Journal of the Mathematical Society of Japan Vol. 8, No. 3, July, 1956

Analogue of a theorem of F. and M. Riesz for minimal surfaces.

By Maxwell O. READE[1]

(Received Dec. 10, 1955)

In a fundamental paper published some years ago, Beckenbach and Radó showed how certain function-theoretic methods and results could be extended to obtain analogous results for functions of class *PL*; they also showed how the new results could be utilized to obtain additional theorems relating to minimal surfaces [1].

In that same spirit, we use a function-theoretic technique to obtain an analogue for functions of class *PL*, of the well-known theorem of F. and M. Riesz [2; p. 46]; then we obtain the corresponding result for minimal surfaces. It should be noted that our first theorem could also be obtained from the deep results due to Littlewood [4] and Deny and Lelong [3].

Let $p(z)\equiv p(x,y)$ be a real-valued function defined for z in the unit disc $\mathfrak{D}$: $|z|<1$. Then $p(z)$ is said to be of class *PL* if and only if the following conditions hold [1; p. 651]: (i) $p(z)$ is continuous, (ii) $p(z)\geqq 0$, and (iii) $\log p(z)$ is subharmonic in that part of $\mathfrak{D}$ where $p(z)>0$.

For functions of class *PL* we have the following result, which generalizes a theorem due to Beckenbach and Radó [1; p. 652].

THEOREM 1. *Let $p(z)$ be of class PL and bounded in $\mathfrak{D}$, and let $E\equiv[\theta \mid \lim_{r\to 1} p(re^{i\theta})=0]$, $z=re^{i\theta}$. If the (linear) measure mE of E is positive, then $p(z)\equiv 0$.*

PROOF. First, E is measurable. Moreover, if $mE=2\pi$, then it follows from Lebesgue's theorem that

$$\lim_{r\to 1}\int_0^{2\pi} p(re^{i\theta})\,d\theta=\int_0^{2\pi}\lim_{r\to 1} p(re^{i\theta})\,d\theta=0\,. \tag{1}$$

1) The research reported here was supported (in part) by a grant from the National Science Foundation (USA).

But the integral mean on the left hand side of (1) is a non-decreasing function of the radius r. Hence the non-negative function $p(z)$ vanishes identically in $\mathfrak{D}$.

Now suppose $0 < mE < 2\pi$. We define the *bounded* harmonic function $u(z) \equiv u(x, y)$, by means of the Poisson integral for $\mathfrak{D}$, with boundary values A/mE on E, and $A/(mE - 2\pi)$ on the complement E' of E; here A is an arbitrary positive constant. We see that $u(0) = 0$ and that $\lim_{r \to 1} u(re^{i\theta})$ exists and equals the assigned boundary values for almost all θ, $0 \leqq \theta \leqq 2\pi$. Now the function $g(z) \equiv e^{u(z)}$ is bounded and of class PL in $\mathfrak{D}$; moreover $g(0) = 1$, $\lim_{r \to 1} g(re^{i\theta}) = e^{A/mE}$ almost everywhere on E, and $\lim_{r \to 1} g(re^{i\theta}) = e^{-A/(2\pi - mE)}$ almost everywhere on E'. The function $p(z)g(z)$ is again bounded and of class PL in $\mathfrak{D}$, since the class of PL functions is closed under multiplication. Since a function of class PL is also subharmonic, we obtain the following set of inequalities;

$$
\begin{aligned}
0 \leqq p(0) = p(0)\, g(0) &\leqq \frac{1}{2\pi} \int_0^{2\pi} p(re^{i\theta})\, g(re^{i\theta})\, d\theta \\
&\leqq \frac{1}{2\pi} \lim_{r \to 1} \int_0^{2\pi} p(re^{i\theta})\, g(re^{i\theta})\, d\theta \\
&\leqq \frac{1}{2\pi} \int_0^{2\pi} \lim_{r \to 1} [p(re^{i\theta})\, g(re^{i\theta})]\, d\theta \\
&\leqq \frac{M}{2\pi}\, mE'\, e^{-\frac{A}{2\pi - mE}},
\end{aligned}
$$

where M is an upper bound for $p(z)$ in $\mathfrak{D}$. Since A was an arbitrary positive constant, it follows that $p(0) = 0$.

Now let z_0 be an arbitrary point in $\mathfrak{D}$. Then there is a linear fractional transformation $z = L(\zeta)$ of $\mathfrak{D}$ onto itself such that $L(0) = z_0$, and such that $p(z)$ is transformed into another function $P(\zeta)$ which is again bounded and of class PL in $\mathfrak{D}$. Moreover, since we have a theorem of Lindelöf type available for bounded functions of class PL [1; p. 652], it follows that $\lim_{\rho \to 1} P(\rho e^{i\varphi}) = 0$, $\zeta = \rho e^{i\varphi}$, for all φ belonging to the image of E under the transformation $z = L(\zeta)$. Hence, as before, we conclude $0=P(0)=p(z_0)$. Now it follows that $p(z)\equiv 0$ in $\mathfrak{D}$.

A simple application of the preceding result is the following;

THEOREM 2. *Let $X_j(z)\equiv X_j(x, y)$, $j=1, 2, 3$, be the coordinate functions of a bounded minimal surface S given in isothermic representation for $z=x+iy$ in the unit disc D, and let* $E_a\equiv\left[\theta \,|\, \lim_{r\to 1}\sum_{j=1}^{3}\{X_j(z)-a_j\}^2=0\right]$, *where a_1, a_2, a_3 are constants. If the (linear) measure mE_a is positive for a fixed triple a_1, a_2, a_3, then $X_j(z)\equiv a_j$, $j=1, 2, 3$ holds in D.*

PROOF. Let a_1, a_2, a_3 denote a fixed triple of constants. Then it is known that under our present hypothesis, the function $p(z)\equiv \sum_{j=1}^{3}[X_j(z)-a_j]^2$ is a (bounded) function of class PL in D [1; p. 654]. The present theorem now follows from the first theorem.

This last result is an analogue of a corresponding one for plane conformal maps. In the plane case, we also have the same sort of conclusion for schlicht maps, and it would be of interest to obtain its generalization to schlicht maps on minimal surfaces.

The Universiy of Michigan
Ann Arbor, Michigan, U. S. A.

Bibliography

[1] E. F. Beckenbach and T. Radó, Subharmonic Functions and Minimal Surfaces, Trans. A. M. S., 35 (1933), 658-661.

[2] C. Carathéodory, Funktionentheorie, v. II, Basel, 1950.

[3] J. Deny and P. Lelong, Etude des Fonctions Sousharmoniques dans un Cylindre ou dans une Cône, Bull. Soc. Math. de France, 75 (1947), 89-112.

[4] J. E. Littlewood, On Functions Subharmonic in a Circle, part II, Proc. L. M. S., 28 (1928), 383-393.