# Proof of a special case of the fundamental conjecture of Takeuti's GLC.

#### By Takakazu Shimauti

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G. Takeuti [2] has generalized G. Gentzen's logic calculus *LK* (cf. [1]) to his *generalized logic calculus GLC*, and enounced the *fundamental conjecture of GLC*: Every provable sequence in *GLC* will be provable without cut. In [2] it is also shown that from this conjecture would follow the consistency of the analysis.

Some special cases of this conjecture have been proved by Takeuti [3], [4], [5], [6]. Another special case will be proved in this paper. After preparations in § 1, we shall formulate our theorem in § 2, and prove it by *reductions* indicated in § 3.

### § 1. Proof-figures in GLC.

We begin with listing the inference-schemata of GLC in a form slightly modified from those given in [2]. The equivalence of this system with that of [2] can be easily verified. For the meaning of terms such as "homologous", "t-variety", "f-variable" etc., we refer to [2], [3].

- 1.1. Inference-schemata
- 1.1.1. Inferences on structure
- "Version"

left: 
$$A, \Gamma \to \Delta$$
  $\Gamma \to A, \Lambda$   $\Gamma \to A, \widetilde{A}$ 

where  $\widetilde{A}$  is a formula homologous to A. "Weakening"

left: 
$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}$$
 right:  $\frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}$ 

"Contraction"

left: 
$$A, A, \Gamma \rightarrow \Delta$$
  
 $A, \Gamma \rightarrow \Delta$  right:  $\Gamma \rightarrow \Delta, A, A$   
 $\Gamma \rightarrow \Delta, A$ 

" Exchange "

left: 
$$\frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta}$$
 right:  $\frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda}$ 

1.1.2. Cut

$$\Gamma \rightarrow \Delta$$
,  $A$   $A$ ,  $\Pi \rightarrow \Lambda$   
 $\Gamma$ ,  $\Pi \rightarrow \Delta$ ,  $\Lambda$ 

1.1.3. Inference on logical symbol

left: 
$$\frac{\Gamma \to \Delta, A}{7A, \Gamma \to \Delta}$$
 right:  $\frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, 7A}$ 

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left: 
$$A, B, \Gamma \rightarrow \Delta$$
  
 $A \land B, \Gamma \rightarrow \Delta$  right:  $\Gamma \rightarrow \Delta, A \qquad \Pi \rightarrow \Lambda, B$   
 $\Gamma, \Pi \rightarrow \Delta, \Lambda, A \land B$ 

" $\forall$  on t-variable"

left: 
$$\frac{F(T), \Gamma \to \Delta}{\forall x F(x), \Gamma \to \Delta}$$
 right: 
$$\frac{\Gamma \to \Delta, F(a)}{\Gamma \to \Delta, \forall x F(x)}$$

where T is an arbitrary t-variety of the same type as x.

"  $\forall$  on f-variable"

left: 
$$F(H), \Gamma \rightarrow \Delta$$
  
 $\forall \varphi F(\varphi), \Gamma \rightarrow \Delta$ 

where H is an arbitrary f-variety of the same type as  $\varphi$ .

where a is a free t-variable of the same type as x, not contained in the lower sequence. (a is called the *eigenvariable* of this inference.)

right: 
$$\Gamma \to \Delta$$
,  $F(\alpha)$   
 $\Gamma \to \Delta$ ,  $\forall \varphi F(\varphi)$ 

where  $\alpha$  is a free f-variable of the same type as  $\varphi$ , not contained in the lower sequence. ( $\alpha$  is called the eigenvariable of this inference.)

- 1.2. In the above schemata, the formulas denoted by A, B, F(T), F(a), F(H) or  $F(\alpha)$  in the upper sequence are called the *subformulas* of the inference, and the formulas denoted by  $A, \widetilde{A}, B, \nearrow A, A \land B, \forall x F(x)$ , or  $\forall \varphi F(\varphi)$  in the lower sequence are called the *chief formulas* of the inference. A subformula of a cut is called a *cut-formula*, and a chief formula of a weakening is called a *weakening formula*.
- 1.3. When a formula C is contained in the upper sequence of an inference which is represented by one of the above inference-schemata, the successor of C is defined as follows: if C is a cut-formula then there is no successor of C; if C is a subformula of an inference other than cut and exchange, then the successor of C is the chief formula of the inference; if C is a subformula denoted by A (resp. B) in the schemata of exchange, then the successor of C is a chief formula denoted by A (resp. B); if C is the k-th formula of C, C, C, C in the upper sequence, then the successor of C is the C-th formula of C, C-th formula of C-th formula is a descendant of the formula; the successor of a descendant of a formula is a descendant of the formula.
- 1.4. A formula in a proof-figure is called *implicit* or *explicit* according as the formula has or has not a descendant, which is a cut-formula of a cut. An inference is called implicit or explicit according as the chief formula of the inference is implicit or explicit.
- 1.5. A sequence in a proof-figure is called *contained in the end-place* of the proof-figure, if and only if there is no implicit logical inference under the sequence. An inference in a proof-figure is called contained in the end-place of the proof-figure, if and only if the lower sequence of the inference is contained in the end-place. An inference in a proof-figure is called to *belong to the boundary* of the end-place, if and only if the lower sequence is contained in the end-place and the upper sequence is not contained in the end-place.
- 1.6. A cut in the end-place is called *suitable*, if and only if each cut-formula of the cut is a descendant of the chief formula of an inference, which belongs to the boundary of the end-place.

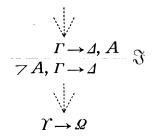
T. Simauti

## $\S$ 2. The formulation of the theorem and the plan of its proof.

- 2.1. THEOREM If a proof-figure  $\mathfrak{P}$  has no implicit contraction, then the end-sequence of  $\mathfrak{P}$  is provable without cut.
- 2.2. In the following, we shall prove this theorem by the mathematical induction on the *grade*, which is the sum of the numbers of cuts and logical inferences contained in the proof-figure.
- 2.3. Let  $\mathfrak{P}$  and  $\mathfrak{Q}$  (resp.  $\mathfrak{Q}_1$  and  $\mathfrak{Q}_2$ ) be proof-figures. We say that  $\mathfrak{P}$  is *reduced* to  $\mathfrak{Q}$  (resp.  $\mathfrak{Q}_1$  and  $\mathfrak{Q}_2$ ), if the following conditions (1), (2), (3) are satisfied: (1)  $\mathfrak{P}$  and  $\mathfrak{Q}$  (resp.  $\mathfrak{Q}_1$  and  $\mathfrak{Q}_2$ ) have no implicit contractions; (2) if the end-sequence of  $\mathfrak{Q}$  (resp.  $\mathfrak{Q}_1$  and  $\mathfrak{Q}_2$ ) is provable without cut then the end-sequence of  $\mathfrak{P}$  is provable without cut; (3) the grade of  $\mathfrak{Q}$  (resp. of  $\mathfrak{Q}_1$  and of  $\mathfrak{Q}_2$ ) is smaller than the grade of  $\mathfrak{P}$ .
- 2.4. Let  $\mathfrak P$  be a proof-figure without implicit contraction with the grade not equal to zero. Our theorem will be proved, if we find a definite way of reduction for any such  $\mathfrak P$ . We may assume thereby, by a wellknown method of changing the free variables, that for every inference on  $\forall$  right its eigenvariable is contained in  $\mathfrak P$  only in sequences above the inference.

### § 3. Reductions

- 3.1. The case, where the end-place of  $\mathfrak{P}$  has an explicit logical inference. Let  $\mathfrak{F}$  be the undermost logical inference contained in the end-place of  $\mathfrak{P}$ .
- 3.1.1. If  $\Im$  is an inference on  $\nearrow$  left, we can assume that  $\Re$  is of the form:



Since there is no logical inference under  $\Im$ ,  $\Upsilon$  contains  $\nearrow A$ . Hence  $\Re$  is reducible to the following proof-figure:

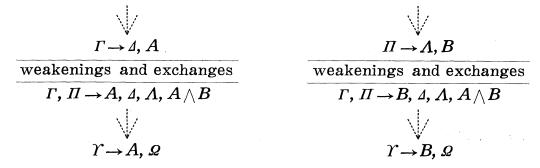
If  $\Im$  is an inference on  $\nearrow$  right, on  $\land$  left or on  $\forall$ , the reduction is similar to the above.

3.1.2. If  $\Im$  is an inference on  $\wedge$  right, and  $\Re$  is of the form

$$\frac{\Gamma \to A, A}{\Gamma, \Pi \to A, \Lambda, A \land B} \Im$$

$$\frac{\Gamma \to A, A}{\Gamma, \Pi \to A, \Lambda, A \land B} \Im$$

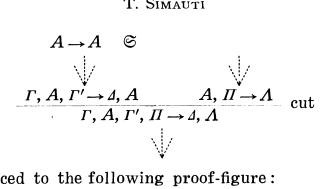
We reduce \$\Pi\$ to the following proof-figures:



3.2. The case where the end-place contains a beginning sequence S and no explicit logical inference. We can assume here that one of the two beginning formulas of S is implicit. In fact, if both of them are explicit, the end-sequence of \$\P\$ is obtained simply by some versions, weakenings and exchanges. Let \$\mathbb{P}\$ be of the form:

140

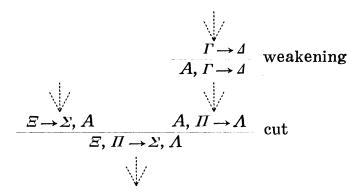
T. SIMAUTI



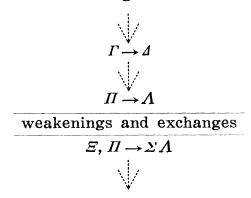
B is then reduced to the following proof-figure:

$$A, \Pi \rightarrow \Lambda$$
weakenings and exchanges
 $\Gamma, A, \Gamma', \Pi \rightarrow A, \Lambda$ 

The case, where the end-place contains an implicit weakening. Let \mathbb{P} be of the form:



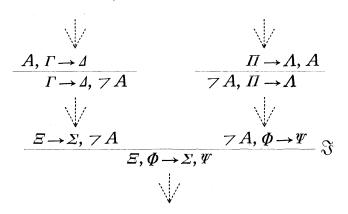
\$\Pi\$ is then reduced to the following:



3.4. The case, where the end-place contains no beginning sequence, no explicit logical inference and no implicit weakening. We can prove by the induction on the number of inferences contained in the end-place that there is a suitable cut. Let  $\Im$  be the lowest cut in the end-place of  $\Re$ , and  $\Re$  be of the form:

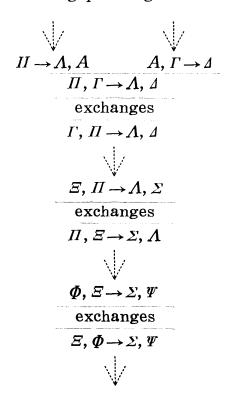
We see easily that the end-place of  $\mathfrak{P}_i$  has a sequence not contained in the end-place of  $\mathfrak{P}$ , if and only if there is an inference, in the boundary of the end-place of  $\mathfrak{P}$ , and a cut-formula of  $\mathfrak{F}$  is a descendant of the chief-formula of this inference. Therefore, if  $\mathfrak{F}$  is not suitable, then the end-place  $\mathfrak{E}$  of  $\mathfrak{P}_i$  or  $\mathfrak{P}_2$  is a subset of the end-place of  $\mathfrak{P}$ . Then, by the hypothesis of the induction, there exists a suitable cut  $\mathfrak{R}$  in  $\mathfrak{E}$ . Clearly  $\mathfrak{R}$  is a suitable cut of  $\mathfrak{P}$ .

Let  $\Im$  be a suitable cut in the end-place of  $\Re$ . 3.4.1. The case, where the outermost logical symbol of the cutformulas of  $\Im$  is  $\nearrow$ . Let  $\Re$  be of the form:



T. Simauti

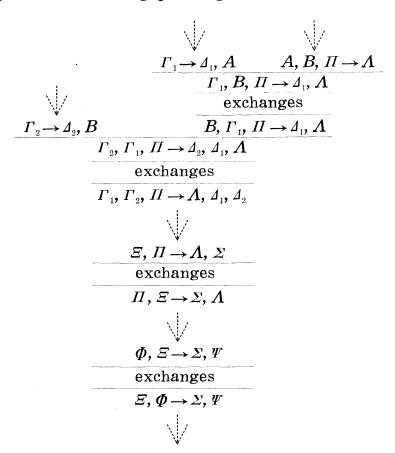
We reduce \$\Pi\$ to the following proof-figure:



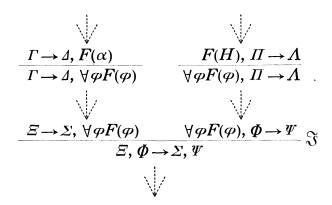
3.4.2. The case, where the outermost logical symbol of the cut-formulas of  $\Im$  is  $\wedge$ . Let  $\Re$  be of the form:

Proof of a special case of the fundamental conjecture of Takeuti's GLC. 143

We reduce \$\Pi\$ to the following proof-figure:

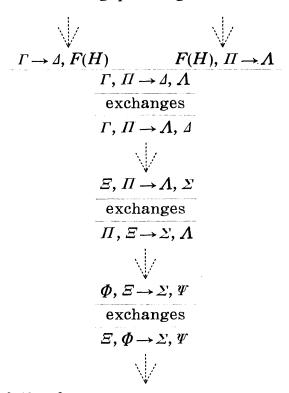


3.4.3. The case, where the outermost logical symbol of the cut-formulas of  $\Im$  is  $\forall$ . Let  $\Re$  be of the form:



T. Simauti

We reduce \$\mathbb{B}\$ to the following proof-figure:



where the part of the form:

$$\bigvee_{\Gamma \to \Delta, F(H)}$$

is obtained from the corresponding part of  $\mathfrak P$  by substituting H for  $\alpha$ .

### References

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