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On the fundamental conjecture of *GLC* III.

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This paper is a continuation of [1] and [2]. We use the same notions and the notations as in these papers. See in particular [1] as to the meaning of the fundamental conjecture. We have proved this conjecture under several conditions in [1], [2]. In this paper, we shall prove it under some other conditions.

§ 1. Formulation of the theorem.

Until at the end of Appendex, the logical symbols $\exists$ and $\vee$ are not used. In this section we introduce some new notions and notations.

1.1. A formula in a proof-figure and a logical symbol in a formula

We shall speak of a 'formula in a proof-figure', when the formula is considered together with the place where it occupies in the proof-figure. Let A and B be two formulas in a proof-figure $\mathfrak{P}$. Then A is equal to B if and only if A is in the same place as B in $\mathfrak{P}$. We shall also speak of logical symbol in a formular or in a proof-figure sequence and inferences etc. in a proof-figure in analogous meanings. We use the symbols $\sharp$, $\natural$ etc. as metamathematical variables to represent logical symbols in a formula or in a proof-figure.

1.2. Semi-formula, quasi-formula.

A figure of the form $H(x,\cdots,y,\varphi,\cdots,\psi)$ with bound variables $x,\cdots,y$ and bound f-variables $\varphi,\cdots,\psi$ is called a semi-formula, if rnd only if $H(a,\cdots,b,\alpha,\cdots,\beta)$ obtained from $H(x,\cdots y,\varphi,\cdots,\psi)$ by substituting free variables $a,\cdots,b$ and free f-variables $\alpha,\cdots\beta$ for $x,\cdots,y$ and $\varphi,\cdots,\psi$ is a formula and $x,\cdots,y$, $\varphi,\cdots,\psi$ are difierent from each other and are not contained in $H(a,\cdots,b,\alpha,\cdots,\beta)$.

If $\{x,\cdots,y\}H(x,\cdots,y)$ is a formula with argument-places, then $H(x,\cdots,y)$ is clearly a semi-formula.

We use the word 'quasi-formula' as the neutral word for 'semi-formula' or 'formula with argument-places'.

1.3.

Let $\sharp$ be a logical symbol in a semi-formula $\mathfrak{A}$. Then we define:

1.3.1. If $\sharp$ is the outermost logical symbol of $\mathfrak{A}$, then $\sharp$ is positive in $\mathfrak{A}$.

1.3.2. Let $\mathfrak{A}$ be of the form $\mathfrak{B} \wedge \mathfrak{C}$. If $\sharp$ is positive in $\mathfrak{B}$ or $\mathfrak{C}$, then $\sharp$ is positive in $\mathfrak{A}$. If $\sharp$ is negative in $\mathfrak{B}$ or $\mathfrak{C}$, then $\sharp$ is negative in $\mathfrak{A}$.

1.3.3. Let $\mathfrak{A}$ be of the form $\neg \mathfrak{B}$ and $\sharp$ be not the outermost logical symbol of $\mathfrak{A}$. Then $\sharp$ is positive or negative in $\mathfrak{A}$, according as $\sharp$ is negative or positive in $\mathfrak{B}$.

1.3.4. Let $\mathfrak{A}$ be of the form $\forall x \mathfrak{B}(x)$ or $\forall \varphi \mathfrak{C}(\varphi)$ and $\sharp$ be not the outermost logical symbol of $\mathfrak{A}$. Then $\sharp$ is positive or negative in $\mathfrak{A}$, according as $\sharp$ is positive or negative in $\mathfrak{B}(x)$ or $\mathfrak{C}(\varphi)$ respectively.

Let $\sharp$ be a logical symbol in a formula with i argument-places $\{x,\cdots,y\}H(x,\cdots,y)$. Then we say that $\sharp$ is positive or negative in $\{x,\cdots,y\}H(x,\cdots,y)$ according as $\sharp$ is positive or negative in $H(x,\cdots,y)$.

Let $\sharp$ and $\natural$ be two logical symbols in a quasi-formula $\mathfrak{A}$. If $\sharp$ and $\natural$ are positive in $\mathfrak{A}$ or $\sharp$ and $\natural$ are negative in $\mathfrak{A}$, then we say that $\sharp$ is positive to $\natural$. If $\sharp$ is not positive to $\natural$, then we say that $\sharp$ is negative to $\natural$.

1.4.

Let $\mathfrak{A}$ be a quasi-formula, and $\mathfrak{B}$ be a semi-formula of the $\forall \varphi \mathfrak{C}(\varphi)$ contained in $\mathfrak{A}$ and, moreover, $\sharp$ be the outermost logical symbol of $\mathfrak{B}$. Then all the variables, f-variables, functions and logical symbols in $\mathfrak{C}(\varphi)$ are said to be 'tied by $\sharp$ in $\mathfrak{A}$'.

Let $\mathfrak{A}$ be a quasi-formula, and $\mathfrak{B}$ be a semi-formula of the form $\forall \varphi \mathfrak{C}(\varphi)$ contained in $\mathfrak{A}$ and, moreover, $\natural$ be a $\forall$ on an f-variable in $\mathfrak{C}(\varphi)$ and $\sharp$ be the outermost logical symbol of $\mathfrak{B}$. Then we say '$\sharp$ affects $\natural$', if and only if $\natural$ ties an f-variable of the form φ.

1.5.

Let $\mathfrak{A}$ be a quasi-formula and $\sharp$ be a logical symbol $\forall$ on an f-variable in $\mathfrak{A}$. $\sharp$ is called 'semi-simple in $\mathfrak{A}$', if and only if the following conditions are fulfilled:

1.5.1. If $\sharp$ ties a $\forall$ on an f-variable denoted by $\natural$, then $\natural$ is positive to $\sharp$.
1.5.2. Let $\natural$ be $\sharp$ itself or be tied by $\sharp$. Then $\natural$ does not affect, and is not affected by any $\forall$ on an f-variable.

A quasi-formula $\mathfrak{A}$ is called 'semi-simple' if and only if every $\forall$ on f-variable in $\mathfrak{A}$ is semi-simple in $\mathfrak{A}$.

Then we prove easily the following lemma by the method of [1].

LEMMA. *The end-sequence of a proof-figure, in which every implicit formula is semi-simple, is provable withbut cut.*

In fact the lemma can be still generalized. The author has in mind to publish a proof of the lemma in its generalized form in a forth coming paper.

1.6.

Let $\mathfrak{A}$ be a quasi-formula and $\sharp$ be a logical symbol $\forall$ on an f-variable in $\mathfrak{A}$. $\sharp$ is called 'simple in $\mathfrak{A}$', if and only if the following conditions are fulfilled:
1.6.1. $\sharp$ is semi-simple in $\mathfrak{A}$.
1.6.2. $\sharp$ ties no free f-variable.

A quasi-formula A is called 'simple' if and only if every $\forall$ on f-variable in $\mathfrak{A}$ is simple in $\mathfrak{A}$.

An inference left on f-variable of the following form

$$\frac{F(H),\ \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is called 'simple', if and only if H is simple.

A proof-figure $\mathfrak{P}$ is called 'simple', if and only if every implicit inference $\forall$ left on f-variable in $\mathfrak{P}$ is simple.

Now the aim of this paper is to prove the following theorem:

THEOREM. *The end-sequence of a simple proof-figure is provable without cut.*

1.7. Grade

Let $\mathfrak{A}$ be a quasi-formula. The first grade of $\mathfrak{A}$ is the number of the logical symbols $\forall$ on f-variables in $\mathfrak{A}$, which are not simple in $\mathfrak{A}$. The second grade of $\mathfrak{A}$ is the number ot the logical symbols in $\mathfrak{A}$. The grade of $\mathfrak{A}$ is the ordinal number $\omega m+n$, there m is the first grade of $\mathfrak{A}$ and n the second grade of $\mathfrak{A}$.

Now, we have several propositions concerning the grade.

1.7.1. Let H be a simple formula with i argument-places and α be a free f-variable with i argument-places. Then the first grade of $F(H)$ is not greater than the first grade of $F(\alpha)$.

PROOF. Let $\sharp$ be a $\forall$ on an f-variable in $F(H)$. If is $\sharp$ contained in H which is indicated in $F(H)$, then clearly $\sharp$ is simple. If $\sharp$ ties a free f-variable in $F(H)$, then clearly the logical symbol $\forall$ in $F(\alpha)$ corresponding to $\sharp$ ties also a free f-variable in $F(\alpha)$. If $\sharp$ affects $\natural$, then the logical symbol $\forall$ corresponding to $\sharp$ in $F(\alpha)$ affects also the $\forall$ corresponding to $\natural$ in $F(\alpha)$. Therefore the proposition is clear.

From 1.7.1 follow immediately 1.7.2. and 1.7.3.

1.7.2. Let H be a simple formula with i argument-places and $F(\alpha)$ be a simple formula and, moreover, α be a free f-variable with i argument-places. Then $F(H)$ is a simple formula.

1.7.3. Let H be a simple formula with i argument-places and $F(\alpha)$ be a not simple formula and, moreover, be a free f-variable with i argument-places. Then the first grade of $\forall\varphi F(\varphi)$ is greater than the first grade of $F(H)$. Therefore the grade of $\forall\varphi F(\varphi)$ is greater than the grade of $F(H)$.

1.7.4. Let A be an implicit simple formula in simple proof-figure $\mathfrak{P}$ and B be an ancestor of A. Then B is a simple formula.

PROOF. Without the loss of generality, we assume that A is a chief-formula of a logical inference $\mathfrak{J}$ and B is a subformula of $\mathfrak{J}$.

If the outermost logical symbol of A is $\neg$, $\wedge$ or $\forall$ on a variable, then the proposition is clear. If the outermost logical symbol of $\mathfrak{A}$ is $\forall$ on an f-variable, then the proposition follows from 1.7.1.

§2. Proof of the theorem.

All the proof-figures considered in this section are simple; we shall not mention it further.

Let $\mathfrak{P}$ be a (simple) proof-figure and $\mathfrak{J}$ be a cut in $\mathfrak{P}$. Then $\mathfrak{J}$ is called 'simple', if and only if the cut-formula of $\mathfrak{J}$ is simple. The grade of $\mathfrak{J}$ is defined as the grade of the cut-formula of $\mathfrak{J}$.

The grade of $\mathfrak{P}$ is defined as the ordinal number $\sum_{\mathfrak{J}} \omega^{\alpha_{\mathfrak{J}}}$, where $\sum$ indicates the natural sum, $\mathfrak{J}$ runs over all the cuts which are not simple in $\mathfrak{P}$, and $\alpha_{\mathfrak{J}}$ is the grade of $\mathfrak{J}$.

If the grade of $\mathfrak{P}$ is zero, then the theorem holds for $\mathfrak{P}$ by the lemma and 1.7.4. Therefore we prove the theorem by the transfinite induction on the grade of the proof-figure. Let the grade of a proof-figure $\mathfrak{P}$ be not zero. Clearly, there exists a cut $\mathfrak{J}$ in $\mathfrak{P}$ which is not simple and such that every cut above $\mathfrak{J}$ is simple. Then, as other cases are easy to treat, we can assume that $\mathfrak{J}$ is of the form

$$\frac{\Gamma \rightarrow \Delta, \forall \varphi F(\varphi) \quad \forall \varphi F(\varphi), \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda} \mathfrak{J}$$

and the proof-figure to $\Gamma, \Pi \rightarrow \Delta, \Lambda$ is denoted by $\mathfrak{P}_0$.

Let A or B be the left or the right cut-formula of $\mathfrak{J}$ respectively. Without the loss of generality, we can assume that every leading formula of A or B is not a beginning formula nor a weakening formula, and moreover the predecessor of every leading formula of A is of the form $F(\alpha)$.

Let $\mathfrak{P}_1$ be obtained from the proof-figure to $\Gamma \rightarrow \Delta, \forall \varphi F(\varphi)$ by substituting $F(\alpha)$ for each formula equivalent to A. Then, the end-sequence of $\mathfrak{P}_1$ is $\Gamma \rightarrow \Delta, F(\alpha)$.

Let $\Pi_1 \rightarrow \Lambda_1$ be an arbitrary sequence above the right upper sequence of $\mathfrak{J}$. Now, we construct, recursively as follows, a proof-figure, whose end-sequence is of the form $\Pi_1^*, \Gamma \rightarrow \Delta, \Lambda_1$ where Π_1^* is obtained from Π_1 by eliminating the formulas equivalent to B.

2.1. If $\Pi_1 \rightarrow \Lambda_1$ is a beginning sequence, then we construct the proof-figure of the form

$$\frac{\Pi_1 \rightarrow \Lambda_1}{\frac{\text{Some weakenings and exchanges}}{\Pi_1, \Gamma \rightarrow \Delta, \Lambda_1}}$$

2.2. Let $\Pi_1 \rightarrow \Lambda_1$ be the lower sequence of an inference $\mathfrak{J}_1$, and the construction of the proof-figure be defined for the upper sequence of $\mathfrak{J}_1$. We must consider the following three cases.

2.2.1. The case, where $\mathfrak{J}_1$ is a weakening, a contraction, a exchange or a cut.

As other cases are to be treated similarly, we assume that $\mathfrak{J}_1$ is of the following form

$$\frac{\Pi_2 \rightarrow \Lambda_2, D \quad D, \Pi_3 \rightarrow \Lambda_3}{\Pi_2, \Pi_3 \rightarrow \Lambda_2, \Lambda_3}$$

where $\Pi_1 \rightarrow \Lambda_1$ is $\Pi_2, \Pi_3 \rightarrow \Lambda_2, \Lambda_3$.

By the assumption, the proof-figure $\mathfrak{Q}_1$ to $\Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2, D$ and the proof-figure $\mathfrak{Q}_2$ to $D, \Pi_3^*, \Gamma \rightarrow \Delta, \Lambda_3$ are defined. Then we construct the proof-figure of the form

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\mathfrak{Q}_1\\ \Downarrow\\ \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2, D\end{array} \qquad \begin{array}{c}\vdots\,\mathfrak{Q}_2\\ \Downarrow\\ D, \Pi_3^*, \Gamma \rightarrow \Delta, \Lambda_3\end{array}}{\Pi_2^*, \Gamma, \Pi_3^*, \Gamma \rightarrow \Delta, \Lambda_2, \Delta, \Lambda_3}}{\dfrac{\text{Some exchanges and contractions}}{\Pi_2^*, \Pi_3^*, \Gamma \rightarrow \Delta, \Lambda_2, \Lambda_3}}$$

2.2.2. The case, where $\mathfrak{J}_1$ is a logical inference and the chief-formula of $\mathfrak{J}_1$ is not equivalent to B.

As other cases are to be treated similaly, we assume that $\mathfrak{J}_1$ is of the following form

$$\frac{G(X), \Pi_2 \rightarrow \Lambda_2}{\forall x G(x), \Pi_2 \rightarrow \Lambda_2}$$

where $\Pi_1 \rightarrow \Lambda_1$ is $\forall x G(x), \Pi_2 \rightarrow \Lambda_2$.

By the assumption, the proof-figure $\mathfrak{Q}_1$ to $G(X), \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2$ is defined. Then we construct the proof-figure of the form

$$\frac{\begin{array}{c}\vdots\,\mathfrak{Q}_1\\ \Downarrow\\ G(X), \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2\end{array}}{\forall x G(x), \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2}$$

2.2.3. The case, where $\mathfrak{J}_1$ is $\forall$ left on f-variable and the chief-formula of $\mathfrak{J}_2$ is equivalent to B.

Without the loss of generality, we assume $\mathfrak{J}_1$ is of the following form

$$\frac{F(H), \Pi_2 \rightarrow \Lambda_2}{\forall \varphi F(\varphi), \Pi_2 \rightarrow \Lambda_2}$$

where $\Pi_1 \rightarrow \Lambda_1$ is $\forall \varphi F(\varphi), \Pi_2 \rightarrow \Lambda_2$.

By the assumption, the proof-figure $\mathfrak{Q}_2$ to $F(H), \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2$ is defined. The we construct the proof-figure of the form

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ \mathfrak{Q}_1 \\ \Gamma \rightarrow \Delta, F(H)\end{array} \qquad \begin{array}{c}\vdots\ \mathfrak{Q}_2 \\ F(H), \Pi_2^*, \Gamma \rightarrow \Delta, \Lambda_2\end{array}}{\Gamma, \Pi_2^*, \Gamma \rightarrow \Delta, \Delta, \Lambda_2}}{\cfrac{\text{Some exchanges and contractions}}{\Pi_2^*, \Gamma \rightarrow, \Delta\ \Lambda_2}}$$

where $\mathfrak{Q}_1$ is obtained from $\mathfrak{P}_1$ by substituting H for α after the necessary changes of eigen-variables in $\mathfrak{P}_1$.

By successive constructions 2.2.1, 2.2.2 and 2.2.3, we can form a proof-figure $\mathfrak{Q}_0$ to $\Pi, \Gamma \rightarrow \Delta, \Lambda$. Now, we construct the proof-figure $\mathfrak{Q}_0'$ of the following form

$$\cfrac{\begin{array}{c}\vdots\ \mathfrak{Q}_0 \\ \Pi, \Gamma \rightarrow \Delta, \Lambda\end{array}}{\cfrac{\text{Some exchanges}}{\Gamma, \Pi \rightarrow \Delta, \Lambda}}$$

Then we see easily by 1.7.3, that the grade of $\mathfrak{Q}_0'$ is less than the grade of $\mathfrak{P}_0$.

Let $\mathfrak{Q}$ be the proof-figure obtained from $\mathfrak{P}$ by substituting $\mathfrak{Q}_0'$ for $\mathfrak{P}_0$. Then clearly $\mathfrak{Q}$ is a simple proof-figure and the grade of $\mathfrak{Q}$ is less than the grade of $\mathfrak{P}$. Therefore the theorem is proved.

§ Appendix

A.1. A function $\gamma(A)$ of the formula or the formula with argument-places taking ordinal numbers as values will be called monotone if it fulfills the following conditions:

A.1.1. $\gamma(\neg A) \geqq (A)$.

A.1.2. $\gamma(A \wedge B) \geqq \max(\gamma(A), \gamma(B))$.

A.1.3. $\gamma(\forall x G(x)) \geqq \gamma(G(X))$.

A.1.4. $\gamma(\{x_L, \cdots, x_i\} H(x_L, \cdots, x_i)) = \gamma(H((X_1, \cdots, X_i))$.

A.1.5. If A is homologous to B, then $\gamma(A)$ is equal to $\gamma(B)$.

A.1.6. If $\gamma(H)=0$ and $\gamma(\forall \varphi F(\varphi))>0$, then $\gamma(\forall \varphi F(\varphi))>\gamma(F(H))$.

We say that A is γ-simple, if and only if $\gamma(A)=0$. An inference $\forall$ left on f-variable

$$\frac{F(H), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$$

is called γ-simple, if H is γ-simple, it is called strictly γ-simple, if H and $\forall \varphi F(\varphi)$ are γ-simple. A proof-figure $\mathfrak{P}$ is called (strictly) γ-simple, if every implicit inference $\forall$ left on f-variable in $\mathfrak{P}$ is (strictly) -simple.

A.2. In the same way as in § 2, we have then the following proposition:

If γ is monotone and the fundamental conjecture is verified for every strictly γ-simple proof-figure, then the fundamental conjecture is verified for every γ-simply proof-figure.

A.3. Let us suppose that a set $\mathfrak{M}$ of formulas and formulas with argument-places is given, and that $\mathfrak{M}$ is 'closed' in the following some.

A.3.1. If $\forall x G(x)$ belongs to $\mathfrak{M}$, then $G(X)$ belongs to $\mathfrak{M}$.

A.3.2. If $B \wedge C$ belongs to $\mathfrak{M}$, then B and C belong to $\mathfrak{M}$.

A.3.3. If $\neg B$ belongs to $\mathfrak{M}$, then B belongs to M.

A.3.4. If $\forall \varphi F(\varphi)$ belongs to $\mathfrak{M}$, then $F(\alpha)$ belongs to $\mathfrak{M}$.

A.3.5. $\{x_1, \cdots, x_i\} H(x_1, \cdots, x_i)$ belongs to $\mathfrak{M}$, if and only if $H(X_1, \cdots, X_i)$ belongs to $\mathfrak{M}$.

A.3.6. If B is homologous to C and B belongs to $\mathfrak{M}$, then C belongs to $\mathfrak{M}$.

A.3.7. If $F(\alpha)$ and H belongs to $\mathfrak{M}$ and the types of α and H are the same, then $F(H)$ belongs to $\mathfrak{M}$.

A.3.8. If A has no logical symbol, then A belongs to $\mathfrak{M}$.

A.4. Now let us define a function γ recursively as follows, and call it 'the function determined by $\mathfrak{M}$':

A.4.1. $\gamma(A)$ is equal to zero, if and only if A belongs to $\mathfrak{M}$.

A.4.2. If A is of the form $\neg B$ and does not belong to $\mathfrak{M}$, then $\gamma(A)$ is equal to $\gamma(B)+1$.

A.4.3. If A is of the form $B \wedge C$ and does not belong to $\mathfrak{M}$, then $\gamma(A)$ is $n+1$, where n is the maximum of $\gamma(B)$ and $\gamma(C)$.

A.4.4. If A is of the form $\forall x G(x)$ and does not belong to $\mathfrak{M}$, then $\gamma(A)$ is equal to $\gamma(G(a))+1$.

A.4.5. If A is of the form $\{x_1, \cdots, x_i\} H(x_1, \cdots, x_i)$, then $\gamma(A)$ is equal to $\gamma(H(a_1, \cdots, a_i))$.

A.4.6. If A is of the form $\forall \varphi F(\varphi)$ and does not belong to $\mathfrak{M}$, then $\gamma(A)$ is equal to $\gamma(F(\alpha))+1$.

A.5. We shall prove the following proposition:

Let $\mathfrak{M}$ be closed and γ be the function determined by $\mathfrak{M}$. If H belongs to $\mathfrak{M}$ and has the same type as α, then $\gamma(F(\alpha))$ is equal to $\gamma(F(H))$.

PROOF. If $\gamma(F(\alpha))=0$, the proposition is clear. Let us proceed by the mathematical induction on $a+b$, where a is $\gamma(F(\alpha))$ and b is the number of logical symbols in $F(\alpha)$. We have several cases according to the kind of the outermost logical symbol of $F(\alpha)$, but, as all cases are treated similarly we deal only with the case, where $F(\alpha)$ is of the form $\forall \varphi G(\varphi, \alpha)$. Then, by the hypothesis of the induction, $\gamma(G(\beta, \alpha))$ is equal to $\gamma(G(\beta, H))$, and we see easily that $\gamma(\forall \varphi G(\varphi, \alpha))$ is equal to $\gamma(\forall \varphi G(\varphi, H))$. Q. E. D.

A.6. From the above proposition follows immediately the following proposition:

Let $\mathfrak{M}$ be closed and γ be the function determined by $\mathfrak{M}$. Then γ is monotone.

A.7. Now we shall give several examples of sets of formulas and formulas with argument-places, which are easily seen to be closed.

A.7.1. The first example $\mathfrak{M}_1$.

We define that belongs to $\mathfrak{M}_1$, if and only if every $\forall$ on f-variable in A affects no $\forall$ on f-variable in A.

A.7.2. The second example $\mathfrak{M}_2$.

We define that A belongs to $\mathfrak{M}_2$, if and only if the following condition is fulfilled:

Let $\sharp$ and $\natural$ be $\forall$ on f-variables in A and let $\sharp$ affect $\natural$. Then $\sharp$ is positive to $\natural$, and, moreover, if $\flat$ is an arbitrary $\forall$ on f-variable, which is tied by $\sharp$ and ties $\natural$, then $\flat$ is positive to $\sharp$.

A.7.3. The third example $\mathfrak{M}_3$.

We define that A belongs to $\mathfrak{M}_3$, if and only if A contains no logical symbol $\forall$ on any variable.

Let γ_3 be the function determined by $\mathfrak{M}_3$. Then from our former paper [2] follows that the fundamental conjecture is verified for the strictly γ_3-simple proof-figure. Therefore by A.2 we have the following theorem:

THEOREM 2. *Let $\mathfrak{P}$ be a proof-figure satisfying the following condition: If*

$$\frac{F(H),\ \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is an implicit $\forall$ left on f-variable in $\mathfrak{P}$, then H has no $\forall$ on variable. Then the end-sequence of $\mathfrak{P}$ is provable without cut.

Hereafter, we use the logical symbol $\exists$ and $\forall$. Accordingly, we define that $\mathfrak{M}$ is closed, if and only if $\mathfrak{M}$ satisfies A.3.1–A.3.8 and the following conditions:

A.3.9. If $B \vee C$ belongs to $\mathfrak{M}$, then B and C belong to $\mathfrak{M}$.

A.3.10. If $\exists x G(x)$ belongs to $\mathfrak{M}$, then $G(X)$ belongs to $\mathfrak{M}$.

A.3.11. If $\exists \varphi F(\varphi)$ belongs to $\mathfrak{M}$, then $F(\alpha)$ belongs to $\mathfrak{M}$.

The concept of 'function determined by $\mathfrak{M}$' should be also modified accordingly.

A.7.4. The fourth example $\mathfrak{M}_4$.

We define that A belong to $\mathfrak{M}_4$, if and only if A does not contain the logical symbol $\neg$.

Let γ_4 be the function determined by $\mathfrak{M}_4$. We see easily that the fundamental conjecture holds for the strictly γ_4-simple proof-figure. (the author intends to prove a theorem, implying this as a special case in a forth coming paper). Therefore by A.2, we have the following theorem:

THEOREM 3. *Let $\mathfrak{P}$ be a proof-figure satisfying the following condition: If*

$$\frac{F(H),\ \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is an implicit $\forall$ left on f-variable in $\mathfrak{P}$, then H has no $\neg$. Then the end-sequence of $\mathfrak{P}$ is provable without cut.

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