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A metamathematical theorem on functions.

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In our former paper [2], [3], we have introduced a logical system *GLC* and a subsystem G^1LC of *GLC*, as generalizations of Gentzen's *LK* (cf. [1]). We have also defined the notion of functions in *GLC* in [2]. This paper is most related to [3], where we have dealt with G^1LC without bound functions. We shall introduce in this paper another logical system called *HLC* ('hierarchical' logic calculus) lying between G^1LC and *LK* (§ 1). We shall define also 'functionals' in generalization of the notion of functions.

The purpose of the present paper is to prove that the consistent system under G^1LC without bound function or under *HLC* remains consistent after 'adjunction' of the concept of functionals, under certain conditions. Our Main Theorem will read as follows:

MAIN THEOREM: *Let Γ_0 be a system of axioms consistent under G^1LC without bound function or under HLC. Suppose Γ_0 contains axioms of equality* (See § 1 for definition), *and let the following sequences be provable.*

$$\Gamma_0 \rightarrow \forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m\exists yF(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,y)$$

$$\Gamma_0 \rightarrow \forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m\forall y\forall z(F(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,y)$$

$$\wedge F(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m z)\vdash y=z)\,.$$

Let M be a functional not contained in Γ_0, and suppose further, in case of *HLC*, that $F(\alpha_1,\cdots,\alpha_n,a_1,\cdots,a_m,b)$ does not contain $\forall$ on *f*-variables. Then Γ_0 and the following axiom are consistent.

$$\forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_mF(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,M(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m))\,.$$

The conclusion of this theorem holds also in *LK* by theorem 2, proved in § 1.

After some preparations in § 1, we shall prove our main theorem in § 2. In § 3 we shall apply this theorem to improve our result in [4] on the theory of ordinal numbers. It allows us replace an axiom by a stronger one. In § 4 we shall prove the consistency of the 'theory of linear continuum'.

§ 1. The logical systems.

We shall begin with generalizing 'G^1LC without bound function' as follows.

We introduce the functional of type $(i_1,\cdots,i_n;m)$, denoted by M, K etc., and add the following rule of construction of the term to the ones given in [3]. 'If H_j is a formula with i_j argument-places for each $j(1\leqq j\leqq n)$ and $T_1,\cdots,T_m$ are terms and K is an arbitrary functional of type $(i_1,\cdots,i_n;m)$, then $K(H_1,\cdots,H_n,T_1,\cdots T_m)$ is a term'.

A function (cf. [3]) may be considered as a special case of functional.

In this paper *LK* is also considered as generalized by introducing functionals as above. Except in § 4, we use only $\neg$, $\wedge$ and $\forall$ as logical symbols. $\vee$, $\vdash$, $\dashv\vdash$ and $\exists$ can be considered as combinations of these symbols.

DEFINITION of *HLC* A proof-figure $\mathfrak{P}$ of G^1LC without bound function is called a proof-figure of *HLC*, if and only if the following condition is fulfilled. In an inference $\forall$ left on *f*-variable of the form

$$\frac{F(H), \Gamma \to \Delta}{\forall \varphi F(\varphi), \Gamma \to \Delta}$$

is used in $\mathfrak{P}$, then H contains no logical symbol $\forall$ on *f*-variable.

We consider also in *HLC* the functionals $M, K,\cdots$ of type $(i_1,\cdots,i_n;m)$ and construct the forms such as $K(H_1,\cdots,H_n,T_1,\cdots,T_m)$ with these functionals. Thereby we shall assume however that $H_1,\cdots,H_n$ contain no logical symbol $\forall$ on *f*-variable.

In the same way as in Gentzen [1], we see the following theorem.

THEOREM 1. *If a sequence $\mathfrak{S}$ is provable in HLC, then $\mathfrak{S}$ is provable without cut in HLC.*

In *LK*, the axiom of mathematical induction is expressed as the system of axioms

$$\forall z_1\forall z_2\cdots\forall z_n\forall x(A(0)\wedge\forall y(A(y)\vdash A(y+1))\vdash A(x)),$$

where $\{x\}A(x)$ runs over all the formulas with an argument-place. More precisely should be written as $\{x\}A(x, z_1,\cdots, z_n)$ and n depends on A. In this paper, such system of the axioms is denoted simply by

$$\forall A\forall x(A(0)\wedge\forall y(A(y)\vdash A(y+1))\vdash A(x)).$$

In the same way, notations such as $\forall A_1\cdots\forall A_n F(A_1,\cdots, A_n)$ will be used, where the number of argument-places of A_i is uniquely determined by F for each $i(1\leqq i\leqq n)$.

Then by theorem 1 the following theorem is easily proved.

THEOREM 2. *The axioms* $A_1,\cdots, A_N$, $\forall A^1_1\cdots\forall A^1_{i_1}F^1(A^1_1,\cdots, A^1_{i_1}),\cdots, \forall A^n_1\cdots\forall A^n_{i_n}F^n(A^n_1,\cdots, A^n_{i_n})$ *are consistent in LK, if and only if* $A_1,\cdots, A_N$, $\forall\varphi^1_1\cdots\forall\varphi^1_{i_1}F^1(\varphi^1_1,\cdots, \varphi^1_{i_1}),\cdots, \forall\varphi^n_1\cdots\forall\varphi^n_{i_n}F^n(\varphi^n_1,\cdots, \varphi^n_{i_n})$ *are consistent in HLC.*

As we have remarked in the introduction, it follows from this theorem, that our main theorem once proved for HLC will imply the same conclusion for LK.

Let A and B be two formulas with i argument-places. Then $A\equiv B$ is an abbreviation of the formula

$$\forall x_1\cdots\forall x_i(A(x_1,\cdots, x_i)\vdash\dashv B(x_1,\cdots, x_i)).$$

Let Γ_0 be a system of axioms in G^1LC without bound functions or in *HLC*. '*Γ_0 contains equality axiom*' means that Γ_0 fulfils the following conditions

1. Γ_0 contains $\forall\varphi\forall x\forall y(x=y\vdash(\varphi[x]\vdash\dashv\varphi[y]))$ and $\forall x(x=x)$
2. If functional K of type $(i_1,\cdots, i_n; m)$ is contained in Γ_0, then Γ_0 contains $\forall\varphi_1\cdots\forall\varphi_n\forall\psi_1\cdots\forall\psi_n\forall x_1\cdots\forall x_n(\varphi\equiv\psi\wedge\cdots\wedge\varphi\equiv\psi\vdash K(\varphi_1,\cdots, \varphi_n, x_1,\cdots, x_m)=K(\psi_1,\cdots, \psi_n, x_1,\cdots, x_m))$.

Then, from the main theorem follows the following theorem

THEOREM ON FUNCTION. *Under the hypothesis of the main theorem the following axioms are consistent*

$$\Gamma_0,$$

$$\forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m F(\varphi_1,\cdots, \varphi_n, x_1,\cdots, x_m, M(\varphi_1,\cdots, \varphi_n, x_1,\cdots, x_m)),$$

$$\forall\varphi_1\cdots\forall\varphi_n\forall\psi_1\cdots\forall\psi_n\forall x_1\cdots\forall x_m(\varphi_1\equiv\psi_1\wedge\cdots\wedge\varphi_n\equiv\psi_n\vdash$$
$$M(\varphi_1,\cdots,, \varphi_n, x_1,\cdots, x_m)=M(\psi_1,\cdots, \psi_n, x_1,\cdots, x_m)).$$

PROOF. We set A_0 as $\forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m F(\varphi_1,\cdots,\varphi_n, x_1,\cdots, x_m, M(\varphi_1,\cdots,\varphi_n, x_1,\cdots, x_m))$. Then we have only to prove that the following sequence is provable

$$\Gamma_0, A_0, \alpha_1\equiv\beta_1,\cdots,\alpha_n\equiv\beta_n\rightarrow M(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m)$$
$$=M(\beta_1,\cdots,\beta_n, a_1,\cdots, a_m)$$

On the other hand, we have

$$A_0\rightarrow F(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m, M(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m))$$
$$\wedge F(\beta_1,\cdots,\beta_n, a_1,\cdots, a_m, M(\beta_1,\cdots,\beta_n, a_1,\cdots, a_m))$$

and

$$\Gamma_0, F(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m, b), F(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m, c)\rightarrow b=c\,.$$

Therefore we have only to prove that the following sequence is provable

$$\Gamma_0, \alpha_1\equiv\beta_1,\cdots,\alpha_n\equiv\beta_n, F(\beta_1,\cdots,\beta_n, a_1,\cdots, a_m, b)$$
$$\rightarrow F(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m, b)\,,$$

which is easily seen.

§ 2. Proof of the main theorem.

In this section, Γ_0 and M fulfil the condition of the main theorem Moreover the functionals except M considered in this section are assumed as contained in Γ_0.

$*$-operation

Let Q be a formula or a term. We define Q^* recursively by the following 1-5. $(Q(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m))^*$ is also denoted by $Q^*(\alpha_1,\cdots,\alpha_n, a_1,\cdots, a_m)$. $(\{x_1,\cdots, x_n\}A(x_1,\cdots, x_n))^*$ is defined by $\{x_1,\cdots, x_n\} A^*(x_1,\cdots, x_n)$.

If Q is a formula, then Q^* is a formula.

If Q is a term, then Q^* is a formula with an argument-place. And in this case, if Q^* is of the form $\{x\}B(x)$, $Q^*(X)$ means $B(X)$.

1. a^* is $\{x\}(x=a)$.
2. If K is a functional other than M, then $(K(A_1,\cdots, A_n, T_1,\cdots, T_m))^*$

is $\{x\}(\forall x_1 \cdots \forall x_m(T_1^*(x_1) \wedge \cdots \wedge T_m^*(x_m) \vdash x = K(A_1^*, \cdots, A_n^*, x_1, \cdots, x_m)))$

3. $(M(A_1 \cdots, A_n, T_1, \cdots, T_m))^*$ is $\{x\}(\forall x_1 \cdots \forall x_m(T_1^*(x_1) \wedge \cdots \wedge T_m^*(x_m) \vdash F(A_1^*, \cdots, A_n^*, x_1, \cdots, x_m, x)))$.

4. $(\alpha[T_1, \cdots, T_n])^*$ is $\forall x_1 \cdots \forall x_n(T_1^*(x_1) \wedge \cdots \wedge T_n^*(x_n) \vdash \alpha[x_1, \cdots, x_n])$.

5. $(\neg A)^*$, $(A \wedge B)^*$, $(\forall x A(x))$ and $(\forall \varphi F(\varphi))^*$ are $\neg A^*$, $A^* \wedge B^*$, $\forall x A^*(x)$ and $\forall \varphi F^*(\varphi)$ respectively.

PROPOSITION 1. *Let T be a term. Then the following sequences are provable*

$$\Gamma_0, T^*(a), T^*(b) \rightarrow a = b$$

and

$$\Gamma_0 \rightarrow \exists x(T^*(x)).$$

PROOF. We prove this by the mathematical induction on the number of stages to construct T. If T is a free variable, then the proposition is clear. Now we consider T is of the form $K(A_1, \cdots, A_n, T_1, \cdots, T_m)$. Then by the hypothesis of the induction, the proposition holds for $T_1, \cdots, T_m$. Therefore

$$\Gamma_0, T^*(a), T^*(b) \rightarrow a = b$$

and

$$\forall x_1 \cdots \forall x_m(T_1^*(x_1) \wedge \cdots \wedge T_m^*(x_m) \vdash a = K(A_1^*, \cdots, A_n^*, x_1, \cdots, x_n))$$

is equivalent to $\exists x_1 \cdots \exists x_m(T_1^*(x_1) \wedge \cdots \wedge T_m^*(x_m) \wedge a = K(a_1^*, \cdots, A_n^*, x_1, \cdots, x_m))$ under Γ_0. Therefore $\Gamma_0 \rightarrow \exists x T^*(x)$ is clear.

PROPOSITION 2. *Let A and T be a formula and a term respectively and M be not contained in A and T. Then the following sequences are provable*

$$\Gamma_0 \rightarrow A^* \leftrightarrow A$$

and

$$\Gamma_0 \rightarrow T^*(a) \leftrightarrow a = T.$$

PROOF. We prove this by the mathematical induction on the number of stages to construct A or T. If T is a free variable, then the proposition is clear. We have now to consider several different cases.

1) Let T be of the form $K(A_1,\cdots,A_n,T_1,\cdots,T_m)$. Then, under Γ_0, the following formula is equivaleht to $T^*(a)$:

$$\forall x_1\cdots\forall x_m(T_1^*(x_1)\wedge\cdots\wedge T_m^*(x_m)\vdash a=K(A_1^*,\cdots,A_m^*,x_1,\cdots,x_m))$$

and this is equivalent to the following (by the hypothesis of induction)

$$\forall x_1\cdots\forall x_m(x_1=T_1\wedge\cdots\wedge x_m=T_m\vdash a=K(A_1,\cdots,A_n,x_1,\cdots,x_m))$$

and this again clearly to $a=T$.

In such cases of 'continued equivalence', we shall hereafter simply when the formulas one after another, in such a way that the equivalence of succesive formulas will be clear to the reader.

2) Let A be $\alpha[T_1,\cdots,T_m]$. Then, holds under Γ_0, the following continued equivalence:

$$A^*$$
$$\forall x_1\cdots\forall x_m(T_1^*(x_1)\wedge\cdots\wedge T_m^*(x_m)\vdash\alpha[x_1,\cdots,x_m])$$
$$\forall x_1\cdots\forall x_m(x_1=T_1\wedge\cdots\wedge x_m=T_m\vdash\alpha[x_1,\cdots,x_m])\,.$$
$$A.$$

3) If A is $\neg B$, $C\wedge B$, $\forall xD(x)$, $\forall\varphi F(\varphi)$, the proposition is clear.

PROPOSITION 3. *The following sequences are provable.*

$$\Gamma_0\rightarrow(F(A))^*\vdash\dashv F^*(A^*)$$

and

$$\Gamma_0\rightarrow(T(A))^*(a)\vdash\dashv T^*(A^*)(a)\,.$$

PROOF. If $T(A)$ and $F(A)$ contain no A, then the proposition is clear. Now we separate the cases.

1) Let $T(A)$ be $K(A_1(A),\cdots,A_n(A),T_m(A),\cdots,T_m(A))$. Then the following continued equivalence holds under Γ_0:

$$(T(A))^*(a)$$
$$\forall x_1\cdots\forall x_m((T_1(A))^*(x_1)\wedge\cdots\wedge(T_m(\mathrm{A}))^*(x_m)\vdash$$
$$a=K((A_1(A))^*,\cdots,(A_n(A))^*,x_1,\cdots,x_m))$$
$$\forall x_1\cdots\forall x_m(T_1^*(A^*)(x_1)\wedge\cdots\wedge T_m^*(A^*)(x_m)\vdash$$
$$a=K(A_1(A^*),\cdots,A_n(A^*),x_1,\cdots,x_m))$$

(by the hypothesis of the induction)

$$T^*(A^*)(a).$$

2) Let $T(A)$ be $M(A_1(A),\cdots,A_n(A),T_1(A),\cdots,T_m(A))$. Then the following continued equivalence holds under Γ_0:

$(T(A))^*(a)$

$\forall x_1 \cdots \forall x_m((T_1(A))^*(x_1) \wedge \cdots \wedge (T_m(A))^*(x_m) \vdash$
$F((A_1(A))^*, \cdots, (A_n(A))^*, x_1, \cdots, x_m, a))$

$\forall x_1 \cdots \forall x_m(T_1^*(A^*)(x_1) \wedge \cdots \wedge T_m^*(A^*)(x_m) \vdash$
$F(A_1^*(A^*), \cdots, A_n^*(A^*), x_1, \cdots, x_m, a))$

$T^*(A^*)(a)$.

3) Let $F(A)$ be $\alpha[T_1(A), \cdots, T_m(A)]$. Then the following continued equivalence holds under Γ_0:

$(F(A))^*$

$\forall x_1 \cdots \forall x_m((T_1(A))^*(x_1) \wedge \cdots \wedge (T_m(A))^*(x_m) \vdash \alpha[x_1, \cdots, x_m])$

$\forall x_1 \cdots \forall x_m(T_1^*(A^*)(x_1) \wedge \cdots \wedge T_m^*(A^*)(x_m) \vdash \alpha[x_1, \cdots, x_m])$

$F^*(A^*)$.

4) The other cases are clear.

PROPOSITION 4. *The following sequences are provable:*

$$\Gamma_0 \rightarrow (A(T))^* \vdash\dashv \forall x(T^*(x) \vdash A^*(x))$$

and

$$\Gamma_0 \rightarrow (T_0(T))^*(a) \vdash\dashv \forall x(T^*(x) \vdash (T_0(x))^*(a)).$$

PROOF. We prove this by the mathematical induction on the number of stages to construct $A(T)$ or $T_0(T)$. We have to consider the following several cases.

1) Let $T_0(T)$ be T itself. In this cases $(T_0(T))^*(a)$ is $T^*(a)$ and $(T_0(x))^*(a)$ is $a=x$. Therefore the proposition is clear by proposition 1.

2) Let $T_0(T)$ be $K(A_1(T), \cdots, A_n(T), T_1(T), \cdots, T_m(T))$: Then the following continued equivalence holds under Γ_0:

$(T_0(T))^*(a)$

$\forall x_1 \cdots \forall x_m((T_1(T))^*(x_1) \wedge \cdots \wedge (T_m(T))^*(x_m) \vdash$
$a = K((A_1(T))^*, \cdots, (A_n(T))^*, x_1, \cdots, x_m))$

$\forall x_1 \cdots \forall x_m(\forall y(T^*(y) \vdash (T_1(y))^*(x_1)) \wedge \cdots \wedge \forall y(T^*(y) \vdash (T_m(y))^*(x_m))$
$\vdash a = K(\forall z(T^*(z) \vdash A_1^*(z)), \cdots, \forall z(T^*(z) \vdash A_n^*(z)), x_1, \cdots, x_m))$

(by the hypothesis of the induction)

$\forall x_1 \cdots \forall x_m(\exists y(T^*(y) \wedge (T_1(y)^*(x_1)) \wedge \cdots \wedge \exists y(T^*(y) \wedge (T_m(y))^*(x_m))$
$\vdash a = K(\forall z(T^*(z) \vdash A_1^*(z)), \cdots, \forall z(T^*(z) \vdash A_n^*(z)), x_1, \cdots, x_m))$

(by the proposition 1)

$$\forall y \forall x_1 \cdots \forall x_m (T^*(y) \wedge (T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash a = K(\forall z(T^*(z) \vdash A_1^*(z)), \cdots, \forall z(T^*(z) \vdash A_n^*(z)), x_1, \cdots, x_m))$$ (By the proposition 1)

On the other hand, $\forall x(T^*(x) \vdash (T(x))^*(a))$ is

$$\forall x(T^*(x) \vdash \forall x \cdots \forall x_m((T_1(x))^*(x_1) \wedge \cdots \wedge (T_m(x))^*(x_m) \vdash a = K(A_1^*(x), \cdots, A_n^*(x), x_1, \cdots, x_m)) ,$$

so it is equivalent to

$$\forall y \forall x_1 \cdots \forall x_m (T^*(y) \wedge (T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash a = K(A_1^*(y), \cdots, A_n^*(y), x_1, \cdots, x_m)) .$$

Therefore we have only to prove

$$\Gamma_0, T^*(b) \rightarrow A_i^*(b) \equiv \forall z(T^*(z) \vdash A_i^*(z))$$

for each i, which is easily proved by proposition 1.

3) Let $T_0(T)$ be $M(A_1(T), \cdots, A_n(T), T_1(T), \cdots, T_m(T))$. In the same way as in the case 2), the following continued equivalence holds under Γ_0:

$$(T_0(T))^*(a)$$

$$\forall x_1 \cdots \forall x_m((T_1(T))^*(x_1) \wedge \cdots \wedge (T_m(T))^*(x_m) \vdash F((A_1(T))^*, \cdots, (A_n(T))^*, x_1, \cdots, x_m, a))$$

$$\forall x_1 \cdots \forall x_m(\forall y(T^*(y) \vdash (T_1(y))^*(x_1)) \wedge \cdots \wedge \forall y(T^*(y) \vdash (T_m(y))^*(x_m)) \vdash F(\forall y(T^*(y) \vdash A_1^*(y), \cdots, \forall y(T^*(y) \vdash A_n^*(y)), x_1, \cdots, x_m, a))$$

$$\forall y \forall x_1 \cdots \forall x_m(T^*(y) \wedge (T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash F(\forall z(T^*(z) \vdash A_1^*(z), \cdots, \forall z(T^*(z) \vdash A_n^*(z)), x_1, \cdots, x_m, a))$$

$$\forall y \forall x_1 \cdots \forall x_m(T^*(y) \wedge (T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash F(A_1^*(y), \cdots, A_n^*(y), x_1, \cdots, x_m, a))$$

$$\forall y(T^*(y) \vdash (T_0(y))^*(a)).$$

4) Let $A(T)$ be $\alpha[T_1(T), \cdots, T_m(T)]$. Then the following continued equivalence holds under Γ_0:

$$(A(T))^*$$

$$\forall x_1 \cdots \forall x_m((T_1(T))^*(x_1) \wedge \cdots \wedge (T_m(T))^*(x_m) \vdash \alpha[x_1, \cdots, x_m])$$

$$\forall x_1 \cdots \forall x_m(\forall y(T^*(y) \vdash (T_1(y))^*(x_1)) \wedge \cdots \wedge \forall y(T^*(y) \vdash (T_m(y))^*(x_m)) \vdash \alpha[x_1, \cdots, x_m])$$

$$\forall y \forall x_1 \cdots \forall x_m (T^*(y) \wedge (T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash \alpha[x_1, \cdots, x_m])$$
$$\forall y (T^*(y) \vdash \forall x_1 \cdots \forall x_m ((T_1(y))^*(x_1) \wedge \cdots \wedge (T_m(y))^*(x_m) \vdash \alpha[x_1, \cdots, x_m])$$
$$\forall y (T^*(y) \vdash A^*(y)).$$

5) The other cases are clear.

PROPOSITION 5. *Let* A_0 *be* $\forall \varphi_1 \cdots \forall \varphi_n \forall x_1 \cdots \forall x_m F(\varphi_1, \cdots, \varphi_n, x_1, \cdots, x_m, M(\varphi_1, \cdots, \varphi_n, x_1, \cdots, x_m))$. *Then* $\Gamma_0 \rightarrow A_0^*$ *is provable.*

PROOF. We have only to prove that

$$\Gamma_0 \rightarrow (F(\alpha_1, \cdots, \alpha_n, a_1, \cdots, a_m, M(\alpha_1, \cdots, \alpha_n, a_1, \cdots, a_m)))^*.$$

To show this by the proposition 2 and 4, we have only to prove

$$\Gamma_0 \rightarrow \forall x((M(\alpha_1, \cdots, \alpha_n, a_1, \cdots, a_n))^*(x) \vdash F(\alpha_1, \cdots, \alpha_n, a_1, \cdots, a_m, x)),$$

which is clear.

PROPOSITION 6. *If* $\Gamma \rightarrow \Delta$ *is provable, then* $\Gamma_0, \Gamma^* \rightarrow \Delta^*$ *is provable, where* Γ^* *means* $A_1^*, \cdots, A_n^*$ *provided that* Γ *is* $A_1, \cdots, A_n$.

PROOF. We prove this by the mathematical induction on the number of inference-figures in the proof-figure to $\Gamma \rightarrow \Delta$. Then, in case of GL^1C without bound function, the proposition is clear by the propositions 2, 3 and 4. In case of *HLC*, we have only to prove the following fact: If A contains no $\forall$ on f-variable, then A^* contains no $\forall$ on f-variable. But this is clear by definition.

On main theorem follows now immediately from Propositions 2, 5, 6.

§ 3. An application.

By the theorem 1, the following proposition follows easily from our former paper [4].

PROPOSITION 7. *The following axioms are consistent in HLC.*

1. $\forall x(x = x)$
2. $0 < \omega$
3. $\forall x \forall y (x < y \vee x = y \vee y < x)$
4. $\forall x \forall y \neg (x = y \wedge x < y)$
5. $\forall x \forall y \neg (x < y \wedge x < y)$
6. $\forall x \forall y \forall z (x < y \wedge y < z \vdash x < z)$
7. $\forall x (0 < x \vee 0 = x)$

8. $\forall x \forall y(x<y \vdash x'=y \vee x'<y)$

9. $\forall x(x<x')$

10. $\forall x \forall y(x'=y' \vdash x=y)$

11. $\forall x(x<\omega \vdash x'<\omega)$

12. $\forall \varphi \forall x \forall y(x=y \vdash (\varphi[x] \mapsto \varphi[y]))$

13. $\forall \varphi \forall x(\varphi[0] \wedge \forall y(\varphi[y] \vdash \varphi[y']) \wedge x<\omega \vdash \varphi[x])$

14. $\forall \varphi \forall x(\varphi[0] \wedge \forall y(\forall u(u<y \vdash \varphi[u]) \vdash \varphi[y] \vdash \varphi[x])$

15. $\forall \varphi_2 \forall u(\forall x \forall y \forall s(\varphi_2[x, s] \wedge \varphi_2[y, s] \vdash x=y)$
$\vdash \exists x \forall y(\exists s(\varphi_2[y, s] \wedge s<u) \dashv y<x))$

16. $\forall u \exists v \forall \varphi_2(\forall x \forall y \forall s(\varphi_2[x, s] \wedge \varphi_2[y, s] \vdash x=y)$
$\vdash \exists x(x<v \wedge \forall y \neg(\varphi_2[x, y] \wedge y<u)))$.

From our main theorem follows now the following theorem.

THEOREM 3. *In the proposition* 7, *the axiom* 14 *can be replaced by*
$\forall \varphi((\forall x \neg \varphi[x] \mapsto \mathrm{Min}(z)\varphi[z]=0) \wedge (\exists z \varphi[x] \vdash \varphi[\mathrm{Min}(z)\varphi[z]])$
$\wedge \forall x(\varphi[x] \vdash x \geqq \mathrm{Min}(z)\varphi[z]))$.

§ 4. A consistency proof of the theory of linear continuum.

We shall mean here by the 'theory of linear continuum' the theory on real numbers, which contains the concepts $=$, $<$, $+$, sup, inf, $\frac{1}{n}(a)$ $(n=2, 3, 4, \cdots)$, but does not contain the concept of multiplication. Here $\frac{1}{2}(a)$, $\frac{1}{3}(a), \cdots$ mean $\frac{a}{2}$, $\frac{a}{3}, \cdots$ respectively and $\frac{1}{2}(*)$, $\frac{1}{3}(*), \cdots$ are considered as functions.

Formally this theory is characterized by the following axioms 4.1.1-4.1.3.

4.1.1. $\forall x(x=x)$
$\forall x \forall y(x=y \vdash y=x)$
$\forall x \forall y \forall z(x=y \wedge y=z \dashv x=z)$
$\forall x \forall y \forall z(x=y \vdash x+z=y+z)$

$$\forall x(0+x=x)$$
$$\forall x\forall y(x+y=y+x)$$
$$\forall x\forall y\forall z((x+y)+z=x+(y+z))$$
$$\forall x\forall y(x=y\rightarrow -x=-y)$$
$$\forall x\forall y\forall z((x=y\wedge y>z\rightarrow x<z)$$
$$\forall x\forall y\forall z(x=y\wedge z<y\rightarrow z<x)$$
$$0<1$$
$$\forall x\forall y(x=y\vee x<y\vee y<x)$$
$$\forall x\forall y\neg(x<y\wedge x=y)$$
$$\forall x\forall y\neg(x<y\wedge y<x)$$
$$\forall x\forall y\forall z(x<y\wedge y<z\rightarrow x<z)$$
$$\forall x\forall y\forall z(x<y\leftrightarrow x+z<y+z)$$

4.1.2. $\forall x\Big(x=\underbrace{\frac{1}{n}(x)+\cdots+\frac{1}{n}(x)}_{n}\Big)$ for each $n=2,3,\cdots$

4.1.3. $\forall A(\forall x\neg A(x)\rightarrow \sup(x)A(x)=0)$

$$\forall A(\forall x\exists y(x\leqq y\wedge A(y))\rightarrow \sup(x)A(x)=0)$$
$$\forall A(\exists xA(x)\wedge\exists x\forall y(A(y)\rightarrow y<x)\rightarrow\forall x(A(x)\rightarrow x\leqq \sup(x)A(x))$$
$$\wedge\forall x(\forall y(A(y)\rightarrow y\leqq x)\rightarrow \sup(x)A(x)\leqq y))\,.$$

The purpose of this paragraph is to give a consistency proof of these axioms. Now 4.1.3. may be replaced by the following weaker axiom 4.1.3′. By our main theorem, the consistency of 4.1.1–4.1.3 follows namely from that of 4.1.1., 4.1.2. and 4.1.3′.

4.1.3′. $\forall A(\exists xA(x)\wedge\exists x\forall y(A(y)\rightarrow y\leqq x)\rightarrow$

$$\exists x(\forall y(\forall(y\rightarrow y\leqq x)\wedge\forall y(\forall z(A(z)\rightarrow z\leqq y)\rightarrow x\leqq y)))\,.$$

Hereafter we assume without loss of generality, that every formula is constructed from logical symbols, free variables, bound variables, $=$, $<$, $+$, $-$, $\frac{1}{n}(\ast)$ $(n=2,3,\cdots)$, 0 and 1. And we denote 4.1.1 and 4.1.2 simply by Γ_a. Then we have the following lemma.

LEMMA. *Let $A(a_1,\cdots,a_i)$ be a formula such that $A(0,\cdots,0)$ does not contain free variables. Then there exists a formula $B(a_1,\cdots,a_i)$, which* *does not contain logical symbols other than* $\wedge$, $\vee$ *and such that the following sequence is provable.*

$$\Gamma_a \rightarrow \forall x_1 \cdots \forall x_i (A(x_1, \cdots, x_i) \mapsto B(x_1, \cdots, x_i)) .$$

PROOF. We shall prove this lemma by the induction on the number of $\forall$ and $\exists$ contained in $A(a_1, \cdots, a_i)$.

If $A(a_1, \cdots, a_i)$ has no $\forall$ nor $\exists$, then the lemma is clear. Therefore we have only to prove the lemma in the case, when $A(a_1, \cdots, a_i)$ is of the form $\exists x A_0(x, a_1, \cdots, a_i)$ and $A_0(a_0, a_1, \cdots, a_i)$ contains no $\forall$ nor $\exists$ nor $\neg$. Moreover, we may assume that $A_0(a_0, a_1, \cdots, a_i)$ is of the form $A_1(a_0, a_1, \cdots, a_i) \vee \cdots \vee A_a(a_0, a_1, \cdots, a_i)$ and $A_j(a_0, a_1, \cdots, a_i)$ $(j=1, \cdots, n)$ has no logical symbol other than $\wedge$.

In this circumstance, we see easily

$$\Gamma_a \rightarrow \forall x_1 \cdots \forall x_i (A(x_1, \cdots, x_i) \mapsto \exists x A_1(x, x_1, \cdots, x_i) \vee \cdots \vee \exists x A_n(x, x_1, \cdots, x_i)) .$$

Hence we have only to prove that there exist formulas $B_j(a_1, \cdots, a_i)$ $(j=1, \cdots, n)$ which have neither $\forall$ nor $\exists$, such that the following sequences are provable for each $j (j=1, \cdots, n)$

$$\Gamma_a \rightarrow \forall x_1 \cdots \forall x_i (\exists x A_j(x, x_1, \cdots, x_i) \mapsto B_j(x_1, \cdots, x_i)) .$$

Here $A_j(a, a_1, \cdots, a_i)$ is a combination of formulas of the form $T_1 = T_2$, $T_1 < T_2$ by $\wedge$ alone.

By simple calculation, we see that formulas of the form are equivalent to some formulas of the form $a = S_1$, $a < S_2$, $S_3 < a$, $S_4 = S_5$ or $S_6 < S_7$ under Γ_a, where $S_1, S_2, \cdots, S_6$ and S_7 are terms without a.

By this reduction we can assume, without loss of generality, that $A_j(a, a_1, \cdots, a_i)$ is a combination of the form

$$a < S, \quad a = S, \quad a > S \quad \text{by} \quad \wedge .$$

Moreover, if $A_j(a, a_1, \cdots, a_i)$ contains a figure of the type $a = S$, say $a = S_0$, then the lemma is obvious; $B_j(a_1, \cdots, a_i)$ is obtained in combining

$$S_0 < S, \quad S_0 = S, \quad S_0 > S \quad \text{by} \quad \wedge .$$

So we may assume that $A_j(a, a_1, \cdots, a_i)$ is a combination of

$$a < S, \quad a > S \quad \text{by} \quad \wedge .$$

So we may assume that $\exists x A_j(x, a_1, \cdots, a_i)$ is of the form

$$\exists x(x < S_1 \wedge \cdots \wedge x < S_n \wedge x > S^1 \wedge \cdots \wedge x > S^m).$$

Let $i_1, \cdots, i_n$ be any permutation of $1, \cdots, n$; and let $j_1, \cdots, j_m$ be any permutation of $1, \cdots, m$. Then we have the sequence

$$\begin{aligned} \Gamma_a \rightarrow \exists x A_j(x, a_1, \cdots, a_i) \vdash \\ &((S_1 \leqq \cdots \leqq S_n \wedge S^1 \geqq \cdots \geqq S^m \wedge \exists x A_j(x, a_1, \cdots, a_i)) \\ &\vee \cdots\cdots \\ &\quad \cdots\cdots \\ &\vee (S_{i_1} \leqq \cdots \leqq S_{i_n} \wedge S^{j_1} \geqq \cdots \geqq S^{j_m} \wedge \exists x A_j(x, a_1, \cdots, a_i)) \\ &\quad \cdots\cdots \\ &\vee \cdots\cdots \end{aligned}$$

Hence we have only to consider the formula

$$S_1 \leqq \cdots \leqq S_n \wedge S^1 \geqq \cdots \geqq S^m \wedge \exists x A_j(x, a_1 \cdots, a_i).$$

This is equivalent to

$$S_1 \leqq \cdots \leqq S_n \wedge S^1 \geqq \cdots \geqq S^m \wedge \exists x(x < S_1 \wedge x > S^1)$$

under Γ_a.

Therefore we may restrict our considerations to the formulas of the following three types:

$$\begin{aligned} &\exists x(x < S) \\ &\exists x(x > S) \\ &\exists x(x < S_1 \wedge x > S_2). \end{aligned}$$

Since the formulas of the first and the second of these types are equivalent to $0=0$ and those of the third type are equivalent to $S_2 < S_1$ under Γ_a, our lemma is proved.

Now we shall prove that the following sequence is provable

$$\Gamma_a \rightarrow 4.1.3'$$

that is, the following sequence is provable

$$\begin{aligned} &\Gamma_a, \exists x A(x), \exists x \forall y(A(y) \vdash y < x) \\ &\quad \rightarrow \exists x(\forall y(A(y) \vdash y \leqq x) \wedge \forall y) \forall z(A(z) \vdash z \leqq y) \vdash x \leqq y)). \end{aligned}$$

We first make use of the lemma and then, in the same way as in the proof of the lemma, transform $A(x)$ to the form $B_1(x) \vee \cdots \vee B_n(x)$, where $B_i(x)$ $(i=1,\cdots,n)$ is of the form $C_i \wedge (x < S_i^1 \wedge x > S_i^2)$ and C_i has no x.

Clearly, we have only to prove that the following sequence is provable for each i $(i=1,\cdots,n)$;

$$\Gamma_a, \exists x B_i(x), \exists x \forall y(B_i(y) \vdash y \leqq x)$$
$$\rightarrow \exists x(\forall y(B_i(y) \vdash y \leqq x) \wedge \forall y(\forall z(B_i(z) \vdash z \leqq y) \vdash x \leqq y).$$

Hence we have only to prove that this sequence is provable in case, when $B_i(x)$ is of the form

$$(x < S), \quad (x > S) \quad \text{or} \quad (x < S_i^1 \wedge x > S_i^2).$$

Since this is clear, the purpose of this paragraph is attained.

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