end{cases}$$

We define now a chain complex $\overline{N}$ by $\overline{N}_0 = 0$, $\overline{N}_1 = \operatorname{Ker}(N_1(\overline{F}) \xrightarrow{\partial_1} N_0(\overline{F}))$, $\overline{N}_n = N_n(\overline{F})$, $n \geqq 2$, the boundary operator of $\overline{N}$ being the same as that of $N(\overline{F})$. Let $\varepsilon'' : \overline{N}_1 \to A \otimes_{\Lambda} \Gamma$ be a map defined by

$$\varepsilon''(x_0' \otimes \xi, x_1) = \varepsilon'(x_0') \otimes \xi,$$

where ε' is the augmentation of X'.

The proof of our theorem is then reduced to that of the following propositions:

(i) $\operatorname{Im} \varepsilon'' = K_A$,

(ii) $\operatorname{Im} \partial_2 = \operatorname{Ker} \varepsilon''$,

(iii) $\overline{N}_1$ is Γ-projective,

(iv) $H(N(\overline{F}) \otimes_{\Gamma} C) \approx H(\overline{N} \otimes_{\Gamma} C)$.

In fact, these imply that $\overline{N}'$ is the Γ-projective resolution of K_A, where $\overline{N}'$ is defined by $\overline{N}'_{n-1} = \overline{N}_n$. We prove now (i),$\cdots$, (iv).

(i) Consider the diagram

$$\begin{tikzcd} \overline{N}_2 \arrow[r] & \overline{N}_1 \arrow[r, "i"] & N_1 \arrow[r, "p"] & X_0' \otimes_{\Lambda} \Gamma \arrow[r, "\varepsilon'"] \arrow[d, "\overline{F}"] & A \otimes_{\Lambda} \Gamma \arrow[d, "g"] \\ & & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\varepsilon"] & A . \end{tikzcd}$$

Since $\partial x_1 + \overline{F}(x_0' \otimes \xi) = 0$ for any $(x_0' \otimes \xi, x_1) \in \overline{N}_1 = \operatorname{Ker} \partial_1$, we have

$$g\varepsilon''(x_0' \otimes \xi, x_1) = \varepsilon'(x_0')\xi = \varepsilon\overline{F}(x_0' \otimes \xi) = -\varepsilon\partial x_1 = 0.$$

Hence $\operatorname{Im}\varepsilon'' \subset K_A$.

Conversely, if $g(a\otimes\xi)=0$, then there exists $x_0'\otimes\xi \in X_0'\otimes_A\Gamma$ such that $\varepsilon'(x_0')\otimes\xi = a\otimes\xi$. Since $\varepsilon\overline{F}(x_0'\otimes\xi)=0$, there exists $x_1\in X_1$ such that $\partial x_1+\overline{F}(x_0'\otimes\xi)=0$. Then $\varepsilon''(x_0'\otimes\xi, x_1)=\varepsilon'(x_0')\otimes\xi=a\otimes\xi$, so that $\operatorname{Im}\varepsilon''=K_A$.

(ii) Since $\varepsilon''\partial_2(x_1'\otimes\eta, x_2)=-(\varepsilon'\partial x_1')\otimes\eta=0$, we have $\operatorname{Im}\partial_2\subset\operatorname{Ker}\varepsilon''$.

Conversely, if $\varepsilon''(x_0'\otimes\xi, x_1)=\varepsilon'(x_0')\otimes\xi=0$, then by the Λ-projectivity of Γ there exists $x_1'\otimes\eta\in X_1'\otimes_A\Gamma$ such that $-\partial x_1'\otimes\eta=x_0'\otimes\xi$. Since $(x_0'\otimes\xi, x_1)\in\overline{N}_1$, we have $\partial x_1=-F(x_0'\otimes\xi)=\partial\overline{F}(x_1'\otimes\eta)$. Hence there exists $x_2\in X_2$ such that $\partial x_2=x_1-\overline{F}(x_1'\otimes\eta)$. Then $\partial_2(x_1'\otimes\eta, x_2)=(-\partial x_1'\otimes\eta, \partial x_2+\overline{F}(x_1'\otimes\eta))=(x_0'\otimes\xi, x_1)$. Thus $\operatorname{Im}\partial_2=\operatorname{Ker}\varepsilon''$.

(iii) For any $x_0\in X_0$, there exists $a\otimes\xi\in A\otimes_A\Gamma$ such that $\varepsilon(x_0)=g(a\otimes\xi)$, and there exists $x_0'\otimes\eta\in X_0'\otimes_A\Gamma$ such that $\varepsilon'(x_0')\otimes\eta=a\otimes\xi$. Since $\varepsilon(\overline{F}(x_0'\otimes\eta)-x_0)=0$, there exists $x_1\in X_1$ such that $-\partial x_1=\overline{F}(x_0'\otimes\eta)-x_0$. Hence $x_0=\partial x_1+\overline{F}(x_0'\otimes\eta)\in\operatorname{Im}\partial_1$. Thus the sequence

$$0\to\operatorname{Ker}\partial_1\to X_0'\otimes_A\Gamma\dot{+}X_1\to X_0\to 0$$

is exact. Since X_0 is Γ-projective, this sequence splits. Hence $\operatorname{Ker}\partial_1$ is Γ-projective.

(iv) We define the chain complex $\overline{\overline{N}}$ by $N(\overline{F})/\overline{N}$, then $\overline{\overline{N}}_n=0$, $n\geqq 2$, $\partial_n=0$, $n\geqq 2$, $\overline{\overline{N}}_1=N_1/\overline{N}_1$, $\overline{\overline{N}}_0=N_0$ and the boundary operator ∂_1 of $\overline{\overline{N}}$ is the isomorphism $\overline{\overline{N}}_1\approx\overline{\overline{N}}_0$. Hence $H(\overline{\overline{N}}\otimes_\Gamma C)=0$. On the other hand, by the arguments in (iii), the exact sequence $0\to\overline{N}\to N\to\overline{\overline{N}}\to 0$ splits, and we have the exact sequence

$$0\to\overline{N}\otimes_\Gamma C\to N\otimes_\Gamma C\to\overline{\overline{N}}\otimes_\Gamma C\to 0\,.$$

By the homology sequence of this, and by the fact that $H(\overline{\overline{N}}\otimes_\Gamma C)=0$, we have the isomorphism $H(\overline{N}\otimes_\Gamma C)\approx H(N\otimes_\Gamma C)$. Q. E. D.

For a left Γ-module A, we can define the φ-relative resolution N, and we obtain:

THEOREM 3.4*. *If Γ be Λ-projective as a right Λ-module, then*

$$\operatorname{Ext}^n_\varphi(A, C)\approx\operatorname{Ext}^{n-1}_\Gamma(K_A, C)\,,$$

where $K_A=\operatorname{Ker}(\Gamma\otimes_A A\to A)$.

Injective φ-relative resolution.

Dually to the above, we can define an *injective φ-relative resolution* of left Γ-module C, by taking Γ- and Λ-injective resolutions Y, Y' and Λ-map $G: Y \to Y'$ over the identity map of C:

$$N^0(\overline{G}) = Y^0, \ \delta y = (Gy, \delta y),$$

$$\begin{cases} N^n(\overline{G}) = \mathrm{Hom}_\Lambda(\Gamma, Y'^{n-1}) \dot{+} Y^n, \text{ (direct sum)}, \\ \delta(f', y) = (-\delta f' + \overline{G}y, \delta y), \ (f', y) \in N^n(\overline{G}), \end{cases}$$

where $\overline{G}: Y \to^{(\varphi)} Y' = \mathrm{Hom}_\Lambda(\Gamma, Y')$ is the Γ-map defined by $(\overline{G}y)(\gamma) = G(\gamma y)$. We can easily set up the dual theorems of the above, of which we mention the following dual of Theorem 3.4:

THEOREM 3.5. *If Γ be Λ-projective as a right Λ-module, then we have the isomorphisms*

$$\mathrm{Ext}^n_\varphi(A, C) \approx \mathrm{Ext}^{n-1}_\Gamma(A, C\sharp), \ n \geqq 1,$$

where $C\sharp = \mathrm{Coker}(C \to \mathrm{Hom}_\Lambda(\Gamma, C))$.

§ 4. Relative group extensions.

Let $\mathfrak{G}$ be a group, $\mathfrak{K}$ a subgroup of $\mathfrak{G}$, Γ the group ring $Z(\mathfrak{G})$ of $\mathfrak{G}$ over the ring Z of integers. Let Λ be the group ring $Z(\mathfrak{K})$. Then the homology and cohomology of $\mathfrak{G}$ are defined by $H_n(\mathfrak{G}: M) = \mathrm{Tor}^\Gamma_n(Z, M)$ and $H^n(\mathfrak{G}: M) = \mathrm{Ext}^n_\Gamma(Z, M)$, for a Γ-module M, and we define the relative homology and relative cohomology of the pair $(\mathfrak{G}, \mathfrak{K})$ by $H_n(\mathfrak{G}, \mathfrak{K}: M) = \mathrm{Tor}^\varphi_n(Z, M)$ and $H^n(\mathfrak{G}, \mathfrak{K}: M) = \mathrm{Ext}^n_\varphi(Z, M)$, for $n \geqq 1$ and a Γ-module M. The definition of the relative ones is due to W. S. Massey [4].

Let X and X' be the standard Γ- and Λ-projective resolutions of Z. We define the cochain complex N by

$$\begin{cases} N^n = \mathrm{Hom}_\Lambda(X'_{n-1}, M) \dot{+} \mathrm{Hom}_\Gamma(X_n, M), \quad \text{(direct sum)}, \\ \delta(f, g) = (-\delta f + \mathrm{res}\, g, \delta g), \quad (f, g) \in N^n, \end{cases}$$

(res = restriction homomorphism), then $H^n(\mathfrak{G}, \mathfrak{K}: M) = H^n(N)$. We shall only compute $H^2(\mathfrak{G}, \mathfrak{K}: M)$ for the present. A systematic analysis of the relative homology will be done on another occasion.

For $(f, g) \in Z^2(N)$ (cocycle), it is necessary and sufficient that

$$\sigma g(\tau, \rho) + g(\sigma, \tau\rho) = g(\sigma, \tau) + g(\sigma\tau, \rho),$$

for any $\sigma, \tau, \rho \in \mathfrak{G}$, and

$$(\mathrm{res}\, g)\,(\kappa, \kappa') = \kappa f(\kappa') - f(\kappa\kappa') + f(\kappa)\,,$$

for any $\kappa, \kappa' \in \mathfrak{K}$. For $(f, g) \in B^2(N)$ (coboundary), it is necessary and sufficient that

$$g(\sigma, \tau) = \sigma h(\tau) - h(\sigma\tau) + h(\sigma),\ \sigma,\ \tau \in \mathfrak{G}\,,$$

$$f(\kappa) = \kappa m - m + \mathrm{res}\, h(\kappa),\ m \in M,\ \kappa \in \mathfrak{K}\,,$$

where h is a fixed element of $\mathrm{Hom}_\Gamma(X_1, M)$ and m is a fixed element of M.

DEFINITION. We define $(\mathfrak{G}, \mathfrak{K})$-extension $(\mathfrak{E}, \overline{\mathfrak{K}}, p)$[2] of $\mathfrak{G}$-module M by the conditions: (1) $\mathfrak{E}$ is a multiplicative group, and $\mathfrak{E}$ contains M as a subgroup.[3] (2) p is an epimorphism $\mathfrak{E} \to \mathfrak{G}$ such that $\mathrm{Ker}\, p = M$. (Hence p induces $i : \mathfrak{E}/M \approx \mathfrak{G}$). (3) $eme^{-1} = p(e)m$, where $e \in \mathfrak{E}$, $m \in M$ and $p(e)$ of the right side is the $\mathfrak{G}$-operator on M. (4) $\overline{\mathfrak{K}}$ is a subgroup of $\mathfrak{E}$ such that $p\,|\,\overline{\mathfrak{K}} : \overline{\mathfrak{K}} \approx \mathfrak{K}$.

Two $(\mathfrak{G}, \mathfrak{K})$-extensions $(\mathfrak{E}, \overline{\mathfrak{K}})$ and $(\mathfrak{E}', \overline{\mathfrak{K}}')$ of M are said to be equivalent, if there exists an isomorphism $t : \mathfrak{E} \approx \mathfrak{E}'$ such that $t\,|\,M =$ identity, $t\,|\,\overline{\mathfrak{K}} : \overline{\mathfrak{K}} \approx \overline{\mathfrak{K}}'$ and $p' t = p$.

Let $(\mathfrak{E}, \overline{\mathfrak{K}})$ be a $(\mathfrak{G}, \mathfrak{K})$-extension of M. Taking a complete system of representatives of $\mathfrak{E}/M$, we define a mapping $q : \mathfrak{G} \to \mathfrak{E}$ which maps $\sigma \in \mathfrak{G}$ to the representative of $i^{-1}(\sigma) \in \mathfrak{E}/M$. Then $p \cdot q =$ identity map of $\mathfrak{G}$, and $\sigma \cdot m = q(\sigma) m q(\sigma)^{-1}$, $\sigma \in \mathfrak{G}$, $m \in M$. The multiplication table of the representatives defines a cocycle $g \in Z^2(\mathfrak{G} : M)$, i. e. g is determined by

$$(*) \qquad q(\sigma) q(\tau) = g(\sigma, \tau) q(\sigma\tau),\ \sigma, \tau \in G,\ g(\sigma, \tau) \in M\,,$$

and satisfies the relation

$$g(\sigma, \tau) + g(\sigma\tau, \rho) = \sigma g(\tau, \rho) + g(\sigma, \tau\rho),\ \sigma,\ \tau,\ \rho \in \mathfrak{G}\,.$$

Since $(\mathfrak{E}, \overline{\mathfrak{K}})$ is a $(\mathfrak{G}, \mathfrak{K})$-extension of M, for every $\kappa \in \mathfrak{K}$ there exists $f(\kappa)$ in M such that $q(\kappa) = f(\kappa)\bar{\kappa}$, where $\kappa = p(\bar{\kappa})$. From the

(2) We abbreviate the notation $(\mathfrak{E}, \overline{\mathfrak{K}}, p)$ as $(\mathfrak{E}, \overline{\mathfrak{K}})$ when it does not cause any confusion.

(3) We express the composition of the elements of M additively in the case of the M-valued cochains and multiplicatively in the case, when M is considered as a subgroup of the multiplicative group $\mathfrak{E}$.

multiplication table (*), f and g are related by:

$$g(\kappa, \kappa')=f(\kappa)+\kappa f(\kappa')-f(\kappa\kappa'), \quad \kappa, \kappa' \in \mathfrak{K}.$$

If we take another system of representatives $r: \mathfrak{G} \rightarrow \mathfrak{E}$ and we define the cocycle $\bar{g} \in Z^2(\mathfrak{G}: M)$ similarly as above, then g and $\bar{g}$ are related by:

$$(**) \qquad \bar{g}(\sigma, \tau)=g(\sigma, \tau)+h(\sigma)+\sigma h(\tau)-h(\sigma\tau), \quad \sigma, \tau \in \mathfrak{G},$$

where h is determined by $r(\sigma)=h(\sigma)q(\sigma)$, $\sigma \in \mathfrak{G}$ and $h(\sigma) \in M$. If t: $(\mathfrak{E}, \overline{\mathfrak{K}}) \approx (\mathfrak{E}', \overline{\mathfrak{K}'})$ be an equivalence of $(\mathfrak{G}, \mathfrak{K})$-extensions of M, and we take $q: \mathfrak{G} \rightarrow \mathfrak{E}$ and $r: \mathfrak{G} \rightarrow \mathfrak{E}'$ similarly as above, then g and $\bar{g}$, which are determined by q and r respectively, must satisfy (**), where h is determined by $r(\sigma)=h(\sigma)t(q(\sigma))$. On the other hand, for a $(\mathfrak{G}, \mathfrak{K})$-extension $(\mathfrak{E}, \overline{\mathfrak{K}})$ of M, we can define an equivalent $(\mathfrak{G}, \mathfrak{K})$-extension $(\mathfrak{E}, \overline{\mathfrak{K}'})$ by putting $\overline{\mathfrak{K}'}=m^{-1}\overline{\mathfrak{K}}m$ for a fixed $m \in M$. Let $t: (\mathfrak{E}, \overline{\mathfrak{K}}) \approx (\mathfrak{E}, \overline{\mathfrak{K}'})$ be the equivalence, then

$$tq(\kappa)=f(\kappa)t(\bar{\kappa})=f(\kappa)m^{-1}\bar{\kappa}m=f(\kappa)m^{-1}\bar{\kappa}m\bar{\kappa}^{-1}\bar{\kappa}=f(\kappa)m^{-1}(\kappa m)\bar{\kappa},$$

i.e., we can take f in $(\mathfrak{E}, \overline{\mathfrak{K}'})$ as

$$\bar{f}(\kappa)=f(\kappa)-m+\kappa m, \quad \kappa \in \mathfrak{K}.$$

But such change of f does not give any influence on the cocycle g, i.e., $g(\kappa, \kappa')=\bar{g}(\kappa, \kappa')$. Thus every equivalence class of $(\mathfrak{G}, \mathfrak{K})$-extension of M determines uniquely a cohomology class in $H^2(\mathfrak{G}, \mathfrak{K}: M)$.

Conversely, we can construct as usual for $(f, g) \in Z^2(\mathfrak{G}, \mathfrak{K}: M)$, a $(\mathfrak{G}, \mathfrak{K})$-extension $(\mathfrak{E}, \overline{\mathfrak{K}})$ by the $\mathfrak{K}$-trivial factor set g, and if (f, g), (f', g') are cohomologous, we obtain equivalent $(\mathfrak{G}, \mathfrak{K})$-extensions $(\mathfrak{E}, \overline{\mathfrak{K}})$, $(\mathfrak{E}' \overline{\mathfrak{K}'})$.

Thus we obtain the following theorem analogously as in the absolute case.

THEOREM 4.1. *Between the relative cohomology group* $H^2(\mathfrak{G}, \mathfrak{K}: M)$ *and the collection of the equivalence classes of* $(\mathfrak{G}, \mathfrak{K})$*-extensions of* M *there holds a one-to-one correspondence.*

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Bibliography

[1] H. Cartan and S. Eilenberg, Homological algebra, Princeton, 1956.

[2] G. Hochschild and J-P. Serre, Cohomology of group extension, Trans. Amer. Math. Soc., 74 (1953), pp. 110–134.

[3] G. Hochschild, Relative homological algebra, Trans. Amer. Math, Soc., 82 (1956), pp. 246–269.

[4] W. S. Massey, Some problems in algebraic topology and the theory of fibre bundles, Ann. of Math., 62 (1955), pp. 327–359.