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# On the representation of complemented modular lattices.

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J. von Neumann [1] has established a beautiful theory of representation of complemented modular lattices, resulting in a generalization of the coordinatization theorem of the projective geometry to the case of the complemented modular lattice with homogeneous basis of degree $\geqq 4$. His theory is presented in a book [2] of F. Maeda, where simplification of proofs obtained by Kodaira and Huruya [3] is taken into accout. However, there still remain considerable difficulties in the construction of the auxiliary ring, and also in the final step of the induction to attain the regular ring representation of the lattice. The purpose of this paper is to simplify further this theory so as to obtain the same results through proofs which present no such difficulties.

Our method is based on the fundamental theorem in § 1 which asserts the existence of the lattice-automorphisms of a certain type. In § 2, we shall construct certain automorphism groups of the lattice and investigate the relations among these groups which will lead us, in § 3, naturally to the definition of the auxiliary ring. In § 4, we shall attain the coordinatization theorem; we shall meet with no 'final step difficulty' (cf. footnote (5)).

To write this paper the author has had frequent consultations with the book of F. Maeda [2]. He also wishes to express his hearty thanks to Professor S. Iyanaga for his encouragement and advices.

## § 1. Fundamental theorem.

Let $L$ be a complemented modular lattice throughout this paper.

First we shall introduce some notions analogous to those used in the combinatorial topology. Let $s$ and $c$ be two elements of $L$ such that $s \neq 1$[1] and $s \geqq c$. These elements $s, c$ will be fixed once for all

1) 1 denotes the maximum element of $L$, and 0 the minimum element.

throughout this paragraph. A finite sequence $a_i$ $(i=1,2,\cdots,p+1)$ of complements of $s$ is said to be a (*oriented*) *p-simplex* if the join of all $a_i$ is orthogonal to $c$, and we denote it by $a_1a_2\cdots a_{p+1}$. (More precisely, this must be called a *p-simplex relative to s, c*, but as we shall consider only such simplexes in this paragraph, we shall not have to add this specification.) A linear combination of $p$-simplexes with integral coefficients is said to be a *p-chain* and the totality of $p$-chains will be considered as usual as an additive group. The boundaries of simplexes or chains are also defined as usual, i.e. $\partial(a_1a_2\cdots a_{p+1})=\sum^{p+1}(-1)^{i+1}(a_1\cdots a_{i-1}a_{i+1}\cdots a_{p+1})$ for $p\geqq 1$; boundaries of 0-chains being 0. A chain whose boundary is 0 is said to be a *cycle* and if a cycle $C$ is the boundary of some chain, then we say that $C$ is *homologous to* 0 and write $C\sim 0$.

Put $S=\{x; x\cap s=0\}$ and $S_x=\{y\in S; y\cap s=x\cap s\}$ for every $x\in S$. Then $S_x\ni y$ is equivalent to $S_x=S_y$. If in particular $x$ is any complement of $s$, then $S_x$ is the totality of complements of $s$. We denote it with $S^0$. The simplexes hitherto considered have their vertices in $S^0$. We shall now consider also simplexes with vertices in $S_x$ (for a fixed $x$) the condition, that the join of vertices should be orthogonal to $c$, remaining as before and the chains formed with these simplexes. They will be called *chains of* $S_x$. (When we say just simplex or chain, we shall mean it in the original sense, i.e. that of $S^0$.)

If any element $a$ of $S^0$ is decomposed in the form $a=a_1\oplus a_2$ [2] then we have a direct sum decomposition $S^0=S_{a_1}\oplus S_{a_2}$, i.e. for every $x\in S^0$, we put $x_1=x\cap(a_1\cap s)$, $x_2=x\cap(a_2\cup s)$ and have $x=x_1\oplus x_2$. $x_i$ is said to be *$a_i$-part of x* $(i=1,2)$. It is evident that $a_1\cup x_2$ and $a_2\cup x_1$ are in $S^0$. Then also every chain $C$ is decomposable in the form $C=C_1\oplus C_2$ where $C_i$ is a chain of $S_{a_i}$. For the purpose, we have only to decompose every vertex of simplexes of $C$. We have then $\partial C=\partial C_1\oplus\partial C_2$. To any decomposition $a=a_1\oplus\cdots\oplus a_k$ of an element $a$ of $S^0$ corresponds thus a decomposition $S^0=S_{a_1}\oplus\cdots\oplus S_{a_k}$ of $S^0$, with which a decomposition of chains $C=C_1\oplus\cdots\oplus C_k$ is associated. A cycle $C$ is said to be *semi-homologous to* 0 and we denote it by $C\approx 0$, if there exists a decomposition

$$C=C_1\oplus C_2\oplus\cdots\oplus C_k,$$

2) $a\oplus b$ means $a\cup b$ and only used in the case $a\cap b=0$.

where all $C_i$ $(i=1, 2, \cdots k)$ are homologous to 0.

Now we shall prove

LEMMA 1. *If there exists an element $p \leqq s$ such that $p$ is perspective to a complement of $s$ and $p \cap c=0$, then every 0-chain with index 0 is homologous to 0.*

PROOF. Let $a \in S^0$ be perspective to $p$ and $x$ an arbitrary element of $S^0$. Put $x_1 = x \cap (a \cup c)$, and let $x_2$ be a relative complement of $x_1$ with respect to $x$, so that we have $x = x_1 \oplus x_2$. Then we obtain the corresponding decomposition $a = a_1 \oplus a_2$ of $a$ and we can see easily that $(a_2 \cup x_1)x$ is a 1-simplex. Since $a$ is perspective to $p$, there exists an element $b \in S^0$ such that we have $a \oplus b = b \oplus p = p \oplus a$ and consequently $ab$ and $b(a_2 \cup x_1)$ are simplexes. Therefore $x-a$ is the boundary of the 1-chain $ab + b(a_2 \cup x_1) + (a_2 \cup x_1)x$ and $x-y=(x-a)-(y-a)$ is homologous to 0 for every $x, y \in S^0$.

COROLLARY. *If we have $a \oplus b \oplus c = 1$ and $a, b$ and $c$ are mutually perspective, then they are also perspective to every complement of $b \cup c$.*

PROOF. In general we can prove easily that $x \cup y \perp z, x \sim y$ and $y \sim z$ imply $x \sim z$. As $s, c$ in the definition of simplexes and chains, we take now $b \cup c$ and $c$. Suppose $xy$ is a simplex. It means then that we have $x \cup y \perp c$, and that $x, y$ are complements of $b \cup c$, so that $x \sim y$. If $x \sim c$, it follows from the above fact that we have also $y \sim c$. Now let $x$ be any complement of $b \cup c$. By our lemma $x-a$ is the boundary of a 1-chain $ax_1 + x_1x_2 + \cdots + x_{r-1}x_r$ with $x_r = x$. As $ax_1$ is a 1-simplex and $a \sim c$ by assumption, we have $x_1 \sim c$. It follows then successively $x_2 \sim c, \cdots, x_r = x \sim c$. $x \sim b$ follows by symmetry of our assumption and $x \sim a$ is obvious.

LEMMA 2. *If there exist $p, q \leqq s$ such that we have $(p, q, c) \perp$*[3] *and both $p$ and $q$ are perspective to a complement of $s$ (hence to all the complements of $s$), then every 1-cycle is semi-homologous to 0.*

PROOF. Every 1-cycle (or more generally 1-cycle of $S_x$) can be represented as a linear combination of cycles of the following type

$$C = xy + yz + zt + tu + \cdots + wx, \tag{1}$$

where the number of vertices $x, y, z, \cdots, w$ is said to be the *rank* of $C$. A cycle $C$ of type (1) is said to be *reducible* if for some decomposition $C = C_1 \oplus \cdots \oplus C_k$ every $C_i$ $(i=1, 2, \cdots, k)$ can be represented as a sum of cycles whose ranks are all less than that of $C$. It is said to

3) $(a, b, c) \perp$ means the independency of $a, b, c$, i.e. that the join of any two among $a, b, c$ is orthogonal to the resting one.

have the property $(p_1)$, if $x\cup c \geqq z$; $(p_2)$ if the rank is $>3$, and $x\cup c \geqq z$, $y\cup c\geqq t$; $(p_3)$ if the rank is $>4$, and $x\cup c\geqq z$, $y\cup c\geqq t$, $z\cup c\geqq u$; and the property $(p)$ if $x\cup y\cup c\geqq z,\cdots, w$. It is to be noticed that $(p_1), (p_2),\cdots, (p_{r-2})$, where $r$ is the rank of $C$, imply $(p)$.

We shall now show that every cycle $C$ of rank $>3$ is decomposable in the form $C=C_1\oplus\cdots\oplus C_k$ where $C_1,\cdots, C_{k-1}$ are reducible and $C_k$ has the property $(p)$. In fact, if the rank of $C$ defined by (1) is $>3$ and $xz$ is a simplex, then we can write

$$C=(xy+yz+zx)+(xz+zt+\cdots+wx),$$

showing that $C$ is reducible. Suppose now $x\cup c \ngeqq z$. Then we have a decomposition $z=z_1\oplus z_2$ with $z_2=(x\cup c)\cap z$. Let $C=C_1\oplus C_2$ be the corresponding decomposition of $C$. We may write $C_i=x_iy_i+y_iz_i+z_it_i+\cdots+w_ix_i$, $i=1,2$. As $z_1\perp z_2$, we have $x\cup c\perp z_1$ and so $x_1\cup c\perp z_1$ i.e. $x_1z_1$ is a simplex. In virtue of what we have proved above, $C_1$ is reducible and $C_2$ has the property $(p_1)$. If $C_2$ has already the properties $(p_2)$, $\cdots, (p_{r-2})$, we have attained our aim. If $C_2$ has not these properties, then we decompose it into a 'reducible factor' and another factor possessing at least the property $(p_2)$, and come to our end after finite number of steps.

Now it is sufficient to prove the lemma for 1-cycle of rank 3 and for 1-cycle with the property $(p)$.

Let $C=xy+yz+zx$ be any 1-cycle of rank 3. We shall show that either $C$ itself has the property $(p)$ or $C$ is decomposable in the form $C_1\oplus C_2$ where $C_1$ has the property $(p)$, and $C_2\sim 0$. In fact, if $x\cup y\cup c \ngeqq z$, we have a decomposition $z=z_1\oplus z_2$ with $z_1=(x\cup y\cup c)\cap z$. It is easy to see that the corresponding decomposition $C=C_1\oplus C_2$ of $C$ has the required property, as $x_2y_2z_2$ is a 2-simplex and $C_2=\partial(x_2y_2z_2)$.

Now let $C$ be a 1-cycle (1) with the property $(p)$. We shall show $C\approx 0$, under the assumptions of our lemma. First assume $x\cup y\cup c\perp p$. As $p$ is perspective to $x$, we can find an axis $a$ of the perspectivity such that $a\leqq p\cup x$. Then we have $x\oplus a=a\oplus p=p\oplus x$, and all $xya$, $yza,\cdots, wxa$ form simplexes, so that $C$ becomes the boundary of a 2-chain $xya+yza+\cdots+wxa$.

Next, assume $(x\cup y)\cap s\leqq p\cup c$. Then we have $x\cup y\perp q$ as $(p, q, c)\perp$, and so $x\cup y\cup c\perp q$. Replacing $p$ by $q$ in the above considerations, we see again $C\sim 0$.

We consider now the general case. Put $y_1=(x\cup p\cup c)\cap y$, $y=y_1\oplus y_2$ and let $C=C_1\oplus C_2$ be the corresponding decomposition of $C$. It is easily seen that $(x_1\cup y_1)\cap s\leqq p\cup c$, $x_2\cup y_2\cup c\perp p$, so that $C_1$, $C_2$ satisfy respectively the second and first of our above assumptions. We have therefore $C_1\sim 0$, $C_2\sim 0$, $C\approx 0$. Q. E. D.

Now we shall prove the following theorem which is fundamental in our theory.

THEOREM 1. *Under the conditions of Lemma 2, there exists for any two complements $a$, $b$ of $s$ such that $a\cup c=b\cup c$ one and only one automorphism $f$ of $L$ which maps $a$ to $b$ and satisfies the following condition:*

$$s\geqq x \text{ or } c\leqq x \text{ implies } f(x)=x. \tag{*}$$

PROOF. We shall devide the proof in several steps.

(i) If $xy$ is a simplex of $S_x$, there exists a perspective isomorphism of $L_{x\cup c}$[4] to $L_{y\cup c}$ by the axis $(x\cup y)\cap s$. Hence if $C=xu+uv+\cdots+wy$ is a chain of $S_x$, we obtain a projective isomorphism $\varphi(C)=\varphi$ of $L_{x\cup c}$ to $L_{y\cup c}$:

$$\varphi=h\circ\cdots\circ g\circ f, \tag{1}$$

where $f, g, \cdots, h$ are the above perspective isomorphisms of $L_{x\cup c}$ to $L_{u\cup c}$, $L_{u\cup c}$ to $L_{v\cup c}, \cdots$ and $L_{w\cup c}$ to $L_{y\cup c}$ respectively.

We shall prove that $\varphi$ is determined by $x$ and $y$ only independently of the choice of $u, v, \cdots$. For that purpose it is sufficient to prove that $\varphi(C)$ is identity if $C$ is a cycle. This is true if $C$ is the boundary of a 2-simplex $xuv$, because then the intermediate perspective mappings have a common axis $(x\cup u\cup v)\cap s$. Therefore $\varphi(C)$ is identity if $C\sim 0$. By Lemma 2, we can decompose any cycle $C$ as $C=C_1\oplus C_2\oplus\cdots\oplus C_k$ such that $C_i\sim 0$ $(i=1, 2, \cdots, k)$. As $\varphi(C)$ is already shown as identity in case $k=1$, we shall consider now the case $k\geqq 2$. Let $x=x_1\oplus x_2\oplus\cdots\oplus x_k$ be the corresponding decomposition of $x$, then $\varphi=\varphi(C)$ is identity on every $L_{x_i\cup c}$. On the other hand, $\varphi$ is also obviously identity on $L_x$. Now $L'=\{y\in L_{x\cup c}; \varphi(y)=y\}$ is a sublattice of $L_{x\cup c}$ including $L_{x_i\cup c}$ $(i=1, 2, \cdots, k)$ and $L_x$. Then we have $L'\supset L_{x_1\cup x_2\cup c}$, because an arbitrary element of $L_{x_1\cup x_2\cup c}$ can be written as $z_1\cup z_2\cup y$ where $z_1\in L_{x_1\cup c}$, $z_2\in L_{x_2\cup c}$ and $y$ is orthogonal to both $x_1\cup c$ and $x_2\cup c$, and hence

$$y=[\{(x_1\cup y)\cap(x_2\cup c)\}\cup x_1]\cap[\{(x_2\cup y)\cap(x_1\cup c)\}\cup x_2]\in L'.$$

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4) $L_x$ means the totality of the element $y\in L$ such that $y\leqq x$.

Thus we have $L'=L_{x\cup c}$ if $k=2$. Similarly we can conclude also in case $k\geqq 3$ that $L'=L_{x\cup c}$, that is, $\varphi$ is identity on whole $L_{x\cup c}$.

The projective isomorphism $\varphi$ defined by (1) is said to be the *canonical isomorphism* (abbr. c. i.) *with respect to* $x\rightarrow y$ *of* $L_{x\cup c}$ *to* $L_{y\cup c}$, or simply *of* $x$ *to* $y$.

(ii) Let $\varphi, \psi$ be the c. i. of $x$ to $y$ and of $y$ to $z$ respectively. Then $\psi\circ\varphi$ is the c. i. of $x$ to $z$. If $L_{x\cup c}=L_{y\cup c}$ the c. i. of $x$ to $y$ will be called a *canonical automorphism* (abbr. c. a.) *of* $L_{x\cup c}$. The totality of c. a. 's of $L_{x\cup c}$ constitutes a group. Now we shall prove that this group is commutative. We can suppose $x$ to be a complement of $s$ without loss of generality. For any two c. a. 's of $L_{x\cup c}$, $\varphi$ and $\psi$, let $\varphi_1, \varphi_2, \psi_1$, and $\psi_2$ be the c. i. of $x$ to $y$, $y$ to $\varphi(x)$, $x$ to $z$ and $z$ to $\psi(x)$ respectively, where $y\,(z)$ is an axis of the perspectivity between $x$ and $p$ $(q)$ such that $y\leqq x\cup p$ $(z\leqq x\cup q)$. Then we have $\varphi=\varphi_2\circ\varphi_1$, $\psi=\psi_2\circ\psi_1$. Since $\psi_1$ and $\psi_2$ are perspective mappings with axis orthogonal to $p\cup c$, we can extend $\psi_1$ $(\psi_2)$ to the perspective isomorphism $\overline{\psi}_1$ $(\overline{\psi}_2)$ of $L_{x\cup p\cup c}$ to $L_{z\cup p\cup c}$ (of $L_{z\cup p\cup c}$ to $L_{\psi(x)\cup p\cup c}$). Then we can extend $\psi$ to the automorphism $\overline{\psi}=\overline{\psi}_2\circ\overline{\psi}_1$ of $L_{x\cup p\cup c}$ $(=L_{\psi(x)\cup p\cup c})$. Since $\varphi_i\,(i=1, 2)$ is a perspective mapping with axis in $L_{p\cup c}$ and $\overline{\psi}$ fixes every element of $L_{p\cup c}$, $\overline{\psi}\circ\varphi_i\circ\overline{\psi}^{-1}$ is also a perspective mapping with the same axis as $\varphi_i$ and hence coincides with $\varphi_i$. Thus we have

$$\psi\circ\varphi\circ\psi^{-1}=\overline{\psi}\circ\varphi_2\circ\overline{\psi}^{-1}\circ\overline{\psi}\circ\varphi_1\circ\overline{\psi}^{-1}=\varphi_2\circ\varphi_1=\varphi\,.$$

(iii) Let $\varphi$ be the c. i. of $x$ to $\varphi(x)$ and suppose $y\perp s$, $x\cup c=y\cup c$. Then $\varphi(x)\leqq x\cup u$ for $u\leqq s$, implies $\varphi(y)\leqq y\cup u$. In fact, if $x\varphi(x)$ is a simplex of $S_x$, then $\varphi$ is a perspective isomorphism and our assertion is trivial. If $\varphi$ is a c. a. of $L_{x\cup c}$, then $\varphi$ commutes with the c. i. $\psi$ of $x$ to $y$, and hence,

$$\varphi(y)=\varphi(\psi(x))=\psi(\varphi(x))\leqq\psi(x\cup(u\cap c))=y\cup(u\cap c)\leqq y\cup u\,.$$

In the general case, decompose $\varphi(x)$ into $\varphi(x)\cap(x\cup c)\oplus$(an element orthogonal to $\varphi(x)\cap(x\cup c)$), and let $x=x_1\oplus x_2$, $y=y_1\oplus y_2$ be the corresponding decompositions of $x$ and $y$. Then we have $\varphi(x_1)=\varphi(x)\cap(x\cup c)$, and the c. i. of $x_1$ to $\varphi(x_1)$ is the restriction of $\varphi$ to $L_{x_1\cup c}$ and a c. a. of $L_{x_1\cup c}$, whereas the c. i. of $x_2$ to $\varphi(x_2)$ is a perspective isomorphism. By what we have seen above, we have $\varphi(y_1)\leqq y_1\cup u$, $\varphi(y_2)\leqq y_2\cup u$ and hence $\varphi(y)\leqq y\cup u$.

(iv) Now we shall define $f(x)$ for every $x\in S$ as follows. Let

$a'$ and $b'$ be the $x$-parts of $a$ and $b$ respectively and $\varphi$ the c. i. of $a'$ to $x$, then we define $f(x)$ to be $\varphi(b')$. If we exchange $a$ and $b$, then we obtain the inverse mapping of $f$. So $f$ is a one-to-one order preserving mapping of $S$ onto $S$. We shall show that $y \leqq x \cup u$ implies $f(y) \leqq f(x) \cup u$ for $x, y \in S$ and $u \leqq s$. Let $x'$ be the $y$-part of $x$, and $\psi$ the c. i. of $x'$ to $y$, then we have $\psi(x') \leqq x' \cup u$, and hence, $\psi(f(x')) \leqq f(x') \cup u$ as proved in (iii). Since we can see easily by the definition of $f$ that $\psi(f(x')) = f(y)$, we have $f(y) \leqq f(x) \cup u$.

(v) Now we shall extend $f$ to the whole $L$. An arbitrary element of $L$ can be written as $x \cup u$ where $x \perp s$ and $u \leqq s$. Then we define $f(x \cup u) = f(x) \cup u$. If $x \cup u \leqq y \cup v$ for another pair $y, v$ such that $y \perp s$, $v \leqq s$, then we have $f(x) \cup u \leqq f(y) \cup v$ as proved in (iv). Thus $f(x \cup u)$ is determined only by $x \cup u$, and $f$ is a one-to-one order preserving mapping of $L$ onto $L$, and hence $f$ is an automorphism of $L$. $f$ satisfies obviously the conditions of our theorem.

(vi) We have nothing more to prove than the uniqueness of $f$. Suppose there exists another automorphism $f'$ satisfying the conditions of the theorem. To see $f = f'$, we have only to prove that $f(x) = f'(x)$ for every complement $x$ of $s$. Let $\varphi$ be the c. i. of $a$ to $x$, then $\varphi$ is also the c. i. of $f'(a) = b$ to $f'(x)$, since every c. i. is defined by perspective mappings between $L_{y \cup c}$ $(y \perp s)$ with axis in $L_s$ and $f'$ keeps these axis invariant. By the construction of $f$, we have

$$f(x) = \varphi(b) \text{ and hence } f(x) = f'(x). \qquad \text{Q. E. D.}$$

An automorphism $f$ of $L$ is said to be *normal for an* element $s \in L$, if $f$ fixes every element $x$ such that $x \leqq s$ or $x \geqq s$, and there exists $c \leqq s$ for which we have $(y \cup f(y)) \cap s = c$ for every complement $y$ of $s$. If $f$ is normal for $s$, then the element $c$ above is said to be the *axis of $f$ in $s$* and we denote $c = \pi_s(f)$ (or simply $\pi(f)$).

Then we have

COROLLARY to Theorem 1. *The automorphism $f$ in Theorem 1 is normal for $s$, and the totality of automorphisms $f$ of $L$ which are normal for $s$ and for which $\pi_s(f) \leqq c$, constitutes a commutative group.*

## § 2. Automorphism groups.

We suppose the existence of a "homogeneous basis" of degree $n \geqq 4$ in the rest of the paper, that is, we suppose that there exist mutually perspective elements $a_i$ $(i = 1, 2, \cdots, n)$ such that we have

$a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1$.

We write $\bar{a}_i$ for $\bigcup_{j \neq i} a_j$, $L_i$ for $L_{a_i}$ and $\overline{L_i}$ for $L_{\bar{a}_i}$. Let $G_{ij}$ be the set of all automorphisms $f$ of $L$ such that $f$ is normal for $\bar{a}_i$ and $\pi(f) \leqq a_j$. If we put $s = \bar{a}_i$ and $c = a_j$ for $s$ and $c$ in § 1, then we see by Theorem 1 that $G_{ij}$ constitutes a commutative group and $\{f(a_i);\ f \in G_{ij}\}$ is the totality of complements of $a_j$ in $L_{a_i \cup a_j}$. If $f \in G_{ij}$ and $g \in G_{ik}$ for $j \neq k$, then we have $fg = gf$. In fact, $fgf^{-1}$ is obviously normal for $a_i$ and we have $\pi(fgf^{-1}) \leqq a_k$ and hence, $fgf^{-1} \in G_{ik}$. Similarly, we have $gf^{-1}g^{-1} \in G_{ij}$. Then $fgf^{-1}g^{-1}$, being an element of the intersection of $G_{ij}$ and $G_{ik}$, is obviously identity.

Therefore the group $G_i$ which is generated by all $G_{ij}$ for $j \neq i$ ($i$ fixed) is commutative.

Now we shall prove

PROPOSITION 1. *For any two complements $x, y$ of $\bar{a}_i$ there exists one and only one automorphism $f$ in $G_i$ which maps $x$ to $y$.*

PROOF. In case $x \cup a_j = y \cup a_j$, our assertion is Theorem 1 itself. If we have $x \cup a_j \cup a_k = y \cup a_j \cup a_k$, then $z = (x \cup a_j) \cap (y \cup a_k)$ is another complement of $a_i$ and we have $x \cup a_j = z \cup a_j$, $y \cup a_k = z \cup a_k$, so that the existence of our automorphism follows from the first case. Similarly we can proceed further and prove the existence of $f$ in the most general case. If $f$ fixes a complement of $a_i$ then it also fixes all the complements of $a_i$ by virtue of the commutativity of $G_i$ and hence it is identity. This shows also the uniqueness of above $f$.

PROPOSITION 2. *$f \in G_i$ is normal for $\bar{a}_i$, and we have $\pi(fg) \leqq \pi(f) \cup \pi(g)$ for every $f, g \in G_i$.*

PROOF. For every two complements $x, y$ such that $y = g(x), g \in G_i$ we have $(x \cup f(x)) \cap a_i = (g(x) \cup gf(x)\} \cap a_i = (y \cup f(y)) \cap a_i$. Thus $f$ is normal for $a_i$. Moreover we have, by virtue of the equality $x \cup g(x) = x \cup \pi(g)$,

$$x \cup fg(x) \leqq x \cup f(x) \cup fg(x) = x \cup f(x \cup \pi(g)) = x \cup \pi(f) \cup \pi(g),$$

and hence $\pi(fg) = (x \cup fg(x)) \cap a_i \leqq \pi(f) \cup \pi(g)$.

It follows from this proposition that $G_{ij}$ is the totality of $f \in G_i$ whose axis $\pi(f)$ is in $L_j$, and hence $G_i$ is a direct sum of $G_{ij}$ ($j \neq i$).

Every automorphism $f$ in $G_i$ preserves clearly the decomposition relation among the complements of $a_i$, i.e. if we have $x = x_1 \oplus x_2$, then $x_1$-part of $f(x)$ is $f(x_1)$. For such decomposition of $x$, there exist $f_1$ and $f_2$ in $G_i$ such that $f_1(x) = f(x_1) \cup x_2$ and $f_2(x) = x_1 \cup f(x_2)$. Then $f_1$ (or

$f_2$) fixes all the $x_2$-part ($x_1$-part), and coincides with $f$ on $x_1$-parts ($x_2$-parts) of elements, and hence we have $f=f_1f_2$. We can see easily that this decomposition of $f$ determines a decomposition of $G_i$ to a direct sum. We call $f_1$ the *$x_1$-part of $f$.* It is determined by $x_1$ and $x_2$, and not by $x_1$ alone, though the $x_1$-parts of elements are determined by $x_1$ alone.

Next we shall investigate the relation between different $G_i$ and $G_j$. We shall make use of the following notations: $b=a_i \cup a_j \cup \cdots \cup a_k$ where $i, j, \cdots, k$ are all different from 1 and 2, and

$$H_1=\{f\in G_1\,;\pi(f)\leqq b\},\ H_2=\{f\in G_2\,;\pi(f)\leqq b\}\,.$$

PROPOSITION 3. *For every $g\in H_2$ and $f\in G_1$ we have*

$$gfg^{-1}\in G_1 \text{ and } \pi(gfg^{-1})=g(\pi(f))\,.$$

PROOF. Let $x$ be any complement of $\bar{a}_1$. Since $g^{-1}(x)$ is also a complement of $\bar{a}_1$, we have

$$x\cup gfg^{-1}(x)=g(g^{-1}(x)\cup fg^{-1}(x))=g(g^{-1}(x)\cup \pi(f))=x\cup g(\pi(f))\,,$$

and hence $gfg^{-1}$ is normal and $\pi(gfg^{-1})=g\pi(f)$. Then applying the uniqueness part of Theorem 1, $\bar{a}_1$ as $s$ and $g(\pi(f))$ as $c$ we see that $gfg^{-1}$ is in $G_1$.

COROLLARY. *Every element of $H_1$ is permutable with that of $H_2$.*

PROOF. If $f$ is in $H_1$ in the proposition, then we have $gfg^{-1}(a_1)=f(a_1)$ since $f(a_1)$ is in $\bar{L}_2$, and hence $gf=fg$.

We write in the sequel $f\otimes g$ for $fgf^{-1}g^{-1}$, then for every $f\in G_1$ and $g\in H_2$, $f\otimes g$ is in $H_1$, because we have $g(x)=g^{-1}(x)=x$ for every $x\geqq b$ and hence $f\otimes g(a_1\cup b)=a_1\cup b$.

In particular we have $G_{ij}\otimes G_{jk}\subset G_{ik}$.

PROPOSITION 4. *For every $f, f'\in G$, and $g, g'\in H_2$, we have*

$$f\oplus gg'=(f\otimes g)\,(f\otimes g')\,,$$

$$ff'\otimes g=(f\oplus g)\,(f'\otimes g)\,.$$

PROOF. Since $fgf^{-1}=(f\otimes g)g$ is permutable with $H_1$ and $H_2$, we have $f\otimes g=g^{-1}fgf^{-1}=f^{-1}g^{-1}fg$, and hence $f(f\otimes g)=g^{-1}fg$. Therefore we have $(f\otimes g)\,(f\otimes g')gg'=(fgf^{-1})\,(fg'f^{-1})=(f\otimes gg')gg'$ and $ff'(f\otimes g)(f'\otimes g)=(g^{-1}fg)\,(g^{-1}f'g)=ff'(ff'\otimes g)$. Q. E. D.

Let $a_1=u\oplus v$. Then we can see easily that if $u$-part of $f$ is $f_1$, then $u$-part of $f\otimes g$ is $f_1\otimes g$.

PROPOSITION 5. *For every $f\in H_1$ and $g\in H_2$, we have* $\pi(f)\leqq\pi(g)$

*if and only if there exists* $h \in G_{12}$ *such that we have* $f = h \otimes g$.

PROOF. Put $f = h \otimes g$, then we have

$$\pi(f) \leqq \pi(h) \cup \pi(gh^{-1}g^{-1}) = \pi(h) \cup g(\pi(h)) \leqq a_2 \cup g(a_2) = a_2 \cup \pi(g),$$

and hence $\pi(f) \leqq (a_2 \cup \pi(g)) \cap b = \pi(g)$. Conversely $\pi(f) \leqq \pi(g)$ implies $(a_1 \cup g(a_2)) \cup a_2 = a_1 \cup g(a_2) \cup \pi(g) \cup a_2 = f(a_1) \cup \pi(f) \cup g(a_2) \cup a_2 = (f(a_1) \cup g(a_2)) \cup a_2$, and we have always $(a_1 \cup g(a_2)) \cap a_2 = g(a_2) \cap a_2 = (f(a_1) \cup g(a_2)) \cap a_2$. Then there exists a perspectivity between $a_1 \cup g(a_2)$ and $f(a_1) \cup g(a_2)$ axis in $L_2$, and we can find $h$ in $G_{12}$ such that $h(a_1)$ coincides with the image of $a_1$ by this perspective mapping, in other words, we have $h(a_1) \cup g(a_2) = f(a_1) \cup g(a_2)$ or $g^{-1}hg(a_1) \cup a_2 = f(a_1) \cup a_2$. Since the left side of the latter equality equals to $h^{-1}g^{-1}hg(a_1) \cup a_2$ we have $(h \otimes g)(a_1) = f(a_1)$, that is, $h \otimes g = f$.

COROLLARY. *If we have* $a_2 \cap g(a_2) = 0$, *then the mapping* $h \to h \otimes g$ *is an isomorphism of* $G_{12}$ *onto* $\{f; \pi(f) \leqq \pi(g)\}$, *a subgroup of* $H_1$.

PROOF. If $h \otimes g$ is identity, then we have $\pi(h) = \pi(ghg^{-1}) = g(\pi(h))$ and hence $\pi(h) = \pi(h) \cap g(\pi(h)) = 0$. This shows that $h$ is identity.

PROPOSITION 6. *If we have* $\pi(h) = a_2$ *and* $h(a_1) \cap a_1 = 0$ *for* $h \in G_{12}$, *then the mapping* $g \to h \otimes g$ *is an isomorphism of* $H_2$ *onto* $H_1$, *and we have* $\pi(g) = \pi(h \otimes g)$.

PROOF. $h = ghg^{-1}$ implies $g(a_2) = g(\pi(h)) = \pi(h) = a_2$, and hence the mapping is an isomorphism. Let $f$ be an arbitrary element of $H_1$. Since $h(a_1)$ is a complement of $\bar{a}_2$ and we have $h(a_1) \cup b = fh(a_1) \cup b$, there exists $g \in H_2$ such that $g^{-1}h(a_1) = fh(a_1)$ $(= hf(a_1))$. Then we have

$$h \otimes g(a_1) = gh^{-1}g^{-1}h(a_1) = gf(a_1) = f(a_1)$$

and hence the mapping is onto. Moreover, since we have

$$\pi(f) = \{x \cup f(x)\} \cap b$$

for every complement $x$ of $a_1$, putting $x = h(a_1)$ we have

$$\pi(f) = (x \cup g^{-1}(x)) \cap b = (x \cup g(x)) \cap b = \pi(g).$$

COROLLARY. *By the perspective isomorphism of* $L_{a_2 \cup b}$ *to* $L_{a_1 \cup b}$ *with axis* $h(a_1)$, *the image of* $g(a_2)$ *is* $h \otimes g(a_1)$.

PROOF. Putting $f = h \otimes g$, we have

$$h(a_1) \cup f(a_1) = g(g^{-1}h(a_1) \cup f(a_1)) = gf(h(a_1) \cup a_1)$$
$$= gf(h(a_1) \cup a_2) = h(a_1) \cup g(a_2).$$

For the associativity of the operation $\otimes$ we shall prove

PROPOSITION 7. *For every* $f \in G_{ij}$, $g \in G_{jk}$ *and* $h \in G_{kl}$ *where* $i, j, k$ *and* $l$ *are all different, we have*

$$(f \otimes g) \otimes h = f \otimes (g \otimes h) .$$

PROOF. Since $h^{-1} f h = f$, we have

$$(f \otimes g)\{(f \otimes g) \otimes h\} = h^{-1}(f \otimes g) h = f \otimes h^{-1} g h$$
$$= f \otimes \{g(g \otimes h)\} = (f \otimes g)\{f \otimes (g \otimes h)\} .$$

## § 3. Auxiliary ring.

To establish definite isomorphisms among all $G_{ij}$, we take first from every $G_{1i}$ $(i = 2, 3, \cdots, n)$ an automorphism $\Gamma_{1i}$ such that $\pi(\Gamma_{1i}) = a_i$ and $\Gamma_{1i}(a_1) \cap a_1 = 0$. As we have shown in § 2, there exist following isomorphisms:

$$G_{ij} \ni f \rightarrow \Gamma_{1i} \otimes f \in G_{1j}, \quad \text{and} \quad G_{i1} \ni f \rightarrow f \otimes \Gamma_{1j} \in G_{ij} .$$

Now we determine $\Gamma_{ij}$ for $i, j \neq 1$ by the equation $\Gamma_{1i} \otimes \Gamma_{ij} = \Gamma_{1j}$. Then we have

$$\Gamma_{ij} \otimes \Gamma_{jk} = \Gamma_{ik} \text{ for every } i, j, k \neq 1 , \qquad (*)$$

because we have $\Gamma_{1i} \otimes (\Gamma_{ij} \otimes \Gamma_{jk}) = (\Gamma_{1i} \otimes \Gamma_{ij}) \otimes \Gamma_{jk} = \Gamma_{1k}$.

Every $\Gamma_{i1}$ is determined by the equation $\Gamma_{i1} \otimes \Gamma_{1j} = \Gamma_{ij}$, where $\Gamma_{i1}$ does not depend on $j$, because we have for another $k$

$$\Gamma_{i1} \otimes \Gamma_{1k} = \Gamma_{i1} \otimes \Gamma_{1j} \otimes \Gamma_{jk} = \Gamma_{ij} \otimes \Gamma_{jk} = \Gamma_{ik} .$$

The equation $\Gamma_{ij} \otimes \Gamma_{j1} = \Gamma_{i1}$ is also valid since we have $(\Gamma_{ij} \otimes \Gamma_{j1}) \otimes \Gamma_{1k} = \Gamma_{ij} \otimes \Gamma_{jk} = \Gamma_{ik}$ where $k$ is different from $1, i, j$. Therefore (*) is true for every different $i, j, k$.

Since by Proposition 6 we have $\pi(\Gamma_{ij}) = a_j$ and $\Gamma_{ij}(a_i) \cap a_i = 0$ for every $i, j$, we obtain a definite isomorphism of $G_{ij}$ to $G_{kj}$: $f \rightarrow \Gamma_{ki} \otimes f$ and that of $G_{ij}$ to $G_{ik}$: $f \rightarrow f \otimes \Gamma_{jk}$.

Now we consider every pair $(i, j)$ $(1 \leq i, j \leq n, i \neq j)$ as a point of the Cartesian plane with co-ordinates $i, j$, and we denote such points by $P, Q, R, S, T, \cdots$. If $P = (i, j)$ we write $G_P$ for $G_{ij}$. If $P \neq Q$ and the line $\overline{PQ}$ is parallel to one of the co-ordinates axis, then we assign to the vector $\overrightarrow{PQ}$ the isomorphism of $G_P$ to $G_Q$ which is defined above. If $\overrightarrow{PQ}, \overrightarrow{QR}, \cdots, \overrightarrow{ST}$ are vectors in succession each of which is parallel to one of the co-ordinate axis, and such that the end point of one vector coincides with the origin of the following vector, then we say $PQR\cdots ST$ forms a *pass*. To every such pass $PQR\cdots ST$ we assign the isomorphism of $G_P$ to $G_T$ composed of the isomorphisms corresponding to the vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \cdots, \overrightarrow{ST}$. We write $PQR\cdots ST \sim PQ'R'\cdots S'T$ if these two passes determine the same isomorphism. We consider also $PP$ as a pass and denote it by 0, and assign to it the identity mapping of $G_P$. If $P$, $Q$ and $R$ are on the same line which is parallel to one of the axis, then we have by (*), $PQR \sim PR$ and $PQPR \sim PQR \sim PR$ and hence $PQP \sim 0$. Moreover every rectangular pass $PQRSP$ is $\sim 0$, since we have $PQR \sim PSR$ as an immediate consequence of the associativity of the operation $\otimes$. Then we can see easily that every closed pass is $\sim 0$. We have namely only to 'decompose' the closed pass into rectangular passes. In doing so, we must take into consideration that $G_P$ is not defined for points $P=(i, i)$. We can, however, easily arrange the 'decomposition', so that these points do not appear as vectors of rectangular passes, since we have $n \geqq 4$ by hypothesis. Thus $P=(i, j)$, $T=(k, l)$ being any two points under cosideration, every pass joining $P$ with $T$ determine the same definite isomorphism of $G_P$ to $G_T$.

Let $R$ be a group which is isomorphic to $G_P = G_{ij}$. The isomorphic image of $\alpha \in R$ in $G_{ij}$ will be denoted by $(\alpha)_{ij}$. By what we have proved, we can take $(\alpha)_{ij}$ for every $i, j$, so that the relations $\Gamma_{ij} \otimes (\alpha)_{jk} = (\alpha)_{ij} \otimes \Gamma_{jk} = (\alpha)_{ik}$ hold.

We write $\alpha+\beta$ for the group multiplication of $\alpha$ and $\beta$ in $R$, and we define the new multiplication $\alpha\beta$ as:

$$(\alpha\beta)_{ik} = (\alpha)_{ij} \otimes (\beta)_{jk} .$$

We can see easily by the associativity of $\otimes$ that $(\alpha)_{ij} \otimes (\beta)_{jk}$ is independent of $j$, so that this definition has a sence, and by virtue of Proposition 4 and 7, $R$ can be considered as a ring by addition and multiplication thus defined, and it has the unit 1 for which we have $(1)_{ij} = \Gamma_{ij}$.

$R$ is said to be the *auxiliary ring of* $L$ determined by the *frame* $\{a_i, \Gamma_{ij}; i, j=1, 2, \cdots, n\}$.

By Proposition 6 $\pi((\alpha)_{ij})$ does not depend on $i$, so we put $\pi_j(\alpha) = \pi((\alpha)_{ij})$. Then we can see easily that the image of $\pi_i(\alpha)$ by the perspective isomorphism of $L_i$ to $L_j$ with axis $\Gamma_{ij}(a_i)$ $(=\Gamma_{ji}(a_j))$ is $\pi_j(\alpha)$. Let $a_i = u \oplus v$ be a decomposition of $a_i$. This induces the decomposition of every element of $G_i$ as defined in § 2, hence in particular of

$\Gamma_{ij}$. Let $\varepsilon_{ij}$ be the $u$-part of $\Gamma_{ij}$, where $\varepsilon$ is an element of $R$. Then for every $\alpha \in R$, the $u$-part of $(\alpha)_{ik} = \Gamma_{ij} \otimes (\alpha)_{jk}$ is clearly $(\varepsilon)_{ij} \otimes (\alpha)_{jk}$ Since decomposition operator is idempotent, $\varepsilon$ is an idempotent element, and the $v$-part of $(\alpha)_{ik}$ is obviously $((1-\varepsilon)\alpha)_{ik}$. Moreover we have $\pi_j(\varepsilon) = (a_i \cup (\varepsilon)_{ij}(a_i)) \cap a_j = (u \cup \Gamma_{ij}(a_i)) \cap a_j$ and hence $\pi_i(\varepsilon) = u$ and similarly $\pi_i(1-\varepsilon) = v$. Let $f \in G_i$, and $f = (\alpha)_{ij}(\beta)_{ik} \cdots$ be the direct sum decomposition of $G_i$ to $\prod_{j \neq i} G_{ij}$, $\alpha, \beta, \cdots$ being elements of $R$. Furthermore, let $\xi$ be any element of $R$. We shall denote with $\xi f$ the element of $G_i$ which has the decomposition $(\xi\alpha)_{ij}(\xi\beta)_{ik} \cdots$. Then $\varepsilon f = (\varepsilon\alpha)_{ij}(\varepsilon\beta)_{ik} \cdots$ is obviously the $u$-part of $f$ and we have $\pi(f) = \pi(\varepsilon f) \cup \pi((1-\varepsilon)f)$ because $\pi(\varepsilon f) \leqq \pi(f)$.

By virtue of Proposition 5 we have $R\alpha \supset R\beta$ if and only if $\pi_i(\alpha) \geqq \pi_i(\beta)$ for any $i$. Hence $R\alpha \rightarrow \pi_i(\alpha)$ gives one-to-one and order-preserving mapping of all principal left ideals of $R$ onto $L_i$. Therefore we see that for every element $\alpha$ of $R$, there exists an idempotent $\varepsilon$ such that we have $R\alpha = R\varepsilon$. A ring which has this property is said to be ***regular***. We can see easily that the totality of the principal left ideals of a regular ring $R$ constitutes a complemented modular lattice, denoting $L(R)$, as a sublattice of the lattice of all left ideals of $R$. (Also the totality of the principal right ideals of $R$ constitutes a complemented modular lattice, but we make exclusively use of $L(R)$.)

Thus we have proved

THEOREM 2. *The auxiliary ring $R$ of $L$ is regular and $L(R)$ is isomorphic to $L_i$ by the correspondence: $R\alpha \leftrightarrows \pi_i(\alpha)$.*

## § 4. Representation.

Let $R$ be a regular ring. $R$ may be considered as a module with the ring mutiplication from the left as operators. We write $R^n$ for the direct sum of $n$ modules which are all isomorphic to $R$. $L(R^n)$ will denote the set of all finitely generated submodules of $R^n$.

Then we shall prove

THEOREM 3. *$L(R^n)$ constitutes a complemented modular lattice as a sublattice of the lattice $\bar{L}(R^n)$ of all submodules of $R^n$ and it has a frame (with a homogeneous basis of degree $n$) which determines (if $n \geqq 4$) the auxiliary ring of $L(R^n)$, which is isomorphic to $R$.*

PROOF. Let $e_i$ $(i=1, 2, \cdots, n)$ be a basis of $R^n$ as $R$-module: $R^n = Re_1 + Re_2 + \cdots + Re_n$. Let $M_n(R)$ be the $n$-square matrix ring over $R$, and $S$ a submodule of $R^n$. We denote with $\bar{S}$ the left ideal of $M_n(R)$ consisting of all matrices whose row-vectors are in $S$. The correspondence $S \leftrightarrows \bar{S}$ gives an isomorphism between the lattice of all submodules of $R^n$ and that of all left ideals of $M_n(R)$.

First we shall prove that every element $S$ of $L(R^n)$ has a complement in $\bar{L}(R^n)$. We prove it by induction. This is true for $n=1$, as $R$ is a regular ring. We suppose that it is true for $R^{n-1}$ and put $R^n = R\boldsymbol{e}_1 + R_2$ where $R_2 = R\boldsymbol{e}_2 + \cdots + R\boldsymbol{e}_n$. Let $S$ be an arbitrary element of $L(R^n)$ with generators $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$. If $S_1$ is the image of $S$ by the projection of $R^n$ to $R\boldsymbol{e}_1$, we can find an idempotent $\varepsilon$ in $R$ such that $S_1 = R\varepsilon\boldsymbol{e}_1$. There exists an element $\boldsymbol{a}$ in $S$ of the form $\boldsymbol{a} = \varepsilon\boldsymbol{e}_1 + \cdots$, so we can determine $\boldsymbol{a}'_i$ $(i=1,2,\cdots,m)$ such that $\boldsymbol{a}'_i$ are in $R_2$ and $\boldsymbol{a}_i = \alpha_i\boldsymbol{a} + \boldsymbol{a}'_i$ for some $\alpha_i$ in $R$, then $\boldsymbol{a}'_i$ generate a submodule $S_2$ which obviously coincides with $S \cap R_2$. Let $S'_1$ be a complement of $S_1$ in $\bar{L}(R\boldsymbol{e}_1)$ and $S'_2$ be that of $S_2$ in $\bar{L}(R_2)$, then we can see easily that $S'_1 \cup S'_2$ is a complement of $S$ in $\bar{L}(R^n)$.

Therefore every principal left ideal of $M_n(R)$ has a complement in the lattice of all left ideals, and hence $M_n(R)$ is a regular ring. Then $L(R^n)$, being isomorphic to $L(M_n(R))$, is a complemented modular lattice.

$R\boldsymbol{e}_i$ $(i=1,2,\cdots,n)$ obviously constitute a homogeneous basis of $L(R^n)$, because $R(\boldsymbol{e}_i + \boldsymbol{e}_j)$ is a complement of both $R\boldsymbol{e}_i$ and $R\boldsymbol{e}_j$ in $R\boldsymbol{e}_i + R\boldsymbol{e}_j$.

For every $\alpha \in R$ there exists an automorphism of $R^n$ (denoted by $(\alpha)_{ij}$) which maps $\boldsymbol{e}_i$ to $\boldsymbol{e}_i - \alpha\boldsymbol{e}_j$ and fixes every $\boldsymbol{e}_k$ $(k \neq i)$. $(\alpha)_{ij}$ can be considered as an automrphism of the lattice $L(R^n)$. We can see easily that $(\alpha)_{ij}$ is normal for $\sum_{j \neq i} R\boldsymbol{e}_j$ and the totality of them, for all $\alpha \in R$, constitues a group isomorphic to the addition group of $R$. This group corresponds to $G_{ij}$ defined in § 2. Moreover by simple calculations we have $(\alpha)_{ij} \otimes (\beta)_{jk} = (\alpha\beta)_{ik}$. Thus $\{R\boldsymbol{e}_i, (1)_{ij}\,;\, i,j=1,2,\cdots n\}$ constitutes a frame, which determines the auxiliary ring isomorphic to $R$.

Thus the proof is completed.

The above theorem, together with the following will complete our theory.

THEOREM 4. *Let $L$ and $L^*$ be complemented modular lattices with homogeneous basis $\{a_i\,;\, i=1,2,\cdots,n\}$, $\{a^*_i\,;\, i=1,2,\cdots,n\}$ of the same degree $n \geqq 4$. If the auxiliary ring $R$ of $L$ determined by a frame $\{a_i, \Gamma_{ij}\,;\, i,j=1,2,\cdots,n\}$ is isomorphic to the auxiliary ring $R^*$ of $L^*$ determined* *by a frame* $\{a_i^*, \Gamma_{ij}^*; i, j=1, 2, \cdots, n\}$, *then* $L$ *is isomorphic to* $L^*$.

PROOF. The isomorphic mapping of $R$ to $R^*$ will be denoted by $*$. This sign $*$ will be also used to indicate the corresponding objects in various senses, easy to be understood in each case. E. g. the homogeneous basis $\{a_i; i=1, 2, \cdots, n\}$ of $L$ will determine the groups $G_{ij}$ introduced at the beginning of § 2. The corresponding groups determined by $\{a_i^*; i=1, 2, \cdots, n\}$ will be denoted by $G_{ij}^*$. Then there exists an isomorphic mapping $(\alpha)_{ij}\to(\alpha^*)_{ij}$ of $G_{ij}$ to $G_{ij}^*$ and this isomorphism can be extended to an isomorphism of $G_i$ to $G_i^*$. Here we remark that if $f\in G_{12}$, $g\in G_2$ and $\pi(g)\leqq a_3\cup\cdots\cup a_n$, then we have obviously $(f\otimes g)^*=f^*\otimes g^*$, and hence

$$(g^{-1}fg)^*=(f(f\otimes g))^*=f^*(f\otimes g)^*=f^*(f^*\otimes g^*)=g^{*-1}f^*g^*.$$

Putting $L_{(i)}=L_{a_i\cup a_{i+1}\cup\cdots\cup a_n}$, $G_{(i)}=\prod_{j=i+1}^{n}G_{ij}$ and $L_{(i)}^*=L_{a_i^*\cup\cdots\cup a_n^*}^*$, $G_{(i)}^*=\prod_{j=i+1}^{n}G_{ij}^*$, we shall prove the following proposition $(P_i)$ for every $i=1, 2, \cdots, n$.

$(P_i)$: There exists an isomorphism: $L_{(i)}\ni x\to x^*\in L_{(i)}^*$ such that we have

(i) $\pi_j(\alpha)^*=\pi_j(\alpha^*)$ for every $j\geqq i$, $\alpha\in R$,

(ii) $f(x)^*=f^*(x^*)$ for every $f\in G_{(i)}$, and

(iii) $\pi(f)^*=\pi(f^*)$ for every $f\in G_{(i)}$.

$(P_n)$ is true, because every element of $L_{(n)}$ may be written as $\pi_n(\alpha)$, and $\pi_n(\alpha)\to\pi_n(\alpha^*)$ gives the desired isomorphism. $(P_i)$, $i=1, 2, \cdots, n-1$ will be proved, if we show $(P_i)\to(P_{i-1})$ for $i=n, \cdots, 2$. We shall show $(P_2)\to(P_1)$, as $(P_i)\to(P_{i-1})$ for other $i$ 's will be shown in the same way.[5)]

Let $x\to x^*$ be an isomorphism of $L_{(2)}$ to $L_{(2)}^*$ satisfying (i), (ii), (iii) (for $i=2$). We shall first prove that $\pi(f)^*=\pi(f^*)$ for $f\in G_1$. If $\pi(f)\leqq a_3\cup a_4\cup\cdots\cup a_n$ then we have $\Gamma_{21}\otimes f\in G_{(2)}$ and hence $\pi(f)^*=\pi(\Gamma_{21}\otimes f)^*=\pi(\Gamma_{21}^*\otimes f^*)=\pi(f^*)$. If there exist $g$ in $G_{(2)}$ and $f_1=(\alpha)_{12}$ in $G_{12}$ such that $f=g^{-1}f_1g$, then by Proposition 3 we have $\pi(f)=g^{-1}\pi(f_1)=g^{-1}(\pi_2(\alpha))$ and hence $\pi(f)^*=g^{*-1}(\pi_2(\alpha^*))=\pi(g^{*-1}f_1^*g^*)=\pi(f^*)$. We can reduce the proof of $\pi(f)^*=\pi(f^*)$ for general $f\in G_1$ to the above two cases. An arbitrary $f\in G_1$ can be written as $f=(\alpha)_{12}\cdot h$

5) In the proof of von Neumann, the step $(P_2)\to(P_1)$ requires special considerations, whereas $(P_i)\to(P_{i-1})$, $i=n, \cdots, 3$ are proved by the same method. In our proof, all steps $(P_i)\to(P_{i-1})$, $i=n, \cdots, 2$ are treated in the same way.

where $\pi(h) \leqq a_3 \cup \cdots \cup a_n$. Let $\xi$ be an element of $R$ such that $\alpha = \alpha\xi\alpha$, then $\varepsilon = \alpha\xi$ is idempotent. Now put $g = \Gamma_{21} \otimes \xi h$, then $g$ is in $G_{(2)}$ and $\varepsilon f = (\varepsilon\alpha)_{12} \cdot \varepsilon h = (\alpha)_{12} \cdot \alpha(\xi h) = (\alpha)_{12} \cdot ((\alpha)_{12} \otimes g) = g^{-1}(\alpha)_{12} g$, and hence $\pi(\varepsilon f)^* = \pi(\varepsilon^* f^*)$. On the other hand we have

$$\pi((1-\varepsilon)f) = \pi((1-\varepsilon)h) \leqq a_3 \cup \cdots \cup a_n \, ,$$

so we have also $\pi((1-\varepsilon)f)^* = \pi((1-\varepsilon^*)f^*)$. Therefore

$$\pi(f)^* = \pi(\varepsilon f)^* \cup \pi((1-\varepsilon)f)^* = \pi(\varepsilon^* f^*) \cup \pi((1-\varepsilon^*)f^*) = \pi(f^*) \, .$$

Now we shall extend the isomorphism $x \rightarrow x^*$ of $L_{(2)}$ to $L^*_{(2)}$ and $\pi_1(\alpha) \rightarrow \pi_1(\alpha^*)$ of $L_1$ to $L_1{}^*$ (we write $x \rightarrow x^*$ also in the latter case) to an isomorphism of $L_{(1)} = L$ to $L^*_{(1)} = L^*$.

Suppose we have proved the equivalency of two inequalities $f(x) \cup u \leqq g(y) \cup v$ and $f^*(x^*) \cup u^* \leqq g^*(y^*) \cup v^*$ for every $f, g \in G_1$, $x, y \leqq a_1$, and $u, v \in L_{(2)}$. Then the element $f^*(x^*) \cup u^*$ of $L^*$ is determined uniquely by an element $f(x) \cup u$ of $L$, independently of its expression. Since every element of $L$ can be written as $f(x) \cup u$, an order-preserving one-to-one mapping of $L$ onto $L^*$ is defined by $f(x) \cup u \rightarrow f^*(x^*) \cup u^*$. For this mapping (i), (ii) and (iii) of $(P_1)$ are obviously satisfied.

Thus we have only to prove that $f(x) \cup u \leqq g(y) \cup v$ implies $f^*(x^*) \cup u^* \leqq g^*(y^*) \cup v^*$, since the converse is then also true by reason of symmetry.

$f(x) \cup u \leqq g(y) \cup v$ implies obviously $x \leqq y$, $u \leqq v$ and hence $x^* \leqq y^*$, $u^* \leqq v^*$. Let $\varepsilon \in R$ be an idempotent such that $\pi_1(\varepsilon) = x$. Then we have

$$\varepsilon g^{-1} f(a_1) = g^{-1} f(x) \leqq y \cup v,$$

$$\pi(\varepsilon g^{-1} f) = (\varepsilon g^{-1} f(a_1) \cup a_1) \cap \bar{a}_1 \leqq (v \cup a_1) \cap \bar{a}_1 = v \, ,$$

and hence $\pi(\varepsilon^* g^{*-1} f^*) \leqq v^*$. From the last inequality, we have

$$g^{*-1} f^*(x^*) = \varepsilon^* g^{*-1} f^*(a_1^*) \leqq a_1^* \cup v^*$$

and also $g^{*-1} f^*(x^*) \leqq (a_1^* \cup v^*) \cap (y^* \cup \bar{a}_1^*) = y^* \cup v^*$. Thus we obtain the inequality $g^{*-1} f^*(x^*) \cup u^* \leqq y^* \cup v^*$ which is equivalent to $f^*(x^*) \cup u^* \leqq g^*(y^*) \cup v^*$.

We have thus proved $(P_1)$ and this implies our theorem as $L_{(1)} = L$, $L^*_{(1)} = L^*$.

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