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On finite-dimensional perturbations of self-adjoint operators.

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§ 1. Introduction. The purpose of the present paper is to show that a perturbation of finite rank does not change the main structure of a self-adjoint operator in a sense to be specified below[1], and to deduce certain asymptotic relationships between the one-parameter continuous groups generated by the unperturbed and perturbed self-adjoint operators.

Let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space, H_0 a (not necessarily bounded) self-adjoint operator in $\mathfrak{H}$ and V a self-adjoint operator with finite rank m. Then $H_1=H_0+V$ is also self-adjoint with domain $\mathfrak{D}$ identical with that of H_0. We assert that H_1 and H_0 are unitarily equivalent to each other except for separable, singular parts with multiplicities not exceeding m. A more precise expression is given by

THEOREM 1. *Let H_0, H_1 be as above. Then there exist two subspaces*[2] $\mathfrak{M}_0$, $\mathfrak{M}_1$ *with respective projections* P_0, P_1 *and a subspace* $\mathfrak{M}_{01}\subset \mathfrak{M}_0\cap\mathfrak{M}_1$ *with the following properties.*

1) $\mathfrak{M}_{01}$ *reduces both* H_0 *and* H_1*; the parts of* H_0 *and* H_1 *in* $\mathfrak{M}_{01}$ *are identical.*

2) $\mathfrak{H}\ominus\mathfrak{M}_{01}$ *is (and hence* $\mathfrak{H}\ominus\mathfrak{M}_0$ *and* $\mathfrak{H}\ominus\mathfrak{M}_1$ *are a fortiori) separable; the parts of* H_0 *and* H_1 *in* $\mathfrak{H}\ominus\mathfrak{M}_{01}$ *have spectra with multiplicities not exceeding* m.

3) $\mathfrak{M}_0$ *and* $\mathfrak{M}_1$ *reduce* H_0 *resp.* H_1*; the parts of* H_0 *and* H_1 *in* $\mathfrak{M}_0$ *resp.* $\mathfrak{M}_1$ *are unitarily equivalent to each other.*

4) *The parts of* H_0 *and* H_1 *in* $\mathfrak{H}\ominus\mathfrak{M}_0$ *resp.* $\mathfrak{H}\ominus\mathfrak{M}_1$ *are singular.*

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1) In his Technical Report 17 "On a problem of Hermann Weyl in the theory of singular Sturm Liouville equations," N. Aronszajn states that he and W. F. Donoghue have obtained results similar to ours. Also M. Rosenblum announces in the abstract 99 in Bull. Amer. Math. Soc. 62 (1956) p. 30 results closely related to ours. (Added in proof) see end of paper.

2) We mean by a subspace a closed linear manifold of $\mathfrak{H}$.

Thus $\mathfrak{M}_0$ *and* $\mathfrak{M}_1$ *contain the absolutely continuous parts of* $\mathfrak{H}$ *with respect to* H_0 *resp.* H_1.

5) *The parts of* H_0 *and* H_1 *in* $\mathfrak{M}_0 \ominus \mathfrak{M}_{01}$ *resp.* $\mathfrak{M}_1 \ominus \mathfrak{M}_{01}$ *are absolutely continuous.*

Note that $\mathfrak{H} = \mathfrak{M}_{01} \oplus [\mathfrak{M}_0 \ominus \mathfrak{M}_{01}] \oplus [\mathfrak{H} \ominus \mathfrak{M}_0]$ and $\mathfrak{H} = \mathfrak{M}_{01} \oplus [\mathfrak{M}_1 \ominus \mathfrak{M}_{01}] \oplus [\mathfrak{H} \ominus \mathfrak{M}_1]$.

Here and in the sequel the terms "absolutely continuous" and "singular" are used in the following sense. With respect to a given self-adjoint operator H with the resolution of identity $\{E(\lambda)\}$, an $x \in \mathfrak{H}$ is said to be absolutely continuous resp. singular if $(E(\lambda)x, x)$ is absolutely continuous resp. singular as function of λ. Let $\mathfrak{H}_a$ and $\mathfrak{H}_s$ be the set of all absolutely continuous resp. singular elements. Then these are subspaces of $\mathfrak{H}$, both reduce H and $\mathfrak{H} = \mathfrak{H}_a \oplus \mathfrak{H}_s$. $\mathfrak{H}_a$ and $\mathfrak{H}_s$ will be called the absolutely continuous resp. singluar parts of $\mathfrak{H}$ with respect to the operator H. The parts of H in $\mathfrak{H}_a$ and $\mathfrak{H}_s$ will be called the absolutely continuous resp. singular parts of H. If $\mathfrak{H}_s = \{0\}$ resp. $\mathfrak{H}_a = \{0\}$, H is said to be absolutely continuous resp. singular, and $\mathfrak{H}$ is said to be absolutely continuous resp. singular with respect to H. Also a subspace of $\mathfrak{H}$ will be said to be absolutely continuous resp. singular with respect to H if all its elements are absolutely continuous resp. singular.

Theorem 1 implies in particular that the absolutely continuous part of a self-adjoint operator is stable under a perturbation of finite rank. This is not necessarily true for perturbations V with infinite rank, even when V belongs to Schmidt class. In fact, it is well known[3] that any self-adjoint operator can be changed into one with a pure point spectrum by such a perturbation V with arbitrarily small Schmidt norm.

On the other hand, it has been shown[4] that there exists a class of perturbations with infinite rank which do not change absolutely continuous spectra extending over an interval. These perturbations are expressed as integral operators with Hölder continuous kernels. In our theorem, however, we do not impose any continuity conditions on the operator V, though V could be given a form of integral

3) J. v. Neumann, Charakterisierung des Spektrums eines Integraloperators, Actualités scientifique et industrielles, 229, Paris, 1935.

4) K. Friedrichs, On the perturbation of continuous spectra, Communications on Pure and Applied Mathematics 1 (1948), pp. 361–406; Ueber die Spektralzerlegung eines Integraloperators, Math. Ann. 115 (1938), pp. 249–272.

operator in some functional realization of $\mathfrak{H}$.

The unitary equivalence of the parts of H_0 and H_1 in $\mathfrak{M}_0$ resp. $\mathfrak{M}_1$ implies the existence of a partially isometric operator[5] U with initial set $\mathfrak{M}_0$ and final set $\mathfrak{M}_1$ which transforms H_0P_0 into H_1P_1. Such a U is by no means uniquely determined, even when $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are given. We shall show, however, that there exist two distinguished operators $U_\pm$ which play the part of U.

THEOREM 2. *The quantities* $\mathfrak{M}_{01}, \mathfrak{M}_0, \mathfrak{M}_1, P_0, P_1$ *of Theorem* 1 *can be chosen in the following way. The strong limits*

$$s-\lim_{t\to\pm\infty} \exp(itH_1)\exp(-itH_0)P_0 = U_\pm \tag{1.1}$$

exist, are partially isometric operators with initial set $\mathfrak{M}_0$ *and final set* $\mathfrak{M}_1$. $U_\pm$ *transform the self-adjoint operator* H_0P_0 *into* H_1P_1, *that is,*

$$H_1P_1 = U_\pm H_0 P_0 U_\pm^*, \qquad H_0P_0 = U_\pm^* H_1 P_1 U_\pm . \tag{1.2}$$

Their adjoints $U_\pm^*$ *are also given by*

$$s-\lim_{t\to\pm\infty} \exp(itH_0)\exp(-itH_1)P_1 = U_\pm^* . \tag{1.3}$$

The operators $U_\pm, U_\pm^*$ *are reduced by* $\mathfrak{M}_{01}$, *and their parts in* $\mathfrak{M}_{01}$ *are equal to the identity. The operators*[6] $S_0 = U_+^* U_-$ *and* $S_1 = U_+ U_-^*$ *are reduced by* $\mathfrak{M}_0$ *resp.* $\mathfrak{M}_1$, *and their respective parts are unitary.* S_0 *and* S_1 *commute with* H_0 *resp.* H_1.

It should be noted that the subspaces $\mathfrak{M}_{01}, \mathfrak{M}_0, \mathfrak{M}_1$ are not necessarily uniquely determined by the given pair H_0, H_1. In particular if we disregard proposition 2) of Theorem 1, $\mathfrak{M}_0$ and $\mathfrak{M}_1$ can be chosen to be the absolutely continuous parts of $\mathfrak{H}$ with respect to H_0 and H_1 respectively. This is seen by the following considerations. Let $\mathfrak{M}_0$ etc. be a set of quantities satisfying these theorems.

The elements of $\mathfrak{M}_0$ singular with respect to H_0 belong to $\mathfrak{M}_{01}$ by 5) and these are mapped onto themselves by $U_\pm$, so that they are at the same time singular with respect to H_1 and belong to $\mathfrak{M}_1$. We denote this set by $\mathfrak{N}$; $\mathfrak{N}$ is obviously a subspace of $\mathfrak{M}_{01}$. Set $\mathfrak{M}_0' = \mathfrak{M}_0 \ominus \mathfrak{N}$, $\mathfrak{M}_1' = \mathfrak{M}_1 \ominus \mathfrak{N}$, $\mathfrak{M}_{01}' = \mathfrak{M}_{01} \ominus \mathfrak{N}$ and denote by P_0', P_1' the projections on $\mathfrak{M}_0', \mathfrak{M}_1'$ respectively. Then both $U_\pm$ map $\mathfrak{M}_0'$ onto $\mathfrak{M}_1'$. Define the

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5) F. J. Murray and J. v. Neumann, On rings of operators, Ann. Math. 37 (1936), pp. 116-229.

6) S_0 corresponds to what is called the scattering operator in quantum mechanics, where H_0 and H_1 represent unperturbed and perturbed Hamiltonians of the mechanical system. Here it is usual that $\mathfrak{M}_0$ coincides with $\mathfrak{H}$ so that S_0 is unitary.

operators $U'_{\pm}$ by

$$U'_{\pm}x=U_{\pm}x \qquad \text{for } x\in\mathfrak{M}'_0 \text{ or } x\in\mathfrak{H}\ominus\mathfrak{M}_0\,,$$
$$U'_{\pm}x=0 \qquad \text{for } x\in\mathfrak{N}\,.$$

Then $U'_{\pm}$ are partially isometric with initial set $\mathfrak{M}'_0$ and final set $\mathfrak{M}'_1$. It is now easily seen that Theorems 1 and 2 are true when $\mathfrak{M}_{01}, \mathfrak{M}_0, \mathfrak{M}_1, P_0, P_1, U_{\pm}$ are replaced by the corresponding quantities with primes, with the single exception of proposition 2) of Theorem 1, and now $\mathfrak{M}'_0$ and $\mathfrak{M}'_1$ are exactly the absolutely continuous parts of $\mathfrak{H}$ with respect to H_0 and H_1 respectively.

§ 2. **Separation of the subspace $\mathfrak{M}_{01}$.** Let $\mathfrak{R}_V$ be the range of V; it is an m-dimensional subspace. Let $\mathfrak{M}(V, H_0)$ and $\mathfrak{M}(V, H_1)$ be the smallest subspaces containing $\mathfrak{R}_V$ and reducing H_0 resp. H_1. Since $V\mathfrak{M}(V, H_0)\subset\mathfrak{R}_V\subset\mathfrak{M}(V, H_0)$ and $V[\mathfrak{H}\ominus\mathfrak{M}(V, H_0)]\subset V(\mathfrak{H}\ominus\mathfrak{R}_V)=\{0\}$ on account of self-adjointness of V, $\mathfrak{M}(V, H_0)$ reduces V, and hence H_1, too. Thus we must have $\mathfrak{M}(V, H_1)\subset\mathfrak{M}(V, H_0)$ and, since the converse relation holds by symmetry, we have the equality $\mathfrak{M}(V, H_0)=\mathfrak{M}(V, H_1)$. We set $\mathfrak{M}_{01}=\mathfrak{H}\ominus\mathfrak{M}(V, H_0)$. $\mathfrak{M}_{01}$ reduces both H_0 and H_1 and it is easily seen that H_0 and H_1 coincides in $\mathfrak{M}_{01}$.

Furthermore, as the subspace $\mathfrak{H}\ominus\mathfrak{M}_{01}$ reducing H_0 is "generated" by some m elements, it is separable and the part of H_0 in it has a spectrum with multiplicity not exceeding m. The same is true for H_1.

Thus we may restrict ourselves to the investigation of the operators H_0 and H_1 in the separable subspace $\mathfrak{H}\ominus\mathfrak{M}_{01}$. For if the theorems have been proved in the space $\mathfrak{H}\ominus\mathfrak{M}_{01}$, we have only to set $U_{\pm}$ equal to identity in the complementary subspace $\mathfrak{M}_{01}$. Without loss of generality, we may therefore assume from the outset that $\mathfrak{H}$ is separable and coincides with $\mathfrak{M}(V, H_0)=\mathfrak{M}(V, H_1)$, and prove the theorems with $\mathfrak{M}_{01}=\{0\}$.

§ 3. **The case $m=1$.** We first prove the theorems for $m=1$; the general case will be reduced to this case in § 6. Thus we have

$$Vx=c(x, \varphi)\varphi\,,$$

where c is a real number and $\varphi\neq 0$ is a fixed element of $\mathfrak{H}$. As remarked in § 2, we may and will assume that $\mathfrak{H}$ coincides with the smallest subspace reducing H_0 and containing φ, and that the same is true with respect to H_1. Thus both H_0 and H_1 have simple spectra, and each $x\in\mathfrak{H}$ admits of two representations by complex-valued Baire functions $f_0(\lambda, x), f_1(\lambda, x)$ on reals in the form

$$(3.1)\qquad x=\int f_j(\lambda, x)dE_j(\lambda)\varphi\,,\qquad ||x||^2=\int |f_j(\lambda, x)|^2 d\rho_j(\lambda)\,,\qquad j=0,1\,,$$

where $\{E_j(\lambda)\}$ is the resolution of identity belonging to H_j and $\rho_j(\lambda)=(E_j(\lambda)\varphi, \varphi)$, $j=0, 1$. The non-decreasing functions ρ_0, ρ_1 determine two measures, again denoted by ρ_0, ρ_1, on Borel sets of real numbers. The functions $f_0(\lambda, x)$, $f_1(\lambda, x)$ are determined by x uniquely up to equivalence with respect to the measures ρ_0 resp. ρ_1. These functions may be called the H_0- and H_1-representations of x, and the following equations hold at least for bounded, complex-valued Baire functions g.

$$(3.2)\qquad f_j(\lambda, g(H_j)x)=g(\lambda)f_j(\lambda, x)\,,\qquad j=0,1\,.$$

We shall now determine the relation between the measures ρ_0 and ρ_1. It follows from $H_1=H_0+V$ that

$$(H_1-\zeta)^{-1}-(H_0-\zeta)^{-1}=-(H_0-\zeta)^{-1}V(H_1-\zeta)^{-1}$$

for any non-real complex number ζ, whence

$$((H_1-\zeta)^{-1}x, \varphi)-((H_0-\zeta)^{-1}x, \varphi)=-c((H_1-\zeta)^{-1}x, \varphi)((H_0-\zeta)^{-1}\varphi, \varphi)\,.$$

On introducing the notations

$$J(\zeta, f, \rho)=\int(\lambda-\zeta)^{-1}f(\lambda)d\rho(\lambda)\,,\quad J(\zeta, \rho)=J(\zeta, 1, \rho)$$

and noting (3.1) and (3.2), the above equation can be written as

$$(3.3)\qquad [1+cJ(\zeta, \rho_0)]J(\zeta, f_1(\ \ , x), \rho_1)=J(\zeta, f_0(\ \ , x), \rho_0)\,.$$

Similarly we deduce

$$(3.4)\qquad [1-cJ(\zeta, \rho_1)]J(\zeta, f_0(\ \ , x), \rho_0)=J(\zeta, f_1(\ \ , x), \rho_1)\,.$$

In particular we have $f_0(\lambda, \varphi)=f_1(\lambda, \varphi)=1$ and these formulas reduce for $x=\varphi$ to

$$(3.5)\qquad [1+cJ(\zeta, \rho_0)][1-cJ(\zeta, \rho_1)]=1\,.$$

The following properties of the function J are well known.[7] If $f\in L_2(\rho)$, the limits

$$J(\lambda\pm i0, f, \rho)=\lim_{\varepsilon\downarrow 0}J(\lambda\pm i\varepsilon, f, \rho)$$

exist and are finite for almost all real λ, and

$$(3.6)\qquad J(\lambda+i0, f, \rho)-J(\lambda-i0, f, \rho)=2\pi i f(\lambda)\rho'(\lambda)\qquad \text{a. e.,}$$

7) See e. g. R. Nevanlinna, Eindeutige analytische Funktion, Berlin, 1936.

where $\rho'(\lambda)=d\rho(\lambda)/d\lambda$ exists a. e. Note that $J(\lambda\pm i0, f, \rho)$ are complex conjugates to each other whenever f is real.

It follows from (3.5) and (3.6) that

$$\rho_1'(\lambda)=|\omega(\lambda\pm i0)|^{-2}\rho_0'(\lambda) \quad \text{a. e.,} \tag{3.7}$$

where

$$\omega(\zeta)=1+cJ(\zeta, \rho_0)=[1-cJ(\zeta, \rho_1)]^{-1} \tag{3.8}$$

and $\omega(\lambda\pm i0)$ are finite and non-vanishing a. e.

(3.7) shows that $\rho_1'(\lambda)=0$ if and only if $\rho_0'(\lambda)=0$ except for a null set of λ, that is, that the absolutely continuous parts of ρ_0 and ρ_1 are equivalent to each other. According to the theory[8] of unitary equivalence of self-adjoint operators with simple spectra, this is sufficient to conclude the unitary equivalence of the absolutely continuous parts of H_0 and H_1. We shall, however, establish this unitary equivalence in the next section by a more explicit construction.

§ 4. **Introduction of $U_\pm$.** In consequence of the equivalence of the absolutely continuous parts of the measures ρ_0 and ρ_1, there exist two mutually disjoint Borel sets A, S of real numbers with the following properties. S is a null set and is a support of the singular parts of ρ_0 and ρ_1. A is a support of the absolutely continuous parts of ρ_0 and ρ_1, and both $\rho_0'(\lambda)$ and $\rho_1'(\lambda)$ exist and are positive for $\lambda\in A$.

We denote by $E_j(X), j=0, 1$, the spectral measure of a Borel set X determined by $\{E_j(\lambda)\}$, and set

$$P_0=E_0(A), \quad P_1=E_1(A), \quad \mathfrak{M}_0=P_0\mathfrak{H}, \quad \mathfrak{M}_1=P_1\mathfrak{H}\,.$$

Then $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are the absolutely continuous parts of $\mathfrak{H}$ with respect to H_0 resp. H_1. It should be noted that $\mathfrak{M}_0$, $\mathfrak{M}_1$ are uniquely determined by these conditions, although the set A is not.

We now define two operators $U_\pm$ by

$$U_\pm x=\int_A \omega(\lambda\pm i0)f_0(\lambda, x)dE_1(\lambda)\varphi\,. \tag{4.1}$$

This definition is permitted because $\omega(\lambda\pm i0)$ is defined a. e., the spectral measure E_1 is absolutely continuous on A and

$$\begin{aligned}||U_\pm x||^2 &=\int_A|\omega(\lambda\pm i0)|^2|f_0(\lambda, x)|^2\rho_1'(\lambda)d\lambda \\ &=\int_A|f_0(\lambda, x)|^2\rho_0'(\lambda)d\lambda=||P_0x||^2\,.\end{aligned} \tag{4.2}$$

8) M. H. Stone, Linear transformations in Hilbert space, New York, 1932,

This shows that $U_{\pm}$ are partially isometric operators with initial set $\mathfrak{M}_0$.

To determine the adjoints $U_{\pm}^*$, we note that

$$(U_{\pm}x, y)=\int_A \omega(\lambda\pm i0)f_0(\lambda, x)f_1(\lambda, y)^*\rho_1'(\lambda)d\lambda$$
$$=\int_A f_0(\lambda, x)\,[\omega(\lambda\pm i0)^{-1}f_1(\lambda, y)]^*\rho_0'(\lambda)d\lambda$$
$$=(x, \int_A \omega(\lambda\pm i0)^{-1}f_1(\lambda, y)dE_0(\lambda)\varphi)\,,$$

where we used * to denote complex conjugate. Thus we obtain

$$U_{\pm}^* y=\int_A \omega(\lambda\pm i0)^{-1}f_1(\lambda, y)dE_0(\lambda)\varphi\,. \tag{4.3}$$

It follows as above that $\|U_{\pm}^* y\|=\|P_1 y\|$, showing that $U_{\pm}^*$ are partially isometric with initial set $\mathfrak{M}_1$. Summing up, we see that $U_{\pm}$ are partially isometric operators with initial set $\mathfrak{M}_0$ and final set $\mathfrak{M}_1$. Thus

$$U_{\pm}^*U_{\pm}=P_0,\ U_{\pm}U_{\pm}^*=P_1,\ U_{\pm}P_0=P_1U_{\pm}=U_{\pm},\ U_{\pm}^*P_1=P_0U_{\pm}^*=U_{\pm}^*\,. \tag{4.4}$$

Let X be any Borel subset of A. Then

$$U_{\pm}E_0(X)x=U_{\pm}\int_X f_0(\lambda, x)dE_0(\lambda)\varphi$$
$$=\int_X \omega(\lambda\pm i0)f_0(\lambda, x)dE_1(\lambda)\varphi=E_1(X)U_{\pm}x\,.$$

Thus we have $U_{\pm}E_0(X)=E_1(X)U_{\pm}$ and, taking the adjoint, $U_{\pm}^*E_1(X)=E_0(X)U_{\pm}^*$. These show explicitly that the parts of H_0 and H_1 in $\mathfrak{M}_0$ resp. $\mathfrak{M}_1$ are unitarily equivalent to each other. Combined with the remark of § 2, this completes the proof of Theorem 1 for $m=1$ with $U=U_{\pm}$, $\mathfrak{M}_{01}=\{0\}$, and also proves (1.2) of Theorem 2.

For later use we shall deduce the relation between the two representations $f_0(\ ,x)$ and $f_1(\ ,x)$ of the same element x. Setting $\zeta=\lambda\pm i0$ in (3.4) and subtracting, we easily find that $f_1(\lambda, x)\rho_1'(\lambda)=\omega(\lambda\mp i0)^{-1}f_0(\lambda, x)\rho_0'(\lambda)-c\rho_1'(\lambda)J(\lambda\pm i0, f_0(\ ,x), \rho_0)$ and hence, by virtue of (3.7), that a. e. on A

$$f_1(\lambda, x)=\omega(\lambda\pm i0)f_0(\lambda, x)-cJ(\lambda\pm i0, f_0(\ ,x), \rho_0)\,. \tag{4.5}$$

In the same way we obtain

$$f_0(\lambda, x)=\omega(\lambda\pm i0)^{-1}f_1(\lambda, x)+cJ(\lambda\pm i0, f_1(\ ,x), \rho_1) \tag{4.6}$$

a. e. on A. These formulas are not quite complete since they are valid only a. e. on A, but they are sufficient to determine $P_1 x$ or $P_0 x$.

§ 5. **Asymptotic properties of unitary groups.** We next prove Theorem 2 for $m=1$ under the same simplifying assumptions as in §§ 3,4. We set

$$U_t = \exp(itH_1)\exp(-itH_0), \qquad -\infty < t < \infty ,$$

and note that for any $x \in \mathfrak{D}$

$$\begin{aligned} dU_t x/dt &= i\exp(itH_1)(H_1 - H_0)\exp(-itH_0)x \\ &= i\exp(itH_1)V\exp(-itH_0)x \\ &= ic(\exp(-itH_0)x, \varphi)\exp(itH_1)\varphi , \end{aligned}$$

for $\exp(-itH_0)$ maps $\mathfrak{D}$ onto $\mathfrak{D}$. Since the right side is strongly continuous in t, we obtain

$$(U_t x, y) - (x, y) = ic\int_0^t (\exp(-itH_0)x, \varphi)(\exp(itH_1)\varphi, y)dt .$$

Although x was assumed to belong to $\mathfrak{D}$ above, this final result is valid for all $x, y \in \mathfrak{H}$.

Introduction of the representations of x and y in conformity with (3.1) and (3.2) leads to

$$\begin{aligned} (U_t x, y) - (x, y) = ic\int_0^t dt\Big[\int \exp(-it\lambda)f_0(\lambda, x)d\rho_0(\lambda)\Big]\cdot \\ \cdot\Big[\int \exp(it\mu)f_1(\mu, y)^* d\rho_1(\mu)\Big]. \end{aligned}$$

We now assume that $x \in \mathfrak{M}_0$ and $y \in \mathfrak{M}_1$. Then we may write $d\rho_0(\lambda) = \rho_0'(\lambda)d\lambda$ and $d\rho_1(\mu) = \rho_1'(\mu)d\mu$ on the right side, and

$$\begin{aligned} &(U_t x, y) - (x, y) \\ &\quad = c\iint \frac{\exp[it(\mu-\lambda)]-1}{\mu-\lambda} f_0(\lambda, x)\rho_0'(\lambda)f_1(\mu, y)^*\rho_1'(\mu)d\lambda d\mu . \end{aligned}$$

This is certainly true at least if $f_0(\ ,x)\rho_0'$ and $f_1(\ ,y)\rho_1'$ belong to L_2 with respect to Lebesgue measure; then the order of integrations is inessential. Moreover, it is well known[9] that for $t \to \pm\infty$

$$\int\Big|\int \frac{\exp[it(\mu-\lambda)]-1}{\mu-\lambda} f_0(\lambda, x)\rho_0'(\lambda)d\lambda - J(\mu \pm i0, f_0(\ ,x), \rho_0)\Big|^2 d\mu \to 0 .$$

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9) E. C. Titchmarsh, Theory of Fourier integrals, Oxford, 1948.

We have therefore

$$\lim_{t\to\pm\infty}(U_t x, y)=(x, y)+c\int J(\mu\pm i0, f_0(\ ,x), \rho_0)f_1(\mu, y)^*\rho_1'(\mu)d\mu\,.$$

Noting (4.5) and $(x,y)=\int f_1(\mu, x)f_1(\mu, y)^*\rho_1'(\mu)d\mu$, this limit is seen to be equal to $\int\omega(\mu\pm i0)f_0(\mu, x)f_1(\mu, y)^*\rho_1'(\mu)d\mu$, which is equal to $(U_\pm x, y)$ by (4.1). In this way we have proved that $(U_t x, y)\to(U_\pm x, y)$ for $t\to\pm\infty$ provided that $x\in\mathfrak{M}_0, y\in\mathfrak{M}_1, f_0(\ ,x)\rho_0'\in L_2, f_1(\ ,y)\rho_1'\in L_2$. But it is easily seen that the set of such x and y are dense in $\mathfrak{M}_0$ and $\mathfrak{M}_1$ respectively. Noting that U_t is uniformly bounded, we thus obtain

$$w\text{-}\lim_{t\to\pm\infty}\exp(itH_1)P_1\exp(-itH_0)P_0=w\text{-}\lim_{t\to\pm\infty}P_1U_tP_0=U_\pm\,, \tag{5.1}$$

where w-lim denotes weak limit.

Actually, however, these limits are strong limits and, moreover, the factor P_1 in (5.1) can be omitted. In fact, for any $x\in\mathfrak{H}$ (5.1) implies

$$\begin{aligned}||U_\pm x||&\leqq \liminf||\exp(itH_1)P_1\exp(-itH_0)P_0x||\\&=\liminf||P_1\exp(-itH_0)P_0x||\leqq||P_0x||\,,\end{aligned}$$

and similar inequalities for lim sup in place of lim inf. But we know that $||U_\pm x||=||P_0x||$. This implies on the one hand that the limits of (5.1) must be strong limits by a well known theorem, and on the other that the equality sign must hold everywhere in the above inequalities. In particular we have $\lim||P_1\exp(-itH_0)P_0x||=||P_0x||$. But as

$$||P_1\exp(-itH_0)P_0x||^2+||(1-P_1)\exp(-itH_0)P_0x||^2=||P_0x||^2\,,$$

it follows that $(1-P_1)\exp(-itH_0)P_0x\to 0$ strongly for $t\to\pm\infty$. Multiplying from left by the uniformly bounded operator $\exp(itH_1)$, we see that $\exp(itH_1)(1-P_1)\exp(-itH_0)P_0x\to 0$. Combined with (5.1) of which strong convergence has already been proved, we obtain finally the desired result

$$s\text{-}\lim_{t\to\pm\infty}\exp(itH_1)\exp(-itH_0)P_0=U_\pm\,. \tag{5.2}$$

This proves the first statement of Theorem 2. The corresponding result (1.3) for $U_\pm^*$ follows from the above by interchange of the roles of H_0 and H_1. The remaining assertions of Theorem 2 are direct consequences of the basic properties of $U_\pm$.

It should be remarked that the above deduction of (5.2) from (5.1) is unnecessarily complicated. But we have given this here because the same argument is useful in the general case $m>1$.

§ 6. **The general case** $m>1$. We turn to the proof of the theorems in the general case. For simplicity we shall give the proof for $m=2$, the general case offering no further difficulty. For convenience we write H_2 in place of H_1 of the theorems. Then H_2-H_0 is of rank 2, and we can find a self-adjoint operator H_1 such that both H_1-H_0 and H_2-H_1 are of rank 1. Again we may assume that $\mathfrak{H}$ is separable and the spectra of H_0 and H_1 have multiplicities $\leqq 2$, and it is sufficient to prove the theorems in which $\mathfrak{M}_{02}=\{0\}$. (We use subscripts 2 for various quantities related to H_2.)

Theorems 1 and 2 have been proved for the pairs H_0, H_1 and H_1, H_2. We shall use the notations $\mathfrak{M}_{01}, \mathfrak{M}_0, \mathfrak{M}_1, P_0, P_1, U_\pm$ of these theorems to describe the quantities related to the pair H_0, H_1 and the notations $\mathfrak{M}_{12}, \mathfrak{M}_1, \mathfrak{M}_2, P_1, P_2, V_\pm$ for the corresponding quantities for the pair H_1, H_2. *The only objection to this would be that the subspace* $\mathfrak{M}_1$ *in the one set need not be identical with* $\mathfrak{M}_1$ *in the other.* Even if this is not the case from the beginning, however, we can always achieve this by suitable modification of these quantities. In fact, as was shown at the end of § 1, we may assume that $\mathfrak{M}_1$ is the set of all absolutely continuous elements with respect to H_1; this set is determined by H_1 itself without reference to any other operator.

Now we have only to take the quantities $\mathfrak{M}_{02}=\{0\}, \mathfrak{M}_0, \mathfrak{M}_2, P_0, P_2$, $W_\pm=V_\pm U_\pm$ for the pair H_0, H_2 in place of $\mathfrak{M}_{01}, \mathfrak{M}_0, \mathfrak{M}_1, P_0, P_1, U_\pm$ of these theorems. The only propositions that possibly need proof are (1.1) and (1.3). To prove (1.1), we note that for $t\to\pm\infty$

$$s\text{-lim}\exp(itH_2)\exp(-itH_1)P_1=V_\pm,\quad s\text{-lim}\exp(itH_1)\exp(-itH_0)P_0=U_\pm$$

have been proved. Multiplication of these two expressions yields

$$s\text{-lim}\exp(itH_2)P_1\exp(-itH_0)P_0=W_\pm,$$

where we have used the fact that H_1 and P_1 commute. Now the elimination of the factor P_1 from this equation can be effected in the same way as we have eliminated the factor P_1 from a similar equation in § 5, thus proving (1.1) in the present case. (1.3) is reduced to (1.1) by exchanging the roles of H_0 and H_2.

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Added in proof. Professor M. Rosenblum has kindly sent to the writer his paper "Perturbation of the continuous spectrum and unitary equivalence", Technical Report, Department of Mathematics, University of California, which is also to be published in Pacific J. Math. In this paper he proves the following theorem. Let H_0 be an absolutely continuous, self-adjoint operator and V a self-adjoint operator belonging to the trace class. Then $H_1=H_0+V$ is unitarily equivalent to H_0 if and only if H_1 is absolutely continuous. He also gives results corresponding to our Theorem 2 with $P_0=P_1=1$ under the above conditions. His theorems neither imply nor are implied by ours. Recently, however, the writer was able to extend the results of the present paper to the case where V may be any self-adjoint operator of the trace class, thus including Rosemblum's results as a special case. Details will be published elsewhere.