Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

On conformally curved Riemann spaces V_n, $n \geqq 6$, admitting a group of motions G_r of order $r > n(n+1)/2-(3n-11)$.

By Yosio MUTŌ

(Received April 12, 1956)

Synopsis

The conformal curvature tensor $C_{\lambda\mu\nu\omega}$ of a Riemann space V_n, $n \geqq 6$, admitting a group of motions of order $r > n(n+1)/2-(3n-11)$ is studied with the use of tensor calculus. The form of $C_{\lambda\mu\nu\omega}$ is obtained by virtue of the fact that the equations $XC_{\lambda\mu\nu\omega}=0$ can contain at most a certain number of linearly independent equations. The $C_{\lambda\mu\nu\omega}$ is in general of the form

$$\begin{aligned} C_{\lambda\mu\nu\omega} &= C[\delta_{\lambda\omega}\delta_{\mu\nu}-\delta_{\lambda\nu}\delta_{\mu\omega}] \\ &\quad -((n-1)/2)C[\delta_{\lambda\omega}(A_\mu A_\nu+B_\mu B_\nu)+\delta_{\mu\nu}(A_\lambda A_\omega \\ &\quad +B_\lambda B_\omega)-\delta_{\lambda\nu}(A_\mu A_\omega+B_\mu B_\omega)-\delta_{\mu\omega}(A_\lambda A_\nu+B_\lambda B_\nu)] \\ &\quad +((n-1)(n-2)/2)C[A_\lambda A_\omega B_\mu B_\nu+A_\mu A_\nu B_\lambda B_\omega \\ &\quad -A_\lambda A_\nu B_\mu B_\omega-A_\mu A_\omega B_\lambda B_\nu] \end{aligned}$$

with $A_\alpha A_\alpha=B_\alpha B_\alpha=1$, $A_\alpha B_\alpha=0$. But for $n=6, 8$ some other form is also possible.

§ 1. Introduction.

It is well known [1, 3, 9][1)] that we have the

THEOREM 1. *If an n-dimensional Riemannian space admits a group of motions of order $n(n+1)/2$, then, the space is of constant curvature.*

As for the Riemannian spaces which are not of constant curvature, we have the following theorems.

THEOREM 2. *An n-dimensional Riemannian space for $n>2, n \neq 4$,*

1) Numbers in brackets refer to the references at the end of the paper.

which is not of constant curvature cannot admit a group of motions of order greater than $n(n-1)/2+1$.

THEOREM 3. *The maximum order of the complete groups of motions in n-dimensional Riemannian spaces which are not Einstein spaces is* $n(n-1)/2+1$.

THEOREM 4. *The order of complete groups of motions of those n-dimensional Riemannian spaces which are different from spaces of constant curvature is not larger than* $n(n-1)/2+2$.

THEOREM 5. *In an n-dimensional Riemannian space for* $n\neq 4$, *there exists no group of motions of order r such that*

$$n(n+1)/2>r>n(n-1)/2+1\,.$$

THEOREM 6. *A necessary and sufficient condition that an n-dimensional Riemannian space* V_n *for* $n>4, n\neq 8$ *admit a group* G_r *of motions of order* $r=n(n-1)/2+1$ *is that the space be the product space of a straight line and an* $(n-1)$*-dimensional Riemannian space of constant curvature (this is equivalent to the fact that the space is conformally flat and admits a parallel vector field) or that the space be of negative constant curvature.*

Theorem 2 is a part of a theorem proved by H. C. Wang [7]. Theorem 3 and Theorem 4 are due to I. P. Egorov [2]. Theorem 5 and Theorem 6 are proved by K. Yano [8]. There are also valuable results obtained by G. Vranceanu and M. Kurita [4] [10], but they might be omitted for lack of space.

In proving Theorem 2, Theorem 5 and Theorem 6, some properties of the transformation groups of spheres discovered by D. Montgomery and H. Samelson are used [5].

Recently H. Wakakuwa [6] used some other results also obtained by D. Montgomery and H. Samelson and proved the

THEOREM 7. *If* $n\neq 5$, *an n-dimensional Riemannian space* V_n *can admit no intransitive group of motions of order r such that*

$$n(n-1)/2>r>(n-1)\,(n-2)/2+3\,.$$

And, except for finite number of n's, V_n *can also admit no transitive group of motions of order r in the above range. In this case if the values of n's are sufficiently large, that is, if* $n>248+1$, *the theorem holds good without exception.*

COROLLARY. *For* $n\neq 5$, *if an n-dimensional Riemannian space* V_n *admits a group of motions of order* $r=(n-1)\,(n-2)/2+2$ *or* $(n-1)$

$(n-2)/2+3$, *then G_r is always transitive.*

Thus, it seems interesting to study the n-dimensional Riemannian spaces admitting a group of motions of order $r=n(n+1)/2-2n+4$. This is the purpose of the present paper. But, unfortunately, and, as it is well known, such problem is difficult when n is small. Moreover, it seems that it is better to begin the study with conformally curved spaces, which are more complicated but more interesting than conformally flat spaces.

Thus, our study is restricted to the case of $n \geqq 6$ and

$$C_{\lambda\mu\nu\omega} \neq 0 ,$$

where $C_{\lambda\mu\nu\omega}$ is the Weyl conformal curvature tensor.

§ 2. The group of motions.

A group of motions in a Riemannian space V_n is characterized by the vector of infinitesimal transformation ξ^κ satisfying the Killing's equations

$$\xi_{\mu;\nu}+\xi_{\nu;\mu}=0 , \tag{1}$$

where $\xi_\mu = g_{\mu\alpha}\xi^\alpha$ and a semicolon denotes covariant differentiation.

The integrability condition of (1) is given by the system of equations

$$XR^{\lambda}_{\cdot\mu\nu\omega}=0 , \tag{2.0}$$

... .

$$X(R^{\lambda}_{\cdot\mu\nu\omega;\sigma_1;\cdots;\sigma_p})=0 , \tag{2.p}$$

... . ,

where X is the symbol of the Lie derivative,

$$XT^{\lambda\cdots}_{\mu\cdots} = T^{\lambda\cdots}_{\mu\cdots;\alpha}\xi^\alpha - T^{\alpha\cdots}_{\mu\cdots}\xi^{\lambda}_{\cdot\alpha} - \cdots$$
$$+T^{\lambda\cdots}_{\alpha\cdots}\xi^{\alpha}_{\cdot\mu} + \cdots ,$$

$$\xi^{\lambda}_{\cdot\mu} = \xi^{\lambda}_{;\mu} ,$$

and $R^{\lambda}_{\cdot\mu\nu\omega}$ is the Riemann-Christoffel's curvature tensor,

$$R^{\lambda}_{\cdot\mu\nu\omega} = \left\{ {\lambda \atop \mu\nu} \right\}_{,\omega} - \left\{ {\lambda \atop \mu\omega} \right\}_{,\nu} + \left\{ {\alpha \atop \mu\nu} \right\}\left\{ {\lambda \atop \alpha\omega} \right\} - \left\{ {\alpha \atop \mu\omega} \right\}\left\{ {\lambda \atop \alpha\nu} \right\} .$$

From (2.0) we easily get

(3) $$XC^{\lambda}_{\cdot\mu\nu\omega}=0\,,$$
that is,

(4) $$C^{\alpha}_{\cdot\mu\nu\omega}\xi^{\lambda}_{\cdot\alpha}-C^{\lambda}_{\cdot\alpha\nu\omega}\xi^{\alpha}_{\cdot\mu}-C^{\lambda}_{\cdot\mu\alpha\omega}\xi^{\alpha}_{\cdot\nu}-C^{\lambda}_{\cdot\mu\nu\alpha}\xi^{\alpha}_{\cdot\omega}=C^{\lambda}_{\cdot\mu\nu\omega;\,\alpha}\xi^{\alpha}\,,$$

for $C^{\lambda}_{\cdot\mu\nu\omega}$ is given by

(5) $$C^{\lambda}_{\cdot\mu\nu\omega}=R^{\lambda}_{\cdot\mu\nu\omega}-\frac{1}{n-2}\,(R_{\mu\nu}\delta^{\lambda}_{\omega}+R^{\lambda}_{\cdot\omega}g_{\mu\nu}-R_{\mu\omega}\delta^{\lambda}_{\nu}-R^{\lambda}_{\cdot\nu}g_{\mu\omega})+\frac{R}{(n-1)\,(n-2)}\,(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,.$$

Let us consider the system of equations (2) or (4) at a point of V_n. If (2) admits r linearly independent solutions ξ^{κ} and $\xi^{\kappa}_{\cdot\tau}$, satisfying

$$\xi_{\mu\nu}+\xi_{\nu\mu}=0\,,$$

where $\xi_{\mu\nu}=g_{\mu\alpha}\xi^{\alpha}_{\cdot\nu}=\xi_{\mu;\,\nu}$, then the V_n admits a group of motions G_r of order r.

From this fact, we see that a necessary condition for a V_n to admit a group of motions of order $r>n(n+1)/2-m$ is that the number of linearly independent equations in $\xi_{\mu\nu}+\xi_{\nu\mu}=0$ and (4) together be less than $n(n+1)/2+m$.

In the present paper we study the case of $m=3n-11$, and determine the form of $C_{\lambda\mu\nu\omega}$.

We assume that the fundamental form is positive definite.

§ 3. The three possible cases to be studied.

As we consider always at a fixed point of V_n, we can take an orthogonal ennuple at the point and consider the components with respect to it. Then, adopting the summation convention, we get the system of equations

(6) $$C_{\alpha\mu\nu\omega}\xi_{\alpha\lambda}+C_{\lambda\alpha\nu\omega}\xi_{\alpha\mu}+C_{\lambda\mu\alpha\omega}\xi_{\alpha\nu}+C_{\lambda\mu\nu\alpha}\xi_{\alpha\omega}\equiv 0 \quad (\mathrm{mod}\ \xi^{\kappa})\,,$$

where (mod ξ^{κ}) means that the equations are valid except for the linear forms of ξ^{κ}.

If we assume (6) to have only $n(n-1)/2$ unknowns $\xi_{\lambda\mu}=-\xi_{\mu\lambda}(\lambda<\mu)$, we see immediately that our problem is to find the form of $C_{\lambda\mu\nu\omega}$

such that the number N of linearly independent equations in (6) satisfies $N<3n-11$.

Now, it is easy to show that we can take an orthogonal ennuple such that we have $C_{2112}\neq 0$.

Because, if C_{2112} is always zero, we have

$$C_{\alpha\beta\gamma\delta}u_\alpha v_\beta v_\gamma u_\delta=0$$

for any choice of the vectors u_λ, v_λ. Then, as we have $C_{\lambda\mu\nu\omega}=C_{\omega\nu\mu\lambda}$, we obtain $C_{\alpha(\mu\nu)\beta}u_\alpha u_\beta=0$, and further $C_{\lambda(\mu\nu)\omega}=0$. On the other hand we have $C_{\lambda\mu(\nu\omega)}=0$ and $C_{\lambda[\mu\nu\omega]}=0$, hence we get $C_{\lambda\mu\nu\omega}=0$, contrary to the assumption $C_{\lambda\mu\nu\omega}\neq 0$.

Assume that the first axis of the orthogonal ennuple is fixed. As we have always $C_{j11k}=C_{k11j}$, we can take other $n-1$ axes in such a way that we have

(7) $$C_{j11k}=C_j\delta_{ik}.$$ [2)]

If we have $C_2=\cdots=C_n$, we get $C_2=0$ because of $C_{\alpha11\alpha}=0$, hence $C_{\lambda11\omega}=0$. Moreover, in this case we have $C_{2112}=0$ for any choice of the axes $2,\cdots,n$.

Thus, we find that we can take an orthogonal ennuple such that we have (7) where not all of C_j are the same.

Let us consider the following cases, by which all possible cases are exhausted.

(8.2) $$C_2\neq C_3=\cdots=C_n\,,$$

(8.3) $$C_2,\ C_3\neq C_4=\cdots=C_n\,,$$

(8.4) $$C_2,\ C_3,\ C_4\neq C_5=\cdots=C_n\,,$$

... ...

(8.i) $$C_2,\cdots,C_i\neq C_{i+1}=\cdots=C_n\,,$$

... ...

(8. $n-1$) $$C_2,\cdots,C_{n-1}\neq C_n\,.$$

As we can interchange the numbers (axes) $2,\cdots,n$, we may assume that we can never find $n-i+1$ equal numbers among the set $C_2,\cdots,C_i$ in (8.i).

If we have (8.i), we get from (6)

2) Throughout the paper, the indices run as follows, if no special remark is made. $\lambda,\mu,\cdots,\alpha,\beta,\cdots=1,\cdots,n;\ i,j,k,\cdots=2,\cdots,n.$

$$(9)\qquad \begin{aligned} &\xi_{2\,i+1}\equiv\cdots\equiv\xi_{2n}\\ &\equiv\xi_{3\,i+1}\equiv\cdots\equiv\xi_{3n}\\ &\quad\cdots\qquad\cdots\\ &\equiv\xi_{i\,i+1}\equiv\cdots\equiv\xi_{in}\equiv 0 \quad (\mathrm{mod}\ \xi^{\kappa},\ \xi_{1\kappa}) \end{aligned}$$

by putting $\mu=\nu=1$ and using (7).

In (9) there are $(i-1)(n-i)$ independent equations. Moreover, if we have $C_p \neq C_q$ for some p, q satisfying $2\leqq p<q\leqq i$, we have another equation

$$\xi_{pq}\equiv 0 \ (\mathrm{mod}\ \xi^{\kappa},\ \xi_{1\kappa})\,.$$

As we are studying the case of $N<3n-11$, $n\geqq 6$, we see, after some elementary calculation, that we have to study only the cases of (8.2), (8.3) and (8.4).

§ 4. The first case.

Let us study the case of (8.2). We can assume that we have (8.2) for any choice of the first axis of the orthogonal ennuple,[3] for, if we have some other, for example, (8.3) by a suitable choice of it, we can study the latter.

Then, we get

$$C_{j11k}=a\delta_{jk}+b\delta_{j2}\delta_{k2}\,,$$

that is,

$$C_{\lambda 11\omega}=a(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1})+b\delta_{\lambda 2}\delta_{\omega 2}\,.$$

Now, we use an arbitrary orthogonal ennuple again, and get

$$(10)\qquad C_{\lambda\alpha\beta\omega}u_{\alpha}u_{\beta}=a(\delta_{\lambda\omega}-u_{\lambda}u_{\omega})+bv_{\lambda}v_{\omega}\,,$$

where a, b, v_{λ} are functions of a unit vector u_{λ}, satisfying

$$(11)\qquad v_{\lambda}v_{\lambda}=1\,,\qquad u_{\lambda}v_{\lambda}=0\,,$$

for, the direction of the second axis of the orthogonal ennuple satisfying (8.2) is a function of the direction of the first axis.

From (10) and $C_{\alpha\mu\nu\alpha}=0$ we get

3) We cannot deny that, when the first axis has some special direction, we may have $C_2=\cdots=C_n=0$. But it does not matter to us in the following discussions, for we can avoid such directions.

(12) $$(n-1)a+b=0\,.$$

We take an orthogonal ennuple such that we have

(13) $$v_\lambda=\delta_{\lambda 2}$$

for $u_\lambda=\delta_{\lambda 1}$. This is possible for an arbitrarily chosen direction of the first axis.

Then, if we put

(14) $$u_\lambda(t)=(1-(t_2)^2-\cdots-(t_n)^2)^{1/2}\delta_{\lambda 1}+t_2\delta_{\lambda 2} +\cdots+t_n\delta_{\lambda n}\,,$$

we obtain the functions $a(t), b(t), v_\lambda(t)$, which we can assume to be analytic for sufficiently small $|t_j|$, for, these functions are determined by the algebraic relations (10) and we can take a suitable direction for the first axis of the ennuple.[3] We easily find that $v_\lambda(t)$ satisfy

(15) $$v_\lambda(0)=\delta_{\lambda 2}\,,$$

(16) $$v_{2/j}(0)\equiv(\partial v_2/\partial t_j)_{t=0}=0\,.$$

Differentiating (10) with respect to t_j, we get for $t_2=\cdots=t_n=0$

(17) $$C_{\lambda 1j\omega}+C_{\lambda j1\omega}=a_{/j}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1})+b_{/j}\delta_{\lambda 2}\delta_{\omega 2} -a(\delta_{\lambda 1}\delta_{\omega j}+\delta_{\omega 1}\delta_{\lambda j})+b(\delta_{\lambda 2}v_{\omega/j}+\delta_{\omega 2}v_{\lambda/j})\,.$$

Putting $\lambda=\omega=j=2$, we get $a_{/2}+b_{/2}=0$. Putting $\lambda=\omega=j=3$, we get $a_{/3}=0$, and so on. But we have $(n-1)a_{/j}+b_{/j}=0$. Hence we get

(18) $$a=C\,,\quad b=-(n-1)C\,,$$

where C is a non-zero constant.

If we consider the quantity

(19) $$M_{\lambda\mu\nu\omega}=-\frac{1}{(n-1)C}\,[C_{\lambda\mu\nu\omega}-C(\delta_{\lambda\omega}\delta_{\mu\nu}-\delta_{\lambda\nu}\delta_{\mu\omega})]\,,$$

we get from (10)

(20) $$M_{\lambda\alpha\beta\omega}u_\alpha u_\beta=v_\lambda v_\omega\,.$$

Now, we can use the

LEMMA 1. *A necessary and sufficient condition for a tensor* $M_{\lambda\mu\nu\omega}$ *satisfying the relations*

(21) $$M_{\lambda\mu\nu\omega}=M_{\omega\nu\mu\lambda},\ M_{\lambda\mu(\nu\omega)}=0,\ M_{\lambda[\mu\nu\omega]}=0$$

to satisfy the equations (20) *for some functions* v_λ *of a vector* u_λ *satisfying* $v_\alpha v_\alpha = u_\alpha u_\alpha$ *is that* n *be even and that the components* $M_{\lambda\mu\nu\omega}$ *with respect to a suitably chosen orthogonal ennuple satisfy*

(22)
$$\begin{cases} M_{2112} = M_{4334} = \cdots = M_{n\, n-1\, n-1\, n} = 1\,, \\ M_{1234} = -2/3, \qquad M_{2314} = 1/3\,, \\ M_{1256} = -2/3, \qquad M_{2516} = 1/3\,, \\ M_{3456} = -2/3, \qquad M_{4536} = 1/3\,, \\ \cdots\cdots \qquad\qquad \cdots\cdots \end{cases}$$

all components except those which are derivable from the ones written above according to (21) *are zero.*

This lemma will be proved in § 8.

As we get from (6) and (19)

$$M_{\alpha\mu\nu\omega}\xi_{\alpha\lambda} + M_{\lambda\alpha\nu\omega}\xi_{\alpha\mu} + M_{\lambda\mu\alpha\omega}\xi_{\alpha\nu} + M_{\lambda\mu\nu\alpha}\xi_{\alpha\omega} \equiv 0 \quad (\mathrm{mod}\ \xi^\kappa)\,,$$

we get for $\lambda=1$, $\mu=2$, $\nu=1$, $\omega=3$

$$\xi_{41} - \xi_{23} \equiv 0 \quad (\mathrm{mod}\ \xi^\kappa)\,,$$

that is,

(23)
$$\xi_{14} \equiv -\xi_{23} \quad (\mathrm{mod}\ \xi^\kappa)\,.$$

Similarly, we get

(24)
$$\xi_{13} \equiv \xi_{24},\ \xi_{15} \equiv \xi_{26},\ \xi_{16} \equiv -\xi_{25}\,,\ \cdots \quad (\mathrm{mod}\ \xi^\kappa)\,,$$

which are $(n/2)(n/2-1)$ linearly independent equations in all. Comparing this number with $3n-11$, we find the

LEMMA 2. *A Riemann space with conformal curvature tensor satisfying* (8.2) *can not admit a group of motions of order* $r > n(n+1)/2 - (3n-11)$ *for* $n > 8$.

§ 5. The second case.

Now we study the case of (8.3). We have to study only the case where we have (8.3) for the general direction of the first axis. As we can put

$$C_{j11k} = a\delta_{jk} + b\delta_{j2}\delta_{k2} + c\delta_{j3}\delta_{k3}\,,$$

that is,

$$C_{\lambda 1 1 \omega}=a(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1})+b\delta_{\lambda 2}\delta_{\omega 2}+c\delta_{\lambda 3}\delta_{\omega 3}\,, \tag{25}$$

and, as the second axis, the third axis, as well as the coefficients a, b, c are dependent upon the direction of the first axis, we get for an arbitrary orthogonal ennuple

$$C_{\lambda\alpha\beta\omega}u_\alpha u_\beta=a(\delta_{\lambda\omega}-u_\lambda u_\omega)+bv_\lambda v_\omega+cw_\lambda w_\omega\,, \tag{26}$$

where $a, b, c, v_\lambda, w_\lambda$ are functions of a unit vector u_α, satisfying

$$(n-1)a+b+c=0\,, \tag{27}$$

$$v_\alpha v_\alpha=w_\alpha w_\alpha=1\,, \tag{28}$$

$$u_\alpha v_\alpha=u_\alpha w_\alpha=v_\alpha w_\alpha=0\,. \tag{29}$$

Moreover, we can assume $a, b, c\neq 0$.

Let us take an orthogonal ennuple such that we have

$$v_\lambda=\delta_{\lambda 2}\,,\qquad w_\lambda=\delta_{\lambda 3}$$

for $u_\lambda=\delta_{\lambda 1}$. If we put

$$u_\lambda=(1-(t_2)^2-\cdots-(t_n)^2)^{1/2}\delta_{\lambda 1}+t_2\delta_{\lambda 2}+\cdots+t_n\delta_{\lambda n}\,, \tag{30}$$

we obtain the functions $v_\lambda(t)$, $w_\lambda(t)$, $a(t)$, $b(t)$, $c(t)$, satisfying

$$v_\lambda(0)=\delta_{\lambda 2}\,,\qquad w_\lambda(0)=\delta_{\lambda 3}\,, \tag{31}$$

$$v_{2/j}(0)=0\,,\qquad w_{3/j}(0)=0\,. \tag{32}$$

As we have (25), where $b, c\neq 0$, we get

$$\xi_{2x}\equiv\xi_{3x}\equiv 0 \pmod{\xi^\kappa, \xi_{1\kappa}}\,,^{4)} \tag{33}$$

which are $2n-6$ linearly independent equations. If we put $\lambda=p$, $\mu=q$, $\nu=r$, $\omega=y$ in (6), we get

$$C_{pqrx}\xi_{xy}\equiv 0 \pmod{\xi^\kappa, \xi_{1\kappa}, \xi_{2\kappa}, \xi_{3\kappa}}\,,$$

which are at least $n-4$ linearly independent equations quite independent of (33) unless we have

$$C_{pqrx}=0\,. \tag{34}$$

But we assumed that $N<3n-11$. Hence we must have (34).

Now consider the equations obtained by substituting (30) into (26),

4) In § 5 the indices run as follows.

$x, y, z, u, v=4,\cdots, n\,;\ p, q, r, s, t=1, 2, 3$

$$C_{\lambda\alpha\beta\omega}u_\alpha(t)u_\beta(t)=a(t)\{\delta_{\lambda\omega}-u_\lambda(t)u_\omega(t)\}$$
$$+b(t)v_\lambda(t)v_\omega(t)+c(t)w_\lambda(t)w_\omega(t)\,. \tag{35}$$

Differentiating (35) with respect to t_2 and putting $t_2=\cdots=t_n=0$, we get

$$C_{\lambda 12\omega}+C_{\lambda 21\omega}=a_{/2}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1})+b_{/2}\delta_{\lambda 2}\delta_{\omega 2}$$
$$+c_{/2}\delta_{\lambda 3}\delta_{\omega 3}-a_0(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2})$$
$$+b_0(\delta_{\lambda 2}v_{\omega/2}+\delta_{\omega 2}v_{\lambda/2})+c_0(\delta_{\lambda 3}w_{\omega/2}+\delta_{\omega 3}w_{\lambda/2})\,. \tag{36}$$

If we put $\lambda=2, \omega=x$, we get $v_{x/2}(0)=0$, for we have (34) and $b_0\neq 0$. If we put $\lambda=3, \omega=x$, we get $w_{x/2}(0)=0$. Similarly, if we differentiate (35) with respect to t_3, we get $v_{x/3}(0)=0$, $w_{x/3}(0)=0$. As we consider the value of such derivatives only for $t_j=0$ in the following, we can omit the symbol (0) and write

$$v_{x/2}=v_{x/3}=w_{x/2}=w_{x/3}=0\,. \tag{37}$$

If we put $\lambda=x, \omega=y$ in (36), we get

$$C_{x12y}+C_{x21y}=a_{/2}\delta_{xy}\,.$$

Then, we can write

$$C_{x12y}+C_{x21y}=2C_{12}\delta_{xy}\,. \tag{38}$$

Similarly, we get

$$C_{x13y}+C_{x31y}=2C_{13}\delta_{xy}\,. \tag{39}$$

Differentiating (35) twice with respect to t_2 and putting $t_2=\cdots=t_n=0$, we get

$$2C_{\lambda 22\omega}-2C_{\lambda 11\omega}=a_{/22}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1})$$
$$+b_{/22}\delta_{\lambda 2}\delta_{\omega 2}+c_{/22}\delta_{\lambda 3}\delta_{\omega 3}$$
$$-2a_{/2}(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2})+2b_{/2}(\delta_{\lambda 2}v_{\omega/2}$$
$$+\delta_{\omega 2}v_{\lambda/2})+2c_{/2}(\delta_{\lambda 3}w_{\omega/2}+\delta_{\omega 3}w_{\lambda/2})$$
$$-a_0(2\delta_{\lambda 2}\delta_{\omega 2}-2\delta_{\lambda 1}\delta_{\omega 1})+b_0(\delta_{\lambda 2}v_{\omega/22}$$
$$+2v_{\lambda/2}v_{\omega/2}+\delta_{\omega 2}v_{\lambda/22})+c_0(\delta_{\lambda 3}w_{\omega/22}$$
$$+2w_{\lambda/2}w_{\omega/2}+\delta_{\omega 3}w_{\lambda/22})\,. \tag{40}$$

Then, putting $\lambda=x, \omega=y$, we get

$$C_{x22y}=\frac{1}{2}a_{/22}\delta_{xy}+a_0\delta_{xy}$$

because of (37). Hence we can put

$$C_{x22y}=C_{22}\delta_{xy}\,.$$

Similarly, we get

$$C_{x23y}+C_{x32y}=2C_{23}\delta_{xy},\quad C_{x33y}=C_{33}\delta_{xy}\,.$$

From these results we find

$$C_{xpqy}+C_{xqpy}=2C_{pq}\delta_{xy}\,. \tag{41}$$

Now, consider the components

$$C_{pqxy}=-C_{xpqy}+C_{xqpy}\,.$$

The equations obtained by putting $\lambda=p$, $\mu=q$, $\nu=x$, $\omega=y$ in (6) give

$$C_{pqzy}\xi_{zx}+C_{pqxz}\xi_{zy}\equiv 0 \pmod{\xi^{\kappa}, \xi_{1\kappa}, \xi_{2\kappa}, \xi_{3\kappa}}\,,$$

and there are at least $n-5$ linearly independent equations in these equations unless $C_{pqxy}=0$. For, if we have $C_{pqxy}\neq 0$, we can assume that we have, for some p, q, $C_{pq45}\neq 0$, $C_{pq46}=\cdots=C_{pq4n}=0$, and get

$$\xi_{56}\equiv\cdots\equiv\xi_{5n}\equiv 0 \pmod{\xi^{\kappa}, \xi_{1\kappa}, \xi_{2\kappa}, \xi_{3\kappa}, \xi_{4\kappa}}$$

by putting $x=4$, $y=6,\cdots, n$. As we assumed that $N<3n-11$, we must have

$$C_{pqxy}=0\,, \tag{42}$$

and, consequently,

$$C_{xpqy}=C_{pq}\delta_{xy}\,, \tag{43}$$

$$C_{pq}=C_{qp}\,. \tag{44}$$

Differentiating (35) with respect to t_x and putting $t_2=\cdots=t_n=0$, we get

$$\begin{aligned} C_{\lambda 1x\omega}+C_{\lambda x1\omega}=&\, a_{/x}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1}) \\ &+b_{/x}\delta_{\lambda 2}\delta_{\omega 2}+c_{/x}\delta_{\lambda 3}\delta_{\omega 3}-a_0(\delta_{\lambda 1}\delta_{\omega x} \\ &+\delta_{\omega 1}\delta_{\lambda x})+b_0(\delta_{\lambda 2}v_{\omega/x}+\delta_{\omega 2}v_{\lambda/x}) \\ &+c_0(\delta_{\lambda 3}w_{\omega/x}+\delta_{\omega 3}w_{\lambda/x})\,. \end{aligned} \tag{45}$$

If we put $\lambda=\omega=x$, we get $a_{/x}=0$. If we put $\lambda=\omega=2$, we get $b_{/x}=0$ by virtue of (34), $a_{/x}=0$ and (32). Similarly, if we put $\lambda=\omega=3$, we get $c_{/x}=0$. Hence (45) becomes

$$C_{\lambda 1 x\omega}+C_{\lambda x 1\omega}=-a_0(\delta_{\lambda 1}\delta_{\omega x}+\delta_{\omega 1}\delta_{\lambda x}) +b_0(\delta_{\lambda 2}v_{\omega/x}+\delta_{\omega 2}v_{\lambda/x}) +c_0(\delta_{\lambda 3}w_{\omega/x}+\delta_{\omega 3}w_{\lambda/x})\,. \tag{46}$$

If we put $\lambda=2$, $\omega=y$ in (46), we get

$$C_{21xy}+C_{2x1y}=b_0 v_{y/x}\,,$$

hence

$$v_{y/x}=-(C_{12}/b_0)\delta_{xy}\,. \tag{47}$$

Similarly, we get

$$w_{y/x}=-(C_{13}/c_0)\delta_{xy}\,. \tag{48}$$

If we put $\lambda=y$, $\omega=z$ in (46), we get $C_{y1xz}+C_{yx1z}=0$, hence $C_{1yzx}+C_{1zyx}=0$. Then, because of $C_{1x(yz)}=0$ and $C_{1[xyz]}=0$ we get

$$C_{1xyz}=0\,. \tag{49}$$

Differentiating (35) with respect to t_2 and then with respect to t_x and putting $t_2=\cdots=t_n=0$, we get

$$\begin{aligned} C_{\lambda 2x\omega}+C_{\lambda x2\omega}=a_{/2x}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1}) &+b_{/2x}\delta_{\lambda 2}\delta_{\omega 2}+c_{/2x}\delta_{\lambda 3}\delta_{\omega 3}-a_{/2}(\delta_{\lambda 1}\delta_{\omega x}\\ &+\delta_{\omega 1}\delta_{\lambda x})-a_{/x}(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2})\\ &+b_{/2}(\delta_{\lambda 2}v_{\omega/x}+\delta_{\omega 2}v_{\lambda/x})+b_{/x}(\delta_{\lambda 2}v_{\omega/2}\\ &+\delta_{\omega 2}v_{\lambda/2})+c_{/2}(\delta_{\lambda 3}w_{\omega/x}+\delta_{\omega 3}w_{\lambda/x})\\ &+c_{/x}(\delta_{\lambda 3}w_{\omega/2}+\delta_{\omega 3}w_{\lambda/2})-a_0(\delta_{\lambda 2}\delta_{\omega x}\\ &+\delta_{\omega 2}\delta_{\lambda x})+b_0(\delta_{\lambda 2}v_{\omega/2x}+v_{\lambda/2}v_{\omega/x}\\ &+v_{\omega/2}v_{\lambda/x}+\delta_{\omega 2}v_{\lambda/2x})+c_0(\delta_{\lambda 3}w_{\omega/2x}\\ &+w_{\lambda/2}w_{\omega/x}+w_{\omega/2}w_{\lambda/x}+\delta_{\omega 3}w_{\lambda/2x})\,. \end{aligned}$$

Then putting $\lambda=\omega=x$, we find $a_{/2x}=0$ by virtue of (37). Putting $\lambda=y$, $\omega=z$, we get $C_{y2xz}+C_{yx2z}=0$, hence $C_{2xyz}=0$.

Similarly, we get $C_{3xyz}=0$, hence

$$C_{pxyz}=0\,. \tag{50}$$

Differentiating (35) with respect to t_x and then with respect to t_y and putting $t_2=\cdots=t_n=0$, we get

$$C_{\lambda xy\omega}+C_{\lambda yx\omega}-2C_{\lambda 11\omega}\delta_{xy}=a_{/xy}(\delta_{\lambda\omega}-\delta_{\lambda 1}\delta_{\omega 1}) \tag{51}$$
$$+b_{/xy}\delta_{\lambda 2}\delta_{\omega 2}+c_{/xy}\delta_{\lambda 3}\delta_{\omega 3}-a_{/x}(\delta_{\lambda 1}\delta_{\omega y}+\delta_{\omega 1}\delta_{\lambda y})$$
$$-a_{/y}(\delta_{\lambda 1}\delta_{\omega x}+\delta_{\omega 1}\delta_{\lambda x})+b_{/x}(\delta_{\lambda 2}v_{\omega/y}+\delta_{\omega 2}v_{\lambda/y})$$
$$+b_{/y}(\delta_{\lambda 2}v_{\omega/x}+\delta_{\omega 2}v_{\lambda/x})+c_{/x}(\delta_{\lambda 3}w_{\omega/y}+\delta_{\omega 3}w_{\lambda/y})$$
$$+c_{/y}(\delta_{\lambda 3}w_{\omega/x}+\delta_{\omega 3}w_{\lambda/x})-a_0(\delta_{\lambda x}\delta_{\omega y}+\delta_{\omega x}\delta_{\lambda y}$$
$$-2\delta_{\lambda 1}\delta_{\omega 1}\delta_{xy})+b_0(\delta_{\lambda 2}v_{\omega/xy}+v_{\lambda/x}v_{\omega/y}$$
$$+v_{\omega/x}v_{\lambda/y}+\delta_{\omega 2}v_{\lambda/xy})+c_0(\delta_{\lambda 3}w_{\omega/xy}$$
$$+w_{\lambda/x}w_{\omega/y}+w_{\omega/x}w_{\lambda/y}+\delta_{\omega 3}w_{\lambda/xy})\,.$$

If we put $\lambda=\omega=x$, we get

$$-2a_0\delta_{xy}=a_{/xy}-2a_0\delta_{xy}+2b_0v_{x/x}v_{x/y}+2c_0w_{x/x}w_{x/y}\,,$$

hence

$$a_{/xy}=-2\{(C_{12})^2/b_0+(C_{13})^2/c_0\}\delta_{xy} \tag{52}$$

because of (47) and (48). If we put $\lambda=z$, $\omega=u$, we get

$$C_{zxyu}+C_{zyxu}=2a_0\delta_{zu}\delta_{xy}+a_{/xy}\delta_{zu}$$
$$+\{-a_0+(C_{12})^2/b_0+(C_{13})^2/c_0\}\,(\delta_{zx}\delta_{uy}+\delta_{zy}\delta_{ux})\,,$$

hence

$$C_{zxyu}+C_{zyxu}=C(2\delta_{zu}\delta_{xy}-\delta_{zx}\delta_{uy}-\delta_{zy}\delta_{ux})\,,$$

where

$$C=a_0-(C_{12})^2/b_0-(C_{13})^2/c_0\,. \tag{53}$$

Moreover, as we have $C_{zx(yu)}=0$, $C_{z[xyu]}=0$, we get

$$C_{xyzu}=C(\delta_{xu}\delta_{yz}-\delta_{yu}\delta_{xz})\,. \tag{54}$$

We are now ready to determine $C_{\lambda\mu\nu\omega}$.

Substituting (43) and (54) into $C_{x\alpha\alpha u}=0$, we get

$$C_{11}+C_{22}+C_{33}+(n-4)C=0\,. \tag{55}$$

On the other hand we have from $C_{p\alpha\alpha q}=0$

$$\begin{cases} C_{1221}+C_{1331}+(n-3)C_{11}=0\,, \\ C_{2112}+C_{2332}+(n-3)C_{22}=0\,, \\ C_{3113}+C_{3223}+(n-3)C_{33}=0\,, \end{cases} \tag{56}$$

$$\text{(57)}\quad \begin{cases} C_{1332}+(n-3)C_{12}=0\,, \\ C_{1223}+(n-3)C_{13}=0\,, \\ C_{2113}+(n-3)C_{23}=0\,, \end{cases}$$

while the results obtained can be summarized as

$$\text{(58)}\quad \begin{cases} C_{pqrx}=0,\ C_{pqxy}=0,\ C_{pxyz}=0,\ C_{pxyq}=C_{pq}\delta_{xy}\,, \\ C_{xyzu}=C(\delta_{xu}\delta_{yz}-\delta_{xz}\delta_{yu})\,. \end{cases}$$

Now, if we put $\lambda=x$, $\mu=y$, $\nu=z$, $\omega=p$ in (6), we get

$$C_{\alpha yzp}\xi_{\alpha x}+C_{x\alpha zp}\xi_{\alpha y}+C_{xy\alpha p}\xi_{\alpha z}+C_{xyz\alpha}\xi_{\alpha p}\equiv 0 \pmod{\xi^\kappa}\,,$$

hence

$$(C_{pq}-C\delta_{pq})\,(\delta_{yz}\xi_{qx}-\delta_{xz}\xi_{qy})\equiv 0 \pmod{\xi^\kappa}$$

by virtue of (58). Putting $y=z$ and summing up, we obtain

$$\text{(59)}\qquad (C_{pq}-C\delta_{pq})\xi_{qx}\equiv 0 \pmod{\xi^\kappa}\,.$$

If we put $\lambda=x$, $\mu=p$, $\nu=q$, $\omega=r$ in (6), we get

$$C_{spqr}\xi_{sx}+C_{xpyr}\xi_{yq}+C_{xpqy}\xi_{yr}\equiv 0 \pmod{\xi^\kappa}\,,$$

hence

$$\text{(60)}\qquad (C_{spqr}+C_{pr}\delta_{sq}-C_{pq}\delta_{sr})\xi_{sx}\equiv 0 \pmod{\xi^\kappa}\,.$$

We obtained (58) by using a special orthogonal ennuple. But these equations are preserved when we take a new orthogonal ennuple obtained from the original one by a transformation for which

$$a_{px}=0\,,\qquad a_{xp}=0\,.$$

Hence we can effect a suitable transformation and get

$$\text{(61)}\qquad C_{pq}=C_p\delta_{pq}$$

without destroying (55), (56), (57), (58).

Then, (59) becomes

$$\text{(62)}\quad \begin{cases} (C_1-C)\xi_{1x}\equiv 0 \pmod{\xi^\kappa}\,, \\ (C_2-C)\xi_{2x}\equiv 0 \pmod{\xi^\kappa}\,, \\ (C_3-C)\xi_{3x}\equiv 0 \pmod{\xi^\kappa}\,. \end{cases}$$

We must study three possible cases,

(63) $$C_1=C_2=C_3=C\,,$$

(64) $$C_1\neq C,\ C_2=C_3=C\,,$$

(65) $$C_1\neq C,\ C_2\neq C,\ C_3=C\,,$$

for, if $C_1\neq C$, $C_2\neq C$, $C_3\neq C$, we have $3n-9$ linearly independent equations in (62), contrary to the assumption $N<3n-11$.

If we have (63), we get from (55) $C=0$, hence $C_{pq}=0$. From (56) we get $C_{1221}=C_{1331}=C_{2332}=0$, while from (57) we get $C_{2113}=C_{1223}=C_{1332}=0$, hence $C_{pqrs}=0$. From (58) we can conclude $C_{\lambda\mu\nu\omega}=0$ contrary to the assumption $C_{\lambda\mu\nu\omega}\neq 0$.

Suppose that we have (64). Putting $p=q=2$, $r=3$ in (60), we get

$$(C_{3223}-C)\xi_{3x}\equiv 0 \pmod{\xi^{\kappa}}$$

because of (57) and $C_{13}=C_{23}=0$. Putting $p=q=3$, $r=2$ in (60), we get

$$(C_{2332}-C)\xi_{2x}\equiv 0 \pmod{\xi^{\kappa}}\,.$$

If $C_{2332}\neq C$, we get $\xi_{3x}\equiv\xi_{2x}\equiv\xi_{1x}\equiv 0 \pmod{\xi^{\kappa}}$, contrary to the relation $N<3n-11$. Hence we get

$$C_{2332}=C\,.$$

Then we get from (56)

$$C_{2112}=C_{3113}=-(n-2)C\,,$$

$$-2(n-2)C+(n-3)C_1=0\,.$$

But we have

$$(n-2)C+C_1=0$$

by virtue of (55). Hence we get $C_1=C=0$ which contradicts the relation $C_1\neq C$.

Thus, we can conclude that (65) is the only one possible case.

As we have

$$\xi_{1x}\equiv\xi_{2x}\equiv 0 \pmod{\xi^{\kappa}}\,,$$

we get

$$(C_{3pqr}+C_{pr}\delta_{3q}-C_{pq}\delta_{3r})\xi_{3x}\equiv 0 \pmod{\xi^{\kappa}}$$

from (60). If the equations

(66) $$C_{3pqr}+C_{pr}\delta_{3q}-C_{pq}\delta_{3r}=0$$

are not satisfied, we have $3n-9$ linearly independent equations $\xi_{1x}\equiv\xi_{2x}\equiv\xi_{3x}\equiv 0 \pmod{\xi^{\kappa}}$, contrary to $N<3n-11$. Hence we have (66).

Putting $p=q=1$, $r=3$ in (66), we get $C_{3113}=C_1$, while putting $p=q=2$, $r=3$ in (66), we get $C_{3223}=C_2$, hence

$$C_{3113}=C_{3223}=C_1=C_2=-\frac{1}{n-2}C_{1221} \tag{67}$$

from (56). Moreover, we get

$$C_3=\frac{2}{(n-2)(n-3)}C_{1221}=C\,. \tag{68}$$

As we have

$$C_{2113}=C_{1223}=C_{1332}=0$$

because of (57) and (61), all components of the conformal curvature tensor are determined.

$$\left\{\begin{aligned}
&C_{2112}=\frac{1}{2}(n-2)(n-3)C\,,\\
&C_{3113}=C_{3223}=-\frac{1}{2}(n-3)C\,,\\
&C_{2113}=C_{1223}=C_{1332}=0\,,\\
&C_{pq}=\delta_{pq}C_p\,,\\
&C_1=C_2=-\frac{1}{2}(n-3)C,\quad C_3=C\,,\\
&C_{pqrx}=C_{pqxy}=C_{pxyz}=0\,,\\
&C_{pxyq}=C_{pq}\delta_{xy}\,,\\
&C_{xyzu}=C(\delta_{xu}\delta_{yz}-\delta_{xz}\delta_{yu})\,.
\end{aligned}\right. \tag{69}$$

If we put

$$\begin{aligned}
C_{\lambda\mu\nu\omega}=&C[\delta_{\lambda\omega}\delta_{\mu\nu}-\delta_{\lambda\nu}\delta_{\mu\omega}]\\
&-\frac{n-1}{2}C[\delta_{\lambda\omega}(A_\mu A_\nu+B_\mu B_\nu)+\delta_{\mu\nu}(A_\lambda A_\omega+B_\lambda B_\omega)\\
&-\delta_{\lambda\nu}(A_\mu A_\omega+B_\mu B_\omega)-\delta_{\mu\omega}(A_\lambda A_\nu+B_\lambda B_\nu)]\\
&+\frac{(n-1)(n-2)}{2}C[A_\lambda A_\omega B_\mu B_\nu-A_\mu A_\omega B_\lambda B_\nu\\
&-A_\lambda A_\nu B_\mu B_\omega+A_\mu A_\nu B_\lambda B_\omega]\,,
\end{aligned} \tag{70}$$

where

$$A_\lambda=\delta^1_\lambda\,,\qquad B_\lambda=\delta^2_\lambda\,,$$

we find that (69) is satisfied. Hence (70) where

(71) $$A_\alpha A_\alpha=B_\alpha B_\alpha=1\,,\qquad A_\alpha B_\alpha=0$$

is the curvature we are looking for.

We thus obtain the

LEMMA 3. *A necessary condition that a Riemann space* $V^n, n\geqq 6$, *with conformal curvature tensor satisfying* (8.3) *admit a group of motions of order* $r>n(n+1)/2-(3n-11)$ *is that the conformal curvature have the form* (70) *with* A_λ *and* B_λ *satisfying* (71).

§ 6. The third case.

Now, we consider the case of (8.4).

As we have $3n-12$ linearly independent equations

(72) $$\xi_{2x}\equiv\xi_{3x}\equiv\xi_{4x}\equiv 0 \pmod{\xi^\kappa, \xi_{1\kappa}}\,,^{5)}$$

we get

$$C_2=C_3=C_4$$

by virtue of the relation $N<3n-11$, for we can not have any equation independent of (72). Because of this fact we have to study only the case $n\geqq 7$.

If we put $\lambda=x$, $\mu=y$, $\nu=z$, $\omega=u$ in (6), we get

$$C_{vyzu}\xi_{vx}+C_{xvzu}\xi_{vy}+C_{xyvu}\xi_{vz}+C_{xyzv}\xi_{vu}\equiv 0 \pmod{\xi^\kappa, \xi_{t\kappa}}\,.^{5)}$$

As we have no equation other than (72), we get

(73) $$C_{xyzu}=C(\delta_{xu}\delta_{yz}-\delta_{xz}\delta_{yu})\,.$$

If we put $\lambda=p$, $\mu=q$, $\nu=r$, $\omega=x$ in (6), we get

$$C_{pqry}\xi_{yx}\equiv 0 \pmod{\xi^\kappa, \xi_{t\kappa}}\,,$$

hence we have

(74) $$C_{pqrx}=0\,.$$

If we put $\lambda=p$, $\mu=q$, $\nu=x$, $\omega=y$ in (6), we get

5) In § 6 the indices run as follows:
$x, y, z, u, v=5,\cdots, n$; $p, q, r, s, t=1, 2, 3, 4$.

$$C_{pqzy}\xi_{zx}+C_{pqxz}\xi_{zy}\equiv 0 \pmod{\xi^{\kappa}, \xi_{t\kappa}},$$

hence we have

(75) $$C_{pqxy}=0$$

and

$$C_{pxyq}=C_{pyxq}=C_{qxyp}.$$

If we put $\lambda=p$, $\mu=x$, $\nu=y$, $\omega=q$ in (6), we get

$$C_{pzyq}\xi_{zx}+C_{pxzq}\xi_{zy}\equiv 0 \pmod{\xi^{\kappa}, \xi_{t\kappa}},$$

hence

$$C_{pzyq}\delta_{xu}+C_{pxzq}\delta_{yu}-C_{puyq}\delta_{xz}-C_{pxuq}\delta_{yz}=0.$$

Then we immediately find

(76) $$C_{pxyq}=C_{pq}\delta_{xy},$$

where

(77) $$C_{pq}=C_{qp}.$$

If we put $\lambda=x$, $\mu=y$, $\nu=z$, $\omega=p$ in (6), we get

$$C_{uyzp}\xi_{ux}+C_{xuzp}\xi_{uy}+C_{xyup}\xi_{uz}\equiv 0 \pmod{\xi^{\kappa}, \xi_{t\kappa}},$$

hence

$$C_{uyzp}\delta_{xv}+C_{xuzp}\delta_{yv}+C_{xyup}\delta_{zv}$$
$$-C_{vyzp}\delta_{xu}-C_{xvzp}\delta_{yu}-C_{xyvp}\delta_{zu}=0.$$

Putting $x=v$ and summing up, we get

$$(n-6)C_{uyzp}+C_{zyup}=-C_{yp}\delta_{zu},$$

where we have put $C_{yp}=C_{xypx}$. Interchanging u and y and subtracting, we get

$$\{2(n-6)+1\}C_{uyzp}=-C_{yp}\delta_{zu}+C_{up}\delta_{zy}$$

because of $C_{[uyz]p}=0$. Putting $u=z$ and summing up, we get

$$-(2n-11)C_{yp}=-(n-5)C_{yp}.$$

As we have $n\geqq 7$, we find $C_{yp}=0$ and

(78) $$C_{xyzp}=0.$$

Now, consider the equations obtained by putting $\lambda=x$, $\mu=y$, $\nu=z$, $\omega=p$ in (6) once more. From (73), (75), (76), (77), (78), we get

$$(C_{pq}-C\delta_{pq})(\delta_{yz}\xi_{qx}-\delta_{xz}\xi_{qy})\equiv 0 \pmod{\xi^\kappa},$$

hence

$$(C_{pq}-C\delta_{pq})\xi_{qx}\equiv 0 \pmod{\xi^\kappa}.$$

If we take an orthogonal ennuple such that we have $C_{pq}=C_p\delta_{pq}$, then, as we have (72) and moreover $N<3n-11$, we get

$$C_{p1}=C\delta_{p1}. \tag{79}$$

If we put $\lambda=p$, $\mu=q$, $\nu=r$, $\omega=s$ in (6), we get

$$C_{tqrs}\xi_{tp}+C_{ptrs}\xi_{tq}+C_{pqts}\xi_{tr}+C_{pqrt}\xi_{ts}\equiv 0 \pmod{\xi^\kappa}. \tag{80}$$

But because of the relation $N<3n-11$ there can be no equation in (80), hence we must have

$$C_{pqrs}=C'(\delta_{ps}\delta_{qr}-\delta_{pr}\delta_{qs}). \tag{81}$$

Substituting (81) and (76) into $C_{p\alpha\alpha q}=0$, we get $3C'\delta_{pq}+(n-4)C_{pq}=0$, hence

$$C_{pq}=-\frac{3}{n-4}C'\delta_{pq}. \tag{82}$$

From (79) we get $C=-(3/(n-4))C'$, hence

$$C_{pq}=C\delta_{pq}. \tag{83}$$

Substituting (73), (76) and (83) into $C_{x\alpha\alpha y}=0$, we get $4C+(n-5)C=0$, hence

$$C=0,\quad C'=0,\quad C_{pq}=0.$$

Then, all components of the conformal curvature vanish because of (81), (74), (75), (76), (78) and (73), contrary to the assumption (8.4).

Thus we get the

LEMMA 4. *A Riemann space $V^n, n\geqq 7$, with conformal curvature tensor satisfying* (8.4) *does not admit a group of motions of order $r>n(n+1)/2-(3n-11)$.*

§ 7. Conclusion.

From the results obtained above we can get the

THEOREM. *A necessary condition that a Riemann space $V^n, n\geqq 6$, admit a group of motions G_r of order $r>n(n+1)/2-(3n-11)$ is that the Weyl conformal curvature tensor satisfy* (22) *where $M_{\lambda\mu\nu\omega}$ is given* *by* (19) *for* $n=6$ *and* $n=8$, *or that it satisfy* (70) *where* A_λ *and* B_λ *satisfy* (71).

§ 8. Proof of Lemma 1.

Let us take an orthogonal ennuple such that we have

$$u_\lambda=\delta_{\lambda 1}, \qquad v_\lambda=\delta_{\lambda 2}$$

for some vectors u_λ, v_λ satisfying (20). Then we get

(84) $$M_{\lambda 11\omega}=\delta_{\lambda 2}\delta_{\omega 2}, \quad M_{2112}=M_{1221}=1.$$

The vector v_λ satisfying (20) for the vector $u_\lambda=\delta_{\lambda 2}$ is obtained from

$$M_{\lambda 22\omega}=v_\lambda v_\omega.$$

We find $(v_1)^2=1$ and get $v_\lambda=\pm\delta_{\lambda 1}$ by virtue of $v_\alpha v_\alpha=1$. Hence we get

(85) $$M_{\lambda 22\omega}=\delta_{\lambda 1}\delta_{\omega 1}.$$

If we put

$$u_\lambda=(1-t^2)^{1/2}\delta_{\lambda 1}+t\delta_{\lambda 2},$$
$$v_\lambda=\delta_{\lambda 2}+v'_\lambda t+\cdots,$$

we get

$$M_{\lambda 12\omega}+M_{\lambda 21\omega}=\delta_{\lambda 2}v'_\omega+\delta_{\omega 2}v'_\lambda$$

from (20). Putting $\lambda=2$, we get

$$M_{212\omega}=v'_\omega,$$

for we have $v'_2=0$. Hence we find

$$v'_\omega=-\delta_{\omega 1}$$

and moreover

(86) $$M_{\lambda 12\omega}+M_{\lambda 21\omega}=-(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2}).$$

Besides, we get

$$v_\lambda=-\sin\theta\delta_{\lambda 1}+\cos\theta\delta_{\lambda 2}$$

for

$$u_\lambda=\cos\theta\delta_{\lambda 1}+\sin\theta\delta_{\lambda 2}.$$

If we take $u_\lambda=\delta_{\lambda 3}$ and consider the vector v_λ determined by

$$v_\lambda v_\omega=M_{\lambda 33\omega},$$

we find $v_1=0$ by putting $\lambda=\omega=1$, for we have (84). We find $v_2=0$, too, because of (85). Besides, we have $v_3=0$. Hence we can take an orthogonal ennuple such that we have

$$v_\lambda=\delta_{\lambda 4}$$

for

$$u_\lambda=\delta_{\lambda 3}\,.$$

Then we get

$$(87)\qquad \begin{cases} M_{\lambda 33\omega}=\delta_{\lambda 4}\delta_{\omega 4}\,, \\ M_{\lambda 44\omega}=\delta_{\lambda 3}\delta_{\omega 3}\,, \\ M_{\lambda 34\omega}+M_{\lambda 43\omega}=-(\delta_{\lambda 3}\delta_{\omega 4}+\delta_{\omega 3}\delta_{\lambda 4})\,. \end{cases}$$

The last formula is obtained by considering (20) for

$$u_\lambda=(1-t^2)^{1/2}\delta_{\lambda 3}+t\delta_{\lambda 4}\,.$$

We can proceed in this way and get pairs of axes (1.2), (3.4), (5.6),$\cdots$, so that we have

$$(88)\qquad \begin{cases} M_{\lambda 11\omega}=\delta_{\lambda 2}\delta_{\omega 2}\,, \\ M_{\lambda 22\omega}=\delta_{\lambda 1}\delta_{\omega 1}\,, \\ M_{\lambda 12\omega}+M_{\lambda 21\omega}=-(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2})\,, \\ \cdots\cdots\cdots \qquad \cdots\cdots\cdots \\ M_{\lambda\, n-1\, n-1\, \omega}=\delta_{\lambda n}\delta_{\omega n}\,, \\ M_{\lambda n n \omega}=\delta_{\lambda\, n-1}\delta_{\omega\, n-1}\,, \\ M_{\lambda\, n-1\, n\, \omega}+M_{\lambda\, n\, n-1\, \omega}=-(\delta_{\lambda\, n-1}\delta_{\omega n}+\delta_{\omega\, n-1}\delta_{\lambda n})\,, \end{cases}$$

if n is even. As an axis is left unpaired if n is odd, we find that n must be even. This is also evident from the topological point of view, for (20) shows that a vector field $v_\lambda(u_\kappa)$ exists on an $(n-1)$-dimensional sphere S^{n-1}.

If we put

$$u_\lambda=(\delta_{\lambda 1}+\delta_{\lambda 3})/\sqrt{2}\,,$$

we get

$$\frac{1}{2}(M_{\lambda 11\omega}+M_{\lambda 33\omega})+\frac{1}{2}(M_{\lambda 13\omega}+M_{\lambda 31\omega})=v_\lambda v_\omega$$

and find

$$v_1=v_3=v_5=v_6=\cdots=v_n=0$$

by virtue of (88). Moreover, we find $v_2=\pm 1/\sqrt{2}$, $v_4=\pm 1/\sqrt{2}$. But we can take the direction of the fourth axis, that is, the sign of v_λ for $u_\lambda=\delta_{\lambda 3}$ in such a way that we have

$$v_\lambda=(\delta_{\lambda 2}+\delta_{\lambda 4})/\sqrt{2}.$$

Then we get

(89) $$M_{\lambda 13\omega}+M_{\lambda 31\omega}=\delta_{\lambda 2}\delta_{\omega 4}+\delta_{\omega 2}\delta_{\lambda 4}$$

and

$$M_{2134}+M_{2314}=1.$$

Similarly, we get

(90) $$M_{\lambda 15\omega}+M_{\lambda 51\omega}=\delta_{\lambda 2}\delta_{\omega 6}+\delta_{\omega 2}\delta_{\lambda 6}.$$

But there is some ambiguity with respect to the sign of

(91) $$M_{\lambda 35\omega}+M_{\lambda 53\omega}=\pm(\delta_{\lambda 4}\delta_{\omega 6}+\delta_{\omega 4}\delta_{\lambda 6}).$$

This is determined when we consider the vector v_λ for

$$u_\lambda=(\delta_{\lambda 1}+\delta_{\lambda 3}+\delta_{\lambda 5})/\sqrt{3}.$$

We get

$$\frac{1}{3}[M_{\lambda 11\omega}+M_{\lambda 33\omega}+M_{\lambda 55\omega}+M_{\lambda 13\omega}+M_{\lambda 31\omega}$$
$$+M_{\lambda 15\omega}+M_{\lambda 51\omega}+M_{\lambda 35\omega}+M_{\lambda 53\omega}]=v_\lambda v_\omega$$

and find $(v_2)^2=(v_4)^2=(v_6)^2=1/3$. Moreover, we find

$$v_2v_4=1/3,$$
$$v_2v_6=1/3,$$
$$v_4v_6=\pm 1/3$$

by virtue of (89), (90), (91). Hence we must have

$$v_4v_6=+1/3$$

and get

(92) $$M_{\lambda 35\omega}+M_{\lambda 53\omega}=+(\delta_{\lambda 4}\delta_{\omega 6}+\delta_{\omega 4}\delta_{\lambda 6}).$$

In this way we can determine all $M_{\lambda\mu\nu\omega}+M_{\lambda\nu\mu\omega}$ for odd μ, ν.

But we can exchange in the suffices odd and even numbers in each pair if we neglect the signs, getting for example

$$M_{\lambda 24\omega}+M_{\lambda 42\omega}=\pm(\delta_{\lambda 1}\delta_{\omega 3}+\delta_{\omega 1}\delta_{\lambda 3}).$$

The sign is found to be positive if we put $\lambda=1$, $\omega=3$, for we have

$$M_{1243}+M_{1423}=M_{2134}+M_{2314}=+1$$

from (89). Similarly, we get

$$M_{\lambda 14\omega}+M_{\lambda 41\omega}=\pm(\delta_{\lambda 2}\delta_{\omega 3}+\delta_{\omega 2}\delta_{\lambda 3}),$$

where the sign is found to be negative if we put $\lambda=2$, $\omega=3$, for we have

$$\begin{aligned} M_{2143}+M_{2413} &= -M_{2134}-M_{2431} \\ &= -M_{2134}+M_{2341}=-M_{2134}-M_{2314}=-1 \end{aligned}$$

because of (88) and (89).

In this way all $M_{\lambda\mu\nu\omega}+M_{\lambda\nu\mu\omega}$ are determined:

$$(93)\quad \begin{cases} M_{\lambda 11\omega}=\delta_{\lambda 2}\delta_{\omega 2}, \quad M_{\lambda 22\omega}=\delta_{\lambda 1}\delta_{\omega 1}, \cdots, \\ 2M_{\lambda(12)\omega}=-(\delta_{\lambda 1}\delta_{\omega 2}+\delta_{\omega 1}\delta_{\lambda 2}), \cdots, \\ 2M_{\lambda(13)\omega}=\delta_{\lambda 2}\delta_{\omega 4}+\delta_{\omega 2}\delta_{\lambda 4}, \cdots, \\ 2M_{\lambda(24)\omega}=\delta_{\lambda 1}\delta_{\omega 3}+\delta_{\omega 1}\delta_{\lambda 3}, \cdots, \\ 2M_{\lambda(14)\omega}=-(\delta_{\lambda 2}\delta_{\omega 3}+\delta_{\omega 2}\delta_{\lambda 3}), \cdots, \\ 2M_{\lambda(23)\omega}=-(\delta_{\lambda 1}\delta_{\omega 4}+\delta_{\omega 1}\delta_{\lambda 4}), \cdots. \end{cases}$$

Then, as we have $M_{\lambda\mu(\nu\omega)}=0$, $M_{\lambda[\mu\nu\omega]}=0$, all components $M_{\lambda\mu\nu\omega}$ are also determined uniquely.

As the $M_{\lambda\mu\nu\omega}$ given in (22) satisfies (93), Lemma 1 is thus proved.

Institute of Mathematics
Yokohama National University

References

[1] E. Cartan, Leçons sur la géométrie des espaces de Riemann, 2d ed., Paris, Gauthier-Villars, 1946.

[2] I. P. Egorov, On a strengthening of Fubini's theorem on the order of the group of motions of a Riemannian space, Doklady Akad. Nauk SSSR. N. S., 66 (1947), pp. 793-796.

[3] L. P. Eisenhart, Continuous groups of transformations, Princeton University Press, 1933.

[4] M. Kurita, On the isometry of a homogeneous Riemann space, Tensor, N. S., 3 (1954), pp. 91-100.

[5] D. Montgomery and H. Samelson, Transformation groups of spheres, Ann. of Math. (2), 44 (1943), pp. 454-470.

[6] H. Wakakuwa, On n-dimensional Riemannian spaces admitting some groups of motions of order less than $n(n-1)/2$, Tôhoku Math. J. Sec. Series, 6 (1954), pp. 121-134.

[7] H. C. Wang, On Finsler spaces with completely integrable equations of Killing, J. London Math. Soc., 22 (1947), pp. 5-9.

[8] K. Yano, On n-dimensional Riemannian spaces admitting a group of motions of order $n(n-1)/2+1$, Trans. Amer. Math. Soc., 74 (1953), pp. 260-279.

[9] K. Yano, Groups of transformations in generalized spaces, Tokyo, 1949.

[10] G. Vranceanu, Groupes de mouvement des espaces à conexion, Acad. Repub. Pop. Române, Studii şi Cercetări Matematice, 2 (1951), pp. 387-444.