Journal of the Mathematical Society of Japan Vol. 8, No. 2, April, 1956

On the Poisson distribution.

By Hisao WATANABE

(Receivd Nov. 25, 1955)

Let $\cdots, x_{-1}, x_0, x_1, \cdots$ be the points on the real line such that $\cdots x_{-1} < x_0 < x_1 \cdots (x_0 \equiv 0)$. Then if $\{x_j - x_{j-1}\}$ $(j=0, 1, 2, \cdots)$ are independent random variables with common distribution function $F(x)$, where $F(x)$ is the distribution function of a non-negative random variable with $F(-0)=0$, $F(\infty)=1$, we shall say that these points are distributed at random according to $F(x)$.

Now consider a system of particles $P_n(n=0, \pm 1, \pm 2, \cdots)$ which start from the above stated random positions $x_n(n=0, \pm 1, \pm 2, \cdots)$. When we denote by $X_n(t)$ the displacement of the n-th particle P_n up to the time t, the coordinate $Y_n(t)$ of the particle at the time t is

$$Y_n(t) = x_n + X_n(t), \qquad X_n(0) = 0, \qquad t \geqq 0.$$

In the following, let us confine ourselves to the discrete time parameter $t=0, 1, 2, \cdots$, and we shall impose the following conditions on the movement of the particles. The random variables $X_n(t) - X_n(t-1)$ are mutually independent for each n, t, $-\infty < n < \infty$, $t \geqq 0$, and obey the same distribution function $G(x)$ for all n, t, moreover, for each $t > 0$ the classes of random variables

$$\{X_n(t), n=0, \pm 1, \pm 2, \cdots\} \qquad \{x_n, n=0, \pm 1, \pm 2, \cdots\}$$

are mutually independent.

By the Fourier analytical method [2], Prof. Maruyama [3] investigated the limiting distribution of the number $N_I(t)$ of particles lying in an interval $I=[a, b]$ at t under the condition that $G(x)$ is a non-lattice distribution function. In this note, we shall discuss the problem when $G(x)$ is a lattice distribution function with maximum span $d > 0$.

THEOREM. *If* $0 < m = \int_{-\infty}^{+\infty} x dF(x) < \infty$, *then we have*

$$\lim_{t\to\infty} Pr\{N_I(t)=k\}=e^{-\lambda}\frac{\lambda^k}{k!} \qquad k=0,1,2,\cdots,$$

with

$$\lambda=\begin{cases} \dfrac{(b-a)}{m} & \text{if } F(x) \text{ is non-lattice and if } F(x) \text{ is lattice with maximum span } d' \text{ and } d/d' \text{ is irrational,} \\ \dfrac{1}{mp}\displaystyle\sum_{r=-\infty}^{\infty} I\left(\frac{r}{p}\right) & \text{if } F(x) \text{ is lattice with maximum span } d' \text{ and } d/d'=q/p, \end{cases}$$

where

$$I(x)=\begin{cases} 1 & x\in[a,b] \\ 0 & x\in[a,b]. \end{cases} \tag{1}$$

PROOF. Our discussion proceeds analogously to [3]. By (1), we can write

$$N_I(t)=\sum_{n=-\infty}^{\infty} I(Y_n(t)). \tag{2}$$

As in [2], we introduce a non-negative smooth function $H(x)$ satisfying the following conditions (i), (ii).

(i) Fourier transformations

$$\int_{-\infty}^{\infty} H(x)\, e^{-itx}dx=h(t),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)e^{itx}dx=H(x)$$

are absolutely convergent and the equalities hold for all t and x.

(ii) $h(t)$ vanishes outside a finite interval $[-c, c]$.

Let us put

$$N_H(t)=\sum_{n=-\infty}^{\infty} H(Y_n(t)).$$

We shall begin with proving the existence of $E\{N_I(t)\}$ for all $t\geq 0$ and $\lim_{t\to\infty} E\{N_I(t)\}$. If we denote the characteristic functions of $F(x)$ and $G(x)$ by $\varphi(u)$ and $\psi(u)$ respectively, we have

$$E\{H(Y_n(t))\}=E\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(x_n+X_n(t))}h(u)du\right\}$$

$$(3) \qquad = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \varphi^n(u)\psi^t(u)h(u)du & n \geqq 0 \\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \varphi^n(-u)\psi^t(u)h(u)du & n \leqq 0 . \end{cases}$$

If we write for $0<\rho<1$,

$$N_{H,\rho}(t) = \sum_{n=-\infty}^{\infty} \rho^{|n|} H(Y_n(t)) ,$$

then

$$\lim_{\rho\to 1} N_{H,\rho}(t) = N_H(t) ,$$

and in view of (3) we get

$$E\{N_{H,\rho}(t)\} = I(\rho, t) + J(\rho, t) + K(t) ,$$

where

$$I(\rho, t) = \frac{1}{\pi}\int_{-e}^{c} \frac{1-\rho}{Q(\rho, u)} h(u)\psi^t(u)du ,$$

$$J(\rho, t) = \frac{\rho}{\pi}\int_{-c}^{e} \frac{1-a(u)}{Q(\rho, u)} h(u)\psi^t(u)du ,$$

$$K(t) = -\frac{1}{2\pi}\int_{-e}^{c} h(u)\psi^t(u)du ,$$

$$a(u) = \int_0^{\infty} \cos ux \, dF(x) \quad \text{and} \quad Q(\rho, u) = |1-\rho\varphi(u)|^2 .$$

First suppose that $F(x)$ is a non-lattice distribution function. Then, by the same analysis as in [3] p. 3, we have

$$\lim_{\rho\to 1-0} I(\rho, t) = \frac{h(0)}{m}$$

and also

$$\lim_{\rho\to 1-0} J(\rho, t) = \frac{1}{\pi}\int_{-c}^{c} \frac{1-a(u)}{Q(1, u)} h(u)\psi^t(u)du .$$

Since $|\psi(u)|<1$ except for $u=2\nu\pi/d$ $(\nu=0, \pm 1, \pm 2,\cdots)$, we have by the convergence theorem of Lebesgue

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} J(\rho, t) = 0 , \quad \lim_{t\to\infty} K(t) = 0 .$$

Combining these we obtain

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} E\{N_{\rho,H}(t)\}=\lim_{t\to\infty} E\{N_H(t)\}=\frac{h(0)}{m}=\frac{1}{m}\int_{-\infty}^{\infty} H(x)dx\,.$$

Next suppose that $F(x)$ is a lattice distribution with maximum span d', in which case the situation is different. We may suppose without loss of generality that $d'\equiv 1$ by changing the scale. Then, $I(\rho, t)$ can be written as

$$I(\rho, t)=\frac{1}{\pi}\sum_{\nu=-\infty}^{\infty}\int_{(2\nu-1)\pi}^{(2\nu+1)\pi} h(u)\psi^t(u)\frac{1-\rho}{Q(\rho, u)}du$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{\nu=-\infty}^{\infty}{}^{1)} h(u+2\nu\pi)\psi^t(u+2\nu\pi)\frac{1-\rho}{Q(\rho, u)}du\,.$$

Since $S(u)=\sum_{\nu=-\infty}^{\infty} h(u+2\nu\pi)\psi^t(u+2\pi)$ is continuous with respect to u, the analysis in [3] p. 5 gives

$$\lim_{\rho\to 1-0} I(\rho, t)=\frac{S(0)}{m}=\frac{1}{m}\sum_{\nu=-\infty}^{\infty} h(2\nu\pi)\psi^t(2\nu\pi)\,.$$

Firstly, let d be irrational. Then, since j/d cannot be an integer for any j $(j=\pm 1, \pm 2,\cdots)$, we have $|\psi(2\nu\pi)|<1$ for any $\nu(\nu=\pm 1, \pm 2,\cdots)$ and

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} I(\rho, t)=\lim_{t\to\infty}\frac{1}{m}\sum_{\nu=-\infty}^{\infty} h(2\nu\pi)\psi^t(2\nu\pi)=\frac{h(0)}{m}\,.$$

On the other hand, when d is rational, we can write $d=q/p$ with relatively prime positive integers p, q. But j/d can be an integer when and only when $j=kq$ $(k=0, \pm 1, \pm 2,\cdots)$, and $\psi(u)$ has $2\pi/d$ as a period. Hence

$$\lim_{t\to\infty}\lim_{\rho\to 1-0} I(\rho, t)=\lim_{t\to\infty}\frac{1}{m}\sum_{\nu=-\infty}^{\infty} h(2\nu\pi)\psi^t(2\nu\pi)$$

$$=\frac{1}{m}\sum_{k=-\infty}^{\infty} h\left(2kq\frac{\pi}{d}\right)=\frac{1}{m}\sum_{k=-\infty}^{\infty} h(2\pi kp)\,.$$

Poisson's summation formula [1]

---

1) Since $h(u)$ vanishes outside $(-c, c)$, Σ is essentially a summation over a finite number of ν.

$$\sum_{k=-\infty}^{\infty} h(2\pi kp) = \sum_{r=-\infty}^{\infty} \frac{r}{p} H\left(\frac{r}{p}\right)$$

applied to the last term gives us

$$\lim_{t\to\infty} \lim_{\rho\to 1-0} I(\rho, t) = \frac{1}{m} \sum_{r=-\infty}^{\infty} \frac{1}{p} H\left(\frac{r}{p}\right).$$

Next we observe that

$$\lim_{\rho\to 1-0} J(\rho, t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{r=\infty}^{\infty} h(u+2\nu\pi)\psi^t(u+2\nu\pi) \frac{1-a(u)}{Q(1, u)} du.$$

Then, since $\lim_{t\to\infty} \sum_{\nu=-\infty}^{\infty} h(u+2\nu\pi)\psi^t(u+2\nu\pi) = 0$ except at most for a finite number of u in $(-\pi, \pi)$ and $1-a(u)/Q(1, u)$ is integrable in that interval, we get

$$\lim_{t\to\infty} \lim_{\rho\to 1-0} J(\rho, t) = 0 ,$$

and also

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} \frac{-1}{\pi} \int_{-c}^{c} h(u)\psi^t(u)du = 0 .$$

Thus, we have established:

(A) If $F(x)$ is a non lattice distribution,

$$\lim_{t\to\infty} E\{N_H(t)\} = \frac{1}{m} \int_{-\infty}^{\infty} H(x)dx .$$

(B) If $F(x)$ is a lattice distribution,

$$\lim_{t\to\infty} E\{N_H(t)\} = \begin{cases} \dfrac{1}{m} \displaystyle\int_{-\infty}^{\infty} H(x)dx & \text{if } d \text{ is irrational} \\[2ex] \dfrac{1}{mp} \displaystyle\sum_{r=-\infty}^{\infty} H\left(\frac{r}{p}\right) & \text{if } d = q/p . \end{cases}$$

In the case of **(A)** and **(B)** with irrational d, we obtain by the argument in [3]

$$\lim_{t\to\infty} E\{\exp(izN_I(t))\} = \exp\left\{\frac{(b-a)}{m}(e^{iz}-1)\right\},$$

and therefore

$$\lim_{t\to\infty} Pr\{N_I(t)=k\}=e^{-\lambda}\frac{\lambda^k}{k!} \qquad (k=0, 1, 2,\cdots)$$

$$\lambda=\frac{(b-a)}{m}.$$

In the remaining case of (**B**), we proceed as in [2], with modifications of the approximation procedures used there. We define a continuous non-negative function $H_0(x)$ which vanishes outside a finite interval $(-k, k)$ and satisfies

$$0<H_0(x)-I(x) \qquad \text{for} \qquad x\in I$$

and for sufficiently small $\eta>0$

$$\sum_{r=-\infty}^{\infty}\left(H_0\left(\frac{r}{p}\right)-I\left(\frac{r}{p}\right)\right)<\eta.$$

Let us write

$$H_0^*(x)=\int_{-\infty}^{\infty} H_0(x-y)K_\lambda(y)dy, \qquad K_\lambda=\frac{\sin^2\lambda x/2}{\pi\lambda x^2/2}.$$

Next take an interval $J\supset I$ and define

$$\begin{aligned} H_1(x)&=0 && \text{if} \quad x\in J \\ &=H_0^*(x)-I(x) && \text{if} \quad x\notin J, \end{aligned} \tag{4}$$

$$H_0^*(x)-I(x)=H_1(x)+H_2(x), \quad -\infty<x<\infty.$$

Then for sufficiently large $\lambda>0$ and with a constant c it holds that

$$H_2(x)\leqq\frac{c}{\lambda}\int_{-\infty}^{\infty}\frac{1}{1+y^2}K_{\lambda'}(x-y)\,dy=\overline{H}_2(x).$$

Further, we define a continuous function $H_3(x)$ such that it vanishes outside a finite interval and

$$0<H_3(x)-H_1(x) \qquad \text{for} \quad x\in J,$$

$$\sum_{-\infty}^{\infty}\left(H_3\left(\frac{r}{p}\right)-H_1\left(\frac{r}{p}\right)\right)<\eta \qquad \text{for a sufficiently small } \eta>0,$$

and let

$$H_3^*(x)=\int_{-\infty}^{\infty} H_3(x-y)\,K_\lambda(y)dy.$$

Then, since $H_0^*(x)$, $H_3^*(x)$ converge uniformly to $H_0(x)$, $H_3(x)$ respectively in any finite interval as $\lambda \to \infty$, there exists λ_0 such that for any $\lambda > \lambda_0$ and $\eta > 0$

$$\left|\sum_{r=-\infty}^{\infty}\left(H_0\left(\frac{r}{p}\right)-H_0^*\left(\frac{r}{p}\right)\right)\right|<\eta,$$

(5)
$$I(x)\leqq H_0^*(x), \quad -\infty<x<\infty,$$

$$\left|\sum_{r=-\infty}^{\infty}\left(H_3\left(\frac{r}{p}\right)-H_3^*\left(\frac{r}{p}\right)\right)\right|<\eta$$

and

(6)
$$H_1(x)\leqq H_3^*(x).$$

Now (4), (5), (6) give us

$$0<N_I(t)\leqq N_{H_0^*}(t),$$
$$0<N_{H_0^*}(t)-N_I(t)=N_{H_0^*-I}(t)$$
$$=N_{H_2}(t)+N_{H_1}(t)\leqq N_{\overline{H}_2}(t)+N_{H_3^*}(t).$$

Remembering that $H_0^*(x)$, $\overline{H}_2(x)$, $H_3^*(x)$ and their Fourier transformations $h_0^*(t)$, $\overline{h}_2(t)$, (h_3^*t) satisfy (i), (ii), we get

(7)
$$\overline{\lim_{t\to\infty}}\ (N_{\overline{H}_2}(t)+N_{H_3^*}(t))\leqq 0(\lambda^{-1})+\frac{4\eta}{mp},$$

(8)
$$\overline{\lim_{t\to\infty}}\ N_{H_0^*}(t)\leqq\frac{1}{mp}\sum_{r=-\infty}^{\infty}I\left(\frac{r}{p}\right)+\frac{2\eta}{mp}.$$

Hence

(9)
$$0<\overline{\lim_{t\to\infty}}\ N_I(t)-\underline{\lim_{t\to\infty}}\ N_I(t)<0(\lambda^{-1})+\frac{4\eta}{mp}.$$

Taking $\lambda^{-1}+\eta$ sufficiently small and taking into account of (7)-(9), we finally obtain

$$E\{N_I(t)\}<\infty \qquad \text{for all } t\geqq 0$$

and

(10)
$$\lim_{t\to\infty}E\{N_I(t)\}=\frac{1}{mp}\sum_{r=-\infty}^{\infty}I\left(\frac{r}{p}\right).$$

By using (10), in the same way as in [3], we can show that

$$\lim_{t\to\infty} Pr\{N_l(t)=k\}=e^{-\lambda}\frac{\lambda^k}{k!} \qquad k=0, 1, 2,\cdots$$

$$\lambda=\frac{1}{mp}\sum_{r=-\infty}^{\infty} I\left(\frac{r}{p}\right).$$

I express my sincerest thanks to Professor G. Maruyama who has suggested this investigation and given valuable advices.

Deffence Academy.

Reference

[1] S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932.
[2] G. Maruyama, Fourier analytic treatment of some prblems on the sums of random variables, Natural Sci. Rep. Ochanomizu Univer, 6 (1955).
[3] G. Maruyama, On the Poisson distribution derived from independent random walks, ibid. 6 (1955).