Journal of the Mathematical Society of Japan Vol. 8, No. 2, April, 1956

On the Eilenberg-MacLane invariants of loop spaces.

By Haruo SUZUKI

(Received Dec. 16, 1954)

1. Let X be a simply connected topological space and let E be the space of all paths in X starting from a fixed point $x_0 \in X$, topologized by compact-open topology. Then E is contractible, and with the projection $\rho: E \rightarrow X$ which associates each path to its terminal point, (E, ρ, X) is a fiber space in the sense of Serre [1], where the fiber at x_0 is the loop space Ω_X of X. It is well known that we have $\pi_i(X) \approx \pi_{i-1}(\Omega_X)$, $i=2, 3, \cdots$.

Fixing integers p, q such that $2<p<q$, we assume in the following that $\pi_i(X)=0$ for $p \neq i<q$ and put $\pi_p(X)=\pi_p$, $\pi_q(X)=\pi_q$. Then $\pi_i(\Omega_X)=0$ for $p \neq i+1<q$, and $\pi_{p-1}(\Omega_X) \approx \pi_p$, $\pi_{q-1}(\Omega_X) \approx \pi_q$. We shall put $\pi_{p-1}(\Omega_X)=\pi_{p-1}$, $\pi_{q-1}(\Omega_X)=\pi_{q-1}$, and consider these groups with the canonical isomorphisms $\pi_p \approx \pi_{p-1}$, $\pi_q \approx \pi_{q-1}$.

Now, the spaces X and Ω_X determine the Eilenberg-MacLane invariants $\boldsymbol{k}_p^{q+1}(X) \in H^{q+1}(\pi_p, p, \pi_q)$ and $\boldsymbol{k}_{p-1}^{q}(\Omega_X) \in H^q(\pi_{p-1}, p-1, \pi_{q-1})$ respectively. As will be shown, the latter invariant $\boldsymbol{k}_{p-1}^{q}(\Omega_X)$ is the image of the former $\boldsymbol{k}_p^{q+1}(X)$ under the suspension homomorphism S of the cohomology groups (Theorem 2 below). Therefore, if we associate to any system $(\pi, \pi', \boldsymbol{k}_p^{q+1})$ consisting of abelian groups π, π' and an element $\boldsymbol{k}_p^{q+1}$ in $H^{q+1}(\pi, p, \pi')$ the system $S^*(\pi, \pi', \boldsymbol{k}_p^{q+1})=(\pi, \pi', S\boldsymbol{k}_p^{q+1})$, then the correspondence S^* has a geometrical meaning.

If q is sufficiently small, we can define the inverse of this operation. When X is a CW-complex the homotopy type of X is determined by that of Ω_X[1] (see §3 Cor. 4 below). There is also an analogous relation about invariants of J. H. C. Whitehead which we shall show for standard complexes and standard loop spaces [5].

1) Conversely, in arcwise connected spaces, homotopy types of loop spaces are determined by those of original spaces (see §5 below).

2. Eilenberg-MacLane complexes $K(\pi_p, p)$. We shall first give a short description of complexes $K(\pi_p, p)$ and suspension homomorphisms of Eilenberg-MacLane cohomology groups [2]. For each positive integer n, let $\varDelta_n$ be a standard n-simplex with ordered vertices $(0,\cdots, n)$. By e_n^i $(i=0,\cdots, n)$ we denote the mapping of $\varDelta_{n-1}$ to $\varDelta_n$ which maps the vertices $0, 1,\cdots, n-1$ of $\varDelta_{n-1}$ on the vertices of $\varDelta_n$, omitting the vertex i of $\varDelta_n$ and preserving their order. The q-cells of $K(\pi_p, p)$ are cocycles of $Z^p(\varDelta_n, \pi_p)$. For each $g\in Z^p(\varDelta_n, \pi_p)$ the mapping e_n^i gives a cocycle $F_i g=ge_n^i\in Z^p(\varDelta_{n-1}, \pi_p)$. We define the boundary of the n-cell g by $\partial g=\sum_{i=0}^{n}(-1)^i F_i g$. The addition in the right hand side of the last equation is to be regarded as a formal sum of cells.

DEFINITION. For each $(p-1)$-cocycle $g\in Z^{p-1}(\varDelta_{n-1}, \pi_{p-1})$ the *suspended p-cocycle Tg* is defined for each p-dimensional ordered simplex $(r_0,\cdots, r_p)$ of $\varDelta_n$ by

$$Tg(r_0,\cdots, r_p)=g(r_0,\cdots, r_{p-1}) \qquad \text{if } r_p=n\,, \tag{1}$$
$$=0 \qquad \text{if } r_p<n\,.$$

If by g_0 we denote the cocycle which is identically zero, in the appropriate dimensions, then the *suspension mapping* S is defined by

$$Sg=Tg-g_0\,. \tag{2}$$

This is a chain transformation (raising dimensions by 1) of $K(\pi_{p-1}, p-1)$ into $K(\pi_p, p)$ and hence induces homomorphisms

$$S:\ H^{p+k}(\pi_p, p\,;G)\rightarrow H^{p-1+k}(\pi_{p-1}, p-1\,;G)$$

between corresponding cohomology groups, where G is any abelian group and $k=0, 1,\cdots$. In the following we take $\pi_q=\pi_{q-1}$ for G.

THEOREM 1. *For $k<p-1$. the suspension homomorphism S is an isomorphism onto. For $k=p-1$ it is an isomorphism into.*

This theorem is proved by the singular cohomology theory of a fiber structure of the path space E, using the theory of spectral sequence (see [1], Proposition 10, p. 483). On the other hand this is shown by a purely algebraic method (see [2], [7], [8]).

3. THEOREM 2. *Let S be the suspension homomorphism of cohomology groups and let $k_p^{q+1}(X)$ and $k_{p-1}^q(\Omega_X)$ be Eilenberg-MacLane*

invariants of spaces X and Ω_X respectively.

We have

(3) $$S(\boldsymbol{k}_p^{q+1}(X))=\boldsymbol{k}_{p-1}^q(\Omega_X).$$

PROOF. Let $M(\Omega_X)$ be a minimal complex (see [3]) in Ω_X based on the constant path $I\to x_0$. Each mapping σ of an $(n-1)$-singular simplex σ of $M(\Omega_X)$ induces a mapping $\overline{T}\sigma:\ \Delta_{n-1}\times I\to X$ defined as

(4) $$\overline{T}\sigma(x,t)=\sigma(x)\,(t)$$

for any $x\in\Delta_{n-1}$ and $t\in I$. We map $\Delta_{n-1}\times I$ onto Δ_n by identifying the set $\Delta_{n-1}\times 1$ to the last vertex of Δ_n. Let i_n be this identification. As $\overline{T}\sigma$ maps the set $\Delta_{n-1}\times 1$ into a point x_0, the composite mapping $\overline{T}\sigma i_n^{-1}$ is a continuous mapping of Δ_n into X. Therefore, it defines a singular n-simplex in X, which we denote by $\overline{S}\sigma$.

We note that a singular simplex which is a constant mapping to x_0 is called *collapsed* and two singular simplexes whose boundary coincide are called *compatible.* If for these simplexes there exists a homotopy which leaves the mapping on the boundary fixed, then we say that they are *homotopic.* Obviously S and its inverse preserve properties stated above. Therefore, because of the definition of the minimal complex, we can take a minimal complex $M(X)$ of X, which contains $S(M(\Omega_X))$ (see [3], § 4). Let κ_X be a natural simplicial transformation of $M(X)$ into $K(\pi_p,p)$ defined in [4] and let κ_Ω be that of $M(\Omega_X)$ into $K(\pi_p,p-1)$. Moreover, let $\bar{\kappa}_X$ be a simplicial transformation of the q-skeleton $K^q(\pi_p,p)$ of $K(\pi_p,p)$ into $M(X)$ which is defined by (3.1), (3.2), (3.3) of [4] and let $\bar{\kappa}_\Omega$ be that of $K^{q-1}(\pi_{p-1},p-1)$ into $M(\Omega_X)$. Composite mappings $\kappa_X\bar{\kappa}_X$ and $\kappa_\Omega\bar{\kappa}_\Omega$ are identities. These transformations can be so constructed that commutativity relation holds in the following diagram:

(5) $$\begin{array}{ccc} M(\Omega_X) & \xrightarrow{\ \overline{S}\ } & M(X) \\ \uparrow{\scriptstyle \bar{\kappa}_\Omega} & & \uparrow{\scriptstyle \bar{\kappa}_X} \\ K^{q-1}(\pi_{p-1},p-1) & \xrightarrow{\ T\ } & K^q(\pi_p,p), \end{array}$$

i. e. we have $\overline{S\kappa_{\Omega}}g=\overline{\kappa}_X Tg$ for any $g\in K^{q-1}(\pi_{p-1}, p-1)$. Let $\Delta_{q+1,q}$ be the q-skeleton of Δ_{q+1}. If we attempt to continue the definition of $\overline{\kappa}_X$ for $(q+1)$-cell g' of $K(\pi_p, p)$, we can only do it for $\Delta_{q+1,q}$ so that the mapping

$$f_X(g')\,;\ \Delta_{q+1,q}\rightarrow X$$

satisfies

$$f_X(g')e^i_{q+1}=\overline{\kappa}_X(F_i g') \qquad i=0,\cdots,q+1\,. \tag{6}$$

Since π_q is not assumed to vanish, the map $f_X(g')$ in general will not be extendible to a mapping of Δ_{q+1} into X. Let $cf_X(g')$ be an element of π_q containing $f_X(g')$. We define a cochain k^{q+1}_p in the complex $K(\pi_p, p)$ by

$$k^{q+1}_p(g')=cf_X(g')\in\pi_q(X)\,. \tag{7}$$

k^{q+1}_p is a $(q+1)$ cocycle. Its cohomology class does not depend on the choice of the simplicial transformation $\overline{\kappa}_X$[2)] and it is denoted by $\boldsymbol{k}^{q+1}_p(X)$. Similarly a q-cocycle k^q_{p-1} and its cohomology class $\boldsymbol{k}^q_{p-1}(\Omega_X)$ are defined. By $f_{\Omega}(g)$ we denote a mapping corresponding to (6) for any q-cell g of $K(\pi_{p-1}, p-1)$. Cohomology classes $\boldsymbol{k}^{q+1}_p(X)$ and $\boldsymbol{k}^q_{p-1}(\Omega_X)$ do not depend on the choice of the minimal complexes (see [4]).

Because of the commutativity of (5), we have

$$\begin{aligned} f_X(Tg)\, e^{q+1}_i &= \overline{\kappa}_X(F_i Tg) \\ &= \kappa_X(TF_i g)=\overline{S\kappa}_{\Omega}(g)\, e^q_i \qquad 0\leqq i\leqq q\,, \end{aligned}$$

and $f_X(Sg)\, e^{q+1}_{q+1}$ is a constant mapping to x_0.
This yields

$$\overline{T}f_{\Omega}(g)\, i^{-1}_q=f_X(Tg)\,.$$

Therefore the element of the group $\pi_q(X)$ containing the mapping $f_X(g)$ is the image of the element of the group $\pi_{q-1}(\Omega_X)$ containing

2) As $\overline{\kappa}_X$ is determined on the $(q-1)$-skeleton $K^{q-1}(\pi_{p-1}, p-1)$ uniquely only $\overline{\kappa}^q_X$ comes into question.

the mapping $f_\Omega(g)$ under the suspension isomorphism Σ of homotopy groups. Since groups $\pi_{q-1}(\Omega_X)$ and $\pi_q(X)$ are identified under the isomorphism in introduction, we have

$$cf_\Omega(g) = cf_X(Tg)\,.$$

Using the relation $cf_X(g_0)=0$, this yields

$$\begin{aligned} k^q_{p-1}(g) &= cf_\Omega(g) \\ &= cf_X(Tg) \\ &= cf_X(Tg) - cf_X(g_0) \\ &= k^{q+1}_p(Tg - g_0) \\ &= k^{q+1}_p(Sg) \\ &= (Sk^{q+1}_p)\,(g)\,, \end{aligned}$$

where S in the last term is a dual cochain transformation of the chain transformation S. Therefore

$$\boldsymbol{k}^q_{p-1}(\Omega_X) = S(\boldsymbol{k}^{q+1}_p(X))$$

holds good. Thus the theorem is proved.

Let K be a standard complex and let $\omega(K)$ be a standard loop space on the complex (see [5]). The injection mapping of the space $\omega(K)$ into the space Ω_K induces isomorphisms of homotopy groups of the two spaces. If we take the minimal complex $M(\Omega_K)$ in $\omega(K)$ particularly, we shall see that Eilenberg-MacLane invariants of two spaces are identified by the isomorphism of injection. Therefore we have the following result:

COROLLARY 3. *The Eilenberg-MacLane invariant* $\boldsymbol{k}^q_{p-1}(\omega(K))$ *of the standard loop space is the image of the invariant* $\boldsymbol{k}^{q+1}_p(K)$ *of the standard complex under the suspension homomorphism* S.

We suppose $q<2p-1$. The Eilenberg-MacLane invariant of the space Ω_X determines that of the original space X, since the homomorphisms S is one-to-one by the Theorem 1. Therefore we obtain the following result:

COROLLARY 4. *Let X, Y be both arcwise connected CW-complexes, such that* $\pi_p(X)\approx\pi_p(Y)$, $\pi_q(X)\approx\pi_q(Y)$ *and* $\pi_i(X)=\pi_i(Y)=0$ *for* $0<i<p$

and $p<i<q$ where $q<2p-1$. If the spaces Ω_X, Ω_Y have the same q-homotopy type, then X, Y have the same $(q+1)$-type.

4. J. H. C. Whitehead invariants (see [6]).

In the following, we restrict our argument to simply connected standard complexes K and their standard loop spaces $\omega(K)$[3] which are also CW-complexes.

Let K^n and $\omega(K)^n$ be n-skeletons of K and $\omega(K)$ respectively. Then the n-th Whitehead invariant $\boldsymbol{l}_n(K)$ is defined as follows: Let j be the homomorphism of $\pi_n(K^n)$ into $\pi_n(K^n, K^{n-1})$ induced by the injection mapping and let $j^\#$ be its $\pi_n(K)$-dual. We denote by $l_n(K)$ the natural homomorphism

$$\pi_n(K^n) \to \pi_n(K^n)/\partial\pi_{n+1}(K^{n+1}, K^n) = \pi_n(K),$$

and we put

$$\boldsymbol{l}_n(K) = \{l_n(K)\}$$
$$\in \Gamma^n(K, \pi_n) = A^\#(\pi_n(K))/j^\# C^\#(\pi_n(K)),$$

where $A^\#(\pi_n(K))$ is the group Ophom $(\pi_n(K^n), \pi_n(K))$[4] and $C^\#(\pi_n(K))$ is the group Ophom $(\pi_n(K^n, K^{n-1}), \pi_n(K))$[4].

Let α be any element of $\pi_n(\omega(K)^n)$ and let f_α be a mapping of (S^n, s) into $(\omega(K)^n, x_0)$ contained in α, where S_n is an n-sphere, s is a point of S^n, and x_0 is the base point of $\omega(K)$, i. e. the constant path $I \to x_0$. A suspended class $\Sigma'\alpha$ of α is the class which contains the mapping Σf_α of an $(n+1)$-sphere into K^{n+1}[5] defined as follows: we identify the subset $S^n \times 0 \cup S^n \times 1 \cup s \times I$ of $S^n \times I$ to a point and denote the identification mapping by h_n. We can take $h_n(S^n \times I)$ for S^{n+1}. The mapping Σf_α is defined by

$$\Sigma f_\alpha(y) = f_\alpha(x)\,(t)$$

for each point $y = h_n(x, t)$. Σ' is a homomorphism of $\pi_n(\omega(K)^n)$ into $\pi_{n+1}(K^{n+1})$. We can extend the definition of Σ' to relative homotopy

3) As for singular polytopes of $M(X)$ and $M(\Omega_X)$, I don't know whether the results of this section hold good or not.

4) Since we assumed $\pi_1(K)=0$, we have Ophom=Hom for the K.

5) This follows from the definition of the complex $\omega(K)$.

groups similarly. It is a homomorphism of the group $\pi_n(\omega(K)^n, \omega(K)^{n-1})$ into the group $\pi_{n+1}(K^{n+1}, K^n)$. Then commutativity relations $\partial\Sigma'=\Sigma'\partial$ and $j\Sigma'=\Sigma'j$ hold good for the boundary operator ∂ and the injection homomorphism j.

The homomorphism of the group $\pi_n(\omega(K))$ into the group $\pi_{n+1}(K)$ induced by Σ' coincides with the suspension isomorphism Σ of homotopy groups. We identify these groups by the isomorphism and denote it by π_{n+1}. In the following diagramm commutativity holds good:

$$\begin{array}{ccc} \pi_n(\omega(K)) & \overset{\Sigma}{=} & \pi_{n+1}(K)\ (=\pi_{n+1}) \\ \uparrow l_n(\omega(K)) & & \uparrow l_{n+1}(K) \\ \pi_n(\omega(K)^n) & \overset{\Sigma'}{\longrightarrow} & \pi_{n+1}(K^{n+1})\,. \end{array}$$

Let $\Sigma^\#$ be a π_{n+1}-dual of Σ' (here we do not consider operations on the groups $\pi_n(\omega(K)^n)$ and $\pi_{n+1}(K^{n+1})$). Then we have

$$\begin{aligned} l_n(\omega(K)) &= l_{n+1}(K) \circ \Sigma' \\ &= \Sigma^\# \, l_{n+1}(K)\,. \end{aligned}$$

Therefore if we denote by Σ^* the homomorphism of the group $\pi^{n+1}(\pi_{n+1}, K)$ into the group $\prod^n(\omega(K), \pi_n)$ induced by $\Sigma^\#$, then we obtain

(9) $$l_n(\omega(K)) = \Sigma^* \, l_{n+1}(K)\,.$$

This relation is analogous to (3).

5. Appendix. Finally we shall show that if two arc-wise connected, simply connected spaces are homotopy equivalent, then their loop spaces have the same property.

LEMMA 5. *Let X, Y be two spaces as above and let f be a continuous mapping of X into Y. f induces a continuous mapping of Ω_X into Ω_Y.*

PROOF. Let τ be an arbitrary element of Ω_X. We define a mapping f_Ω of Ω_X into Ω_Y by

$$(f_\Omega \tau)(t) = f(\tau(t))$$

for each $t \in I$. Let C be a compact set of $I=[0,1]$ and let U be an open set of Y. If by (C, U) we denote an element of the open base of Ω_Y, determined by C and U, then $(C, f^{-1}(U))$ is an open set of Ω_X. Since we have $f_\Omega^{-1}(C, U)=(C, f^{-1}(U))$, f_Ω is continuous.

LEMMA 6. *Let F: $X \times I \rightarrow Y$ be a homotopy. This mapping induces a homotopy of mappings of Ω_X into Ω_Y defined by the above lemma.*

PROOF. Let τ be an arbitrary element of Ω_X. We define a mapping F_Ω of $\Omega_X \times I$ into Ω_Y by

$$F_\Omega(\tau, s)(t)=F(\tau(t), s)$$

for any t and s in I. Let (C, U) be an arbitrary element of the open base in Ω_Y which contains $F(\tau, s)$. An inverse image $F^{-1}(U)$ of U is an open set in $X \times I$ and it contains $(\tau(C), s)$. $(\tau(C), s)$ is a compact set, therefore there exists a finite open covering $(U_i, (s-\varepsilon_i, s+\varepsilon_i))$ contained in $F^{-1}(U)$ for each i, where ε_i is a positive real number. We put $\operatorname{Min} \varepsilon_i=\varepsilon>0$ and $V=((C, \bigcup_i U_i), (s-\varepsilon, s+\varepsilon))$. We have $(\tau, s) \in V$ and $F_\Omega(V) \subset (C, U)$. Therefore F_Ω is continuous.

THEOREM 7. *If two spaces X and Y are homotopy equivalent then their loop spaces have the same property.*

This is an immediate consequence of the two lemmas stated above.

Mathematical Institute, Tohoku University.

References

1. J.-P. Serre, Homologie singulière des espace fibrés et Applications, Ann. of Math., 54 (1951), pp. 425-505.
2. S. Eilenberg and S. MacLane, Cohomology theory of abelian groups and homotopy theory I-III, Proc. N. A. S., U. S. A., 35 (1950), pp. 443-447 and 657-663, 37 (1951), pp. 307-310.
3. S. Eilenberg and J. A. Zilber, Semi-simplicial complexes and singular homology, Ann. of Math., 51 (1950), pp. 499-513.
4. S. Eilenberg and S. MacLane, Relations between homology and homotopy groups of spaces II, Ann. of Math., 51 (1950), pp. 514-533.
5. H. Toda, Topology of standard path spaces and homotopy theory I, Proc. of Japan Acad., 29, pp. 229-304.

6. J. H. C. Whitehead, The G-dual of semi-exact couple, Proc. of the London Math. Soc., 3rd Ser. vol. III (1953), 385-416.
7. S. Eilenberg and S. MacLane, On the groups $H(\pi, n)$ I, Ann. of Math., 58 (1953), pp. 55-166.
8. H. Cartan, Sur les groupes d'Eilenberg-MacLane $H(\pi, n)$ I, Methode des construction, Proc. N. A. S., U. S. A., 40 (1954), pp. 467-471.