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Some classes of meromorphic functions with assigned zeros and poles.

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§ I. Introduction.

Recently Goodman [1], investigating the class of functions $T(p)$, the set of typically-real functions of order p, has obtained the following result:

THEOREM A. *Let*

(1.1) $$f(z)=z^{q}+\sum_{n=q+1}^{\infty}c_{n}z^{n}$$

be a function of the class $T(p)$. *Suppose that in addition to the q-th order zero at* $z=0$, $f(z)$ *has exactly* l *zeros,* $a_{1}, a_{2},\cdots, a_{l}$, *such that* $0<|a_{j}|<1$, $j=1,2,\cdots, l$. *Then*

(1.2) $$f(z)\ll\frac{z^{q}}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^{2[(t+1)/2]}\cdot N(z),$$

(1.3) $$|f(re^{i\theta})|\leqq\frac{r^{q}}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^{2[(t+1)/2]}\cdot N(r), \qquad 0\leqq r<1,$$

where

(1.4) $$t=p-(q+l),$$

(1.5) $$N(z)=\prod_{j=1}^{l}(1+|a_{j}|z)\left(1+\frac{z}{|a_{j}|}\right).$$

Relating to this theorem Umezawa [2] has shown that when $t=0$, the estimates (1.2), (1.3) hold also for the functions belonging to a wider class of functions $D(p)$, which is the set of regular functions of order p in the direction of a diametral line. But he has not touched the case $t>0$. In this paper we shall show that also for the case $t>0$ much the same estimates as (1.2), (1.3) hold. But they will be given as corollaries of results for certain more general functions. (Corollaries 2, 2a, 2b).

The purpose of this note is to introduce some new classes of functions related to $T(p)$, $D(p)$ and to investigate their properties similar to (1.2), (1.3).

§ 2. Functions of order p in all in the directions of two rays. Main theorem.

DEFINITION 1. Let $f(z)$ be meromorphic in $|z|\leqq 1$ and let $f(z)\neq 0, \infty$ for $|z|=1$. If there exist such two rays starting from the origin as are crossed by $f(e^{i\theta})$ $2p$ times in all as $e^{i\theta}$ traverses the boundary of the unit circle, then $f(z)$ is said to be of order p in all in the directions of the two rays. If further the minor angle formed by the two rays is equal to α, then $f(z)$ is said to belong to the class $M(p, \alpha)$.

DEFINITION 2. In Definition 1 a point $f(e^{i\theta_0})$ lying on either of the two rays is called a positively or negatively cutting point according as at the point the angular velocity of $f(e^{i\theta})$ about the origin is positive or negative when $e^{i\theta}$ traverses the boundary of the unit circle in the positive direction.

DEFINITION 3. In Definition 1 if particularly $f(z)$ is regular in $|z|\leqq 1$ and the $2p$ points at which $f(e^{i\theta})$ crosses the two rays are all positively cutting points, then $f(z)$ is said to be starlike of order p in the directions of the two rays. If further the minor angle formed by the two rays is equal to α, then $f(z)$ is said to belong to the class $S(p, \alpha)$.

The two special cases that $\alpha=\pi$ and $\alpha=0$ will be discussed particularly in detail in §§ 3 and 4.

LEMMA. *Let $f(z)$ be a function of the class $M(p, \alpha)$ which has exactly j zeros and exactly k poles in $|z|<1$ and let $t=p-(j-k)$, then $f(z)$ has exactly $2p-t$ positively cutting points and exactly t negatively cutting points.*

PROOF. Let μ, ν denote the numbers of the positively and the negatively cutting points respectively. Then it can easily be seen that the increment of $\arg f(z)$ is $(\mu-\nu)\pi$ as z traverses the circle

$|z|=1$ in the positive direction. On the other hand, as is well known, the same quantity is equal to $2(j-k)\pi$. Consequently

$$\mu-\nu=2(j-k), \qquad \mu+\nu=2p.$$

Hence $$\mu=p+(j-k)=2p-t, \qquad \nu=p-(j-k)=t.$$

THEOREM. *Let $f(z)$ be a function of the class $M(p, \lambda\pi)$ having the expansion*

$$f(z)=z^q+\sum_{n=q+1}^{\infty} c_n z^n \quad (q: \text{ an integer, positive, negative or zero}) \tag{2.1}$$

in a neighbourhood of the origin. Suppose that $f(z)$ has, in $0<|z|<1$, exactly l zeros, $a_1, a_2, \cdots, a_l$, and exactly m poles, $b_1, b_2, \cdots, b_m$. Finally let $f(e^{i\theta_\mu})$, $\mu=1, 2, \cdots, 2p-t$, and $f(e^{i\varphi_\nu})$, $\nu=1, 2, \cdots, t$, denote the positively and the negatively cutting points respectively, where

$$t=p-(q+l-m) \tag{2.2}$$

necessarily holds in virtue of the lemma. Then

$$r^q\left(\frac{1-r}{1+r}\right)^{2-\lambda}\min_{|z|=r}|A(z)B(z)| \leqq |f(re^{i\theta})| \leqq r^q\left(\frac{1+r}{1-r}\right)^{2-\lambda}\max_{|z|=r}|A(z)B(z)|, \tag{2.3}$$

$$0\leqq r<1,$$

$$\{z^{-q}f(z)/A(z)B(z)\}^{1/(2-\lambda)} \ll (1+z)/(1-z), \tag{2.4}$$

where

$$A(z)=\prod_{j=1}^{l}(1-\bar{a}_j z)\left(1-\frac{z}{a_j}\right)\Big/\prod_{k=1}^{m}(1-\bar{b}_k z)\left(1-\frac{z}{b_k}\right), \tag{2.5}$$

$$B(z)=\prod_{\nu=1}^{t}(e^{i\varphi_\nu}-z)\Big/\prod_{\mu=1}^{2p-t}(e^{i\theta_\mu}-z). \tag{2.6}$$

PROOF. Let us suppose that $f(z)$ is of order p in all in the directions of two rays OX, OY such that $\angle XOY=\lambda\pi$. Set

$$g(z)=f(z)\left(\prod_{k=1}^{m} b_k\Big/\prod_{j=1}^{l} a_j\right)(-z)^{l-m}/A(z)$$

$$=f(z)\prod_{j=1}^{l}\frac{z}{(z-a_j)(1-\bar{a}_j z)}\prod_{k=1}^{m}\frac{(z-b_k)(1-\bar{b}_k z)}{z},$$

then $g(z)$ is meromorphic in $|z|\leqq 1$ and has no zeros and no poles in $0<|z|\leqq 1$. Moreover $\arg g(z)=\arg f(z)$ for $|z|=1$, because

$$\frac{z}{(z-a_j)(1-\bar{a}_j z)}=\frac{1}{|1-\bar{a}_j z|^2}>0\,,\quad \frac{(z-b_k)(1-\bar{b}_k z)}{z}=|1-\bar{b}_k z|^2>0$$

$$\text{on } |z|=1\,.$$

Therefore $g(z)$ is also of order p in all in the directions of OX, OY and further the points $g(e^{i\theta_\mu})$, $\mu=1,2,\cdots,2p-t$, and the points $g(e^{i\varphi_\nu})$, $\nu=1,2,\cdots,t$, are also the positively and the negatively cutting points of $g(z)$ respectively. Next set

$$h(z)=(-1)^{p-t}\exp\left[\frac{i}{2}\left(\sum_{\nu=1}^{t}\varphi_\nu-\sum_{\mu=1}^{2p-t}\theta_\mu\right)\right]z^{t-p}/B(z)\,,$$

then, since

$$h(e^{i\theta})=2^{2(p-t)}\prod_{\mu=1}^{2p-t}\sin\frac{\theta_\mu-\theta}{2}\prod_{\nu=1}^{t}\operatorname{cosec}\frac{\varphi_\nu-\theta}{2}\,,$$

$h(z)$ is real on $|z|=1$ except at each $e^{i\varphi_\nu}$, at which $h(z)=\infty$, and changes its sign at each of $e^{i\theta_\mu}$, $e^{i\varphi_\nu}$ as z traverses the circle $|z|=1$. We can choose therefore a suitable sign σ, $+$ or $-$, so that $\sigma h(z)$ may be positive for every z ($|z|=1$) at which $g(z)$ takes a value in the major angle XOY and may be negative for every z ($|z|=1$) at which $g(z)$ takes a value in the minor angle XOY.

Let us lastly, for this σ, put

$$F(z)=\sigma h(z)\,g(z)\,,$$

then $F(z)$ is regular and has no zeros in $|z|<1$, and further it is also regular on $|z|=1$ except at each $e^{i\varphi_\nu}$ and takes no values interior to the minor angle XOY. Moreover the image curve of the circle $|z|=1$ by $F(z)$ (hereafter we denote it by C) touches OX or OY at the points $F(e^{i\theta_\mu})=0$, $\mu=1,2,\cdots,2p-t$, and at the points $F(e^{i\varphi_\nu})=\infty$, $\nu=1,2,\cdots,t$. To see the manner of C touching at these points, we prolong OX, OY in the opposite directions and denote these prolongations by OX', OY', and further denote the four minor angles XOY, YOX', $X'OY'$, $Y'OX$ by D_1, D_2, D_3, D_4 respectively. We may here assume that D_1, D_2, D_3 and D_4 lie in the counter-clockwise order.

Under this arrangement we can easily see, from the above-mentioned properties of $g(z)$ and $h(z)$, that
(1) if $g(e^{i\theta_\mu})$ lies on OX, then when $e^{i\theta}$ passes $e^{i\theta_\mu}$, C runs from D_4 into D_3, touching XX' at the origin,
(2) if $g(e^{i\theta_\mu})$ lies on OY, then when $e^{i\theta}$ passes $e^{i\theta_\mu}$, C runs from D_3 into D_2, touching $Y'Y$ at the origin,
(3) if $g(e^{i\varphi_\nu})$ lies on OX, then when $e^{i\theta}$ passes $e^{i\varphi_\nu}$, C runs from D_3 into D_4, touching $X'X$ at the point at infinity,
(4) if $g(e^{i\varphi_\nu})$ lies on OY, then when $e^{i\theta}$ passes $e^{i\varphi_\nu}$, C runs from D_2 into D_3, touching YY' at the point at infinity,
provided that $e^{i\theta}$ traverses the circle $|z|=1$ in the positive direction.

Therefore, for every $\rho(<1)$ sufficiently close to 1, $F(\rho e^{i\theta})$ lies within the major angle XOY, which is equal to $(2-\lambda)\pi$. Consequently it holds for a branch of $F(z)^{1/(2-\lambda)}$ and for a real α suitably chosen that

$$\Re\{e^{i\alpha}F(\rho e^{i\theta})^{1/(2-\lambda)}\}>0$$

for every $\rho(<1)$ sufficiently close to 1. On the other hand $\Re\{e^{i\alpha}F(z)^{1/(2-\lambda)}\}$ is harmonic in $|z|<1$ because of the properties of $F(z)$ stated above. Hence

$$\Re\{e^{i\alpha}F(z)^{1/(2-\lambda)}\}>0 \qquad \text{for} \qquad |z|<1,$$

which yields

$$|F(0)|^{1/(2-\lambda)}\frac{1-r}{1+r}\leqq|F(re^{i\theta})|^{1/(2-\lambda)}\leqq|F(0)|^{1/(2-\lambda)}\frac{1+r}{1-r}, \quad 0\leqq r<1, \tag{2.7}$$

$$F(z)^{1/(2-\lambda)}\ll|F(0)|^{1/(2-\lambda)}\frac{1+z}{1-z}. \tag{2.8}$$

Thus we obtain (2.3), (2.4) respectively by substituting

$$F(z)=(-1)^q\sigma\exp\left[\frac{i}{2}\left(\sum_{\nu=1}^{t}\varphi_\nu-\sum_{\mu=1}^{2p-t}\theta_\mu\right)\right]\left(\prod_{k=1}^{m}b_k\Big/\prod_{j=1}^{l}a_j\right)z^{-q}f(z)/A(z)\,B(z)$$

in (2.7), (2.8). This proves the theorem.

COROLLARY 1. *Under the hypotheses of the theorem we have*

$$(2.9)\quad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^{t+2-\lambda}\frac{|N(-r)|}{P(r)}\leqq|f(re^{i\theta})|$$

$$\leqq\frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^{t+2-\lambda}\frac{N(r)}{|P(-r)|},\qquad 0\leqq r<1,$$

$$(2.10)\quad |c_{q+1}|\leqq 2(p+2-\lambda)+\sum_{j=1}^{l}\left(|a_j|+\frac{1}{|a_j|}\right)+\sum_{k=1}^{m}\left(|b_k|+\frac{1}{|b_k|}\right),$$

where

$$(2.11)\quad N(z)=\prod_{j=1}^{l}(1+|a_j|z)\left(1+\frac{z}{|a_j|}\right),\quad P(z)=\prod_{k=1}^{m}(1+|b_k|z)\left(1+\frac{z}{|b_k|}\right).$$

PROOF. It is clear that

$$|N(-r)|/P(r)\leqq|A(re^{i\theta})|\leqq N(r)/|P(-r)|,$$

$$(1-r)^t/(1+r)^{2p-t}\leqq|B(re^{i\theta})|\leqq(1+r)^t/(1-r)^{2p-t}.$$

Hence (2.9) follows from (2.3). Next when $|z|=r\to 0$, we have

$$|f(z)/z^q|=|1+c_{q+1}z|+O(r^2).$$

On the other hand from (2.9)

$$|f(z)/z^q|\leqq 1+\left\{2(p+2-\lambda)+\sum_{j=1}^{l}\left(|a_j|+\frac{1}{|a_j|}\right)+\sum_{k=1}^{m}\left(|b_k|+\frac{1}{|b_k|}\right)\right\}r+O(r^2).$$

Hence (2.10) holds.

COROLLARY 1a. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty}c_nz^n$ *be a function of the class* $S(p,\lambda\pi)$, *then*

$$(2.12)\quad \frac{r^p}{(1+r)^{2p}}\left(\frac{1-r}{1+r}\right)^{2-\lambda}\leqq|f(re^{i\theta})|\leqq\frac{r^p}{(1-r)^{2p}}\left(\frac{1+r}{1-r}\right)^{2-\lambda},\quad 0\leqq r<1,$$

$$(2.13)\quad |c_{p+1}|\leqq 2(p+2-\lambda).$$

PROOF. In this case $f(z)$ has no zeros and no poles in $0<|z|\leqq 1$, and $q=p$. Therefore both $N(z)$ and $P(z)$ disappear and t vanishes. Thus (2.12), (2.13) follow at once from (2.9), (2.10).

3. Functions of order p in the direction of one straight line.

DEFINITION 4. Let $f(z)$ be a function of the class $M(p, \pi)$, then $f(z)$ is said to be of order p in the direction of one straight line. Because in this case the two rays in Definition 1 form one whole straight line.

The classes $T(p)$, $D(p)$ are special ones of $M(p, \pi)$, and the idea of $T(p)$ was first introduced by Rogosinski [12] for $p=1$ and extended to the case of general p by Robertson [4, 5], and that of $D(p)$ was first introduced by Ozaki [9] for $p=1$ and extended to the case of general p by Ozaki, Umezawa, and Takatsuka [6], [13]. Moreover, as is well known, a function $f(z)$ belonging to the class $S(p, \pi)$ is said to be starlike of order p in the direction of one straight line, and the idea of $S(p, \pi)$ was first introduced by Robertson [3] for $p=1$ and extended to the case of general p by him [4].

DEFINITION 5. Let $f(z)$ be meromorphic in $|z|\leqq 1$, and let a straight line $\overline{f(\zeta), f(-\zeta)}$, $|\zeta|=1$, be a diametral line of $f(z)$, then the segment $[f(\zeta), f(-\zeta)]$ is called a diametral segment and the points $f(\zeta), f(-\zeta)$ are called the ends of the diametral segment.

REMARK. The idea of diametral line was introduced by Ozaki [9, 10] and DeBruijn [11] and was studied in detail by Ozaki, Umezawa, and Takatsuka [6] and Umezawa [2]. It was shown by them that if $f(z)$ is regular in $|z|\leqq 1$, then there exists a point $\zeta(|\zeta|=1)$ for which the three points $f(\zeta)$, 0, and $f(-\zeta)$ lie on one straight line. Similarly we can easily see that there exists such a diametral line also for a function meromorphic in $|z|\leqq 1$.

COROLLARY 2. *In the theorem, let $f(z)$ be a function of the class $M(p, \pi)$ then*

$$\text{(3.1)} \quad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^{t+1}\frac{|N(-r)|}{P(r)} \leqq |f(re^{i\theta})|$$

$$\leqq \frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^{t+1}\frac{N(r)}{|P(-r)|}, \quad 0\leqq r<1,$$

$$\text{(3.2)} \quad f(z) \ll \frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^{t+1}\frac{N(z)}{P(-z)},$$

where t, $N(z)$, *and* $P(z)$ *are defined by* (2.2), (2.11), *and* $f(z)$ *of* (3.2) *denotes the expansion* (2.1).

A part of this corollary has been obtained by Ito [8].

PROOF. Substituting $\lambda=1$ in (2.9), we immediately obtain (3.1). Next, from (2.4) we have

$$f(z)\ll z^q\frac{1+z}{1-z}\operatorname{maj}[A(z)\,B(z)].$$

where $\operatorname{maj}[A(z)\,B(z)]$ denotes a majorant of $[A(z)\,B(z)]$. On the other hand obviously

$$A(z)\ll N(z)/P(-z)\,,\qquad B(z)\ll(1+z)^t/(1-z)^{2p-t}.$$

Hence (3.2) holds.

COROLLARY 2a. *In the theorem let* $f(z)$ *be of order* p *in the direction of a diametral line and let further the ends of the diametral segment be both positively cutting points, then*

$$\text{(3.3)}\qquad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^t\frac{1-r^2}{1+r^2}\frac{|N(-r)|}{P(r)}\leqq|f(re^{i\theta})|$$

$$\leqq\frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^t\frac{N(r)}{|P(-r)|}\,,\qquad 0\leqq r<1\,,$$

$$\text{(3.4)}\qquad f(z)\ll\frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^t\frac{N(z)}{P(-z)}\,.$$

PROOF. Let $f(e^{i\theta_{\mu_1}})$, $f(e^{i\theta_{\mu_2}})$ be the ends of the diametral segment, then $e^{i\theta_{\mu_1}}=-e^{i\theta_{\mu_2}}$. Therefore in this case $B(z)$ has the following form:

$$B(z)=\prod_{\nu=1}^{t}(e^{i\varphi_\nu}-z)/(-e^{2i\theta_{\mu_1}}+z^2)\prod_{\substack{\mu=1\\ \mu\neq\mu_1,\mu_2}}^{2p-t}(e^{i\theta_\mu}-z)\,.$$

Hence

$$\frac{(1-r)^t}{(1+r^2)(1+r)^{2p-t-2}}\leqq|B(re^{i\theta})|\leqq\frac{(1+r)^t}{(1-r^2)(1-r)^{2p-t-2}}\,,$$

$$B(z)\ll\frac{(1+z)^t}{(1-z^2)\,(1-z)^{2p-t-2}}\,.$$

Applying these inequalities to (2.3), (2.4), with λ's replaced by 1, we obtain (3.3), (3.4) respectively.

Similarly we have the following corollary.

COROLLARY 2b. *In the theorem let $f(z)$ be of order p in the direction of a diametral line and let further the ends of the diametral segment be both negatively cutting points, then*

$$(3.5)\qquad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^t \frac{|N(-r)|}{P(r)} \leqq |f(re^{i\theta})|$$

$$\leqq \frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^t \frac{1+r^2}{1-r^2}\frac{N(r)}{|P(-r)|}, \qquad 0\leqq r<1,$$

$$(3.6)\qquad f(z)\ll \frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^t \frac{1+z^2}{1-z^2}\frac{N(z)}{P(-z)}.$$

COROLLARY 2c. *In the theorem let $f(z)$ be starlike of order p in the direction of a diametral line, then*

$$(3.7)\qquad \frac{r^q}{(1+r)^{2p}}\frac{1-r^2}{1+r^2}|N(-r)|\leqq|f(re^{i\theta})|\leqq\frac{r^q}{(1-r)^{2p}}N(r),\ 0\leqq r<1,$$

$$(3.8)\qquad f(z)\ll\frac{z^q}{(1-z)^{2p}}N(z).$$

(3.8) and the right-hand inequality of (3.7) are the results of Umezawa [2] mentioned in § 1.

PROOF. In this case $f(z)$ is regular for $|z|\leqq 1$ and has exactly $p-q$ zeros, $a_1, a_2,\cdots, a_{p-q}$, in $0<|z|<1$, Therefore $P(z)$ disappears and t vanishes. Moreover both the ends of the diametral segment are positively cutting points. Thus by Corollary 2a we obtain (3.7), (3.8).

Let us now compare the result of Theorem A with ours.

In Theorem A, $f(z)$ is of order p in the direction of the real axis, which is a diametral line of $f(z)$, and the points $f(1)$, $f(-1)$ are the ends of the diametral segment. It can easily be verified that when t is odd, one of the points $f(1)$, $f(-1)$ is positively cutting point and the other is a negatively cutting one, and when t is even, the points $f(1)$, $f(-1)$ are both positively cutting ones or are both negatively cutting ones.

Therefore (1.2), (1.3) for an odd t are contained in our result, Corollary 2. Moreover when t is even and both the points $f(1)$, $f(-1)$

are positively cutting ones, (1.2) and (1.3) are contained in our result, Corollary 2*a*. But when t is even and both the points $f(1)$, $f(-1)$ are negatively cutting ones, (1.2) and (1.3) are not contained in our result, Corollary 2b.

The following two corollaries are special ones of Corollaries 2 and 2c.

COROLLARY 2d. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty}c_n z^n$ *be a function of the class* $S(p,\pi)$, *then*

$$\frac{r^p}{(1+r)^{2p}}\frac{1-r}{1+r}\leqq|f(re^{i\theta})|\leqq\frac{r^p}{(1-r)^{2p}}\frac{1+r}{1-r},\quad 0\leqq r<1, \tag{3.9}$$

$$f(z)\ll\frac{z^p}{(1-z)^{2p}}\frac{1+z}{1-z}. \tag{3.10}$$

This is the result due to Robertson [4] and Ito [7].

COROLLARY 2e. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty}c_n z^n$ *be starlike of order* p *in the direction of a diametral line, then*

$$\frac{r^p}{(1+r)^{2p}}\frac{1-r^2}{1+r^2}\leqq|f(re^{i\theta})|\leqq\frac{r^p}{(1-r)^{2p}},\quad 0\leqq r<1, \tag{3.11}$$

$$f(z)\ll\frac{z^p}{(1-z)^{2p}}. \tag{3.12}$$

(3.12) is the result due to Ozaki, Umezawa, and Takatsuka [6].

4. Functions of order p in the direction of one ray.

DEFINITION 6. Let $f(z)$ be a function of the class $M(p,0)$, then $f(z)$ is said to be of order p in the direction of one ray. Because in this case the two rays in Definition 1 coincide with each other and $f(e^{i\theta})$ crosses the ray exactly p times as $e^{i\theta}$ traverses the circle $|z|=1$. Similarly a function $f(z)$ belonging to the class $S(p,0)$ is said to be starlike of order p in the direction of one ray.

COROLLARY 3. *In the theorem, let* $f(z)$ *be a function of the class* $M(p,0)$ *then*

$$(4.1)\qquad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^{t+2}\frac{|N(-r)|}{P(r)}\leqq|f(re^{i\theta})|$$

$$\leqq\frac{r^p}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^{t+2}\frac{N(r)}{|P(-r)|},\quad 0\leqq r<1,$$

$$(4.2)\qquad f(z)\ll\frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^{t+2}\frac{N(z)}{P(-z)},$$

where $N(z)$, $P(z)$ *are defined by* (2.11).

PROOF. Substituting $\lambda=0$ in (2.9), we immediately obtain (4.1). Next, from (2.4)

$$f(z)\ll z^q\left(\frac{1+z}{1-z}\right)^2\operatorname{maj}[A(z)\,B(z)],$$

which yields (4.2).

COROLLARY 3a. *In the theorem let* $f(z)$ *be of order* p *in the direction of a ray containing a diametral segment and let further the ends of the diametral segment be both positively cutting points, then*

$$(4.3)\qquad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^{t}\left(\frac{1-r^2}{1+r^2}\right)^{2}\frac{|N(-r)|}{P(r)}\leqq|f(re^{i\theta})|$$

$$\leqq\frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^{t}\frac{N(r)}{|P(-r)|},\qquad 0\leqq r<1,$$

$$(4.4)\qquad f(z)\ll\frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^{t}\frac{N(z)}{P(-z)}.$$

PROOF. In this case, since the two rays in the theorem coincide with each other, we now have

$$e^{i\theta_\mu}=e^{i\theta_{\mu+1}},\quad \mu=1,3,\cdots,2p-t-1,$$

$$e^{i\varphi_\nu}=e^{i\varphi_{\nu+1}},\quad \nu=1,3,\cdots,t-1.$$

Therefore, if $f(e^{i\theta_{\mu_1}})$, $f(e^{i\theta_{\mu_2}})$ are the ends of the diametral segment, then

$$e^{i\theta_{\mu_1}}=e^{i\theta_{\mu_1+1}}=-e^{i\theta_{\mu_2}}=-e^{i\theta_{\mu_2+1}}.$$

Accordingly $B(z)$ has the following form:

$$B(z)=\prod_{\nu=1}^{t}(e^{i\varphi_\nu}-z)/(-e^{2i\theta_{\mu_1}}+z^2)^2 \prod_{\substack{\mu=1\\ \mu\neq\mu_1,\mu_1+1,\mu_2,\mu_2+1}}^{2p-t}(e^{i\theta_\mu}-z).$$

Hence

$$\frac{(1-r)^t}{(1+r^2)^2(1+r)^{2p-t-4}}\leqq|B(re^{i\theta})|\leqq\frac{(1+r)^t}{(1-r^2)^2(1-r)^{2p-t-4}},$$

$$B(z)\ll\frac{(1+z)^t}{(1-z^2)^2(1-z)^{2p-t-4}}.$$

Thus from (2.3), (2.4), with λ's replaced by 0, we obtain (4.3), (4.4) respectively.

REMARK. It is worth noting that (4.4) and the right-hand inequality of (4.3) coincide with the corresponding results of Corollary 2a.

Similarly we have the following corollary.

COROLLARY 3b. *In the theorem let $f(z)$ be of order p in the direction of a ray containing a diametral segment and let further the ends of the diametral segment be both negatively cutting points, then*

$$(4.5)\qquad \frac{r^q}{(1+r)^{2(p-t)}}\left(\frac{1-r}{1+r}\right)^t\frac{|N(-r)|}{P(r)}\leqq|f(re^{i\theta})|$$

$$\leqq\frac{r^q}{(1-r)^{2(p-t)}}\left(\frac{1+r}{1-r}\right)^t\left(\frac{1+r^2}{1-r^2}\right)^2\frac{N(r)}{|P(-r)|},\quad 0\leqq r<1,$$

$$(4.6)\qquad f(z)\ll\frac{z^q}{(1-z)^{2(p-t)}}\left(\frac{1+z}{1-z}\right)^t\left(\frac{1+z^2}{1-z^2}\right)^2\frac{N(z)}{P(-z)}.$$

We here notice that the left-hand inequality of (4.5) also coincides with the corresponding one of Corollary 2b.

COROLLARY 3c. *In the theorem let $f(z)$ be starlike of order p in the direction of a ray containing a diametral segment, then*

$$(4.7)\qquad \frac{r^q}{(1+r)^{2p}}\left(\frac{1-r^2}{1+r^2}\right)^2|N(-r)|\leqq|f(re^{i\theta})|\leqq\frac{r^q}{(1-r)^{2p}}N(r),\quad 0\leqq r<1,$$

$$(4.8)\qquad f(z)\ll\frac{z^q}{(1-z)^{2p}}N(z).$$

PROOF. Just as in the proof of Corollary 2c we obtain (4.7), (4.8) respectively from (4.3), (4.4).

The following two corollaries are special ones of Corollaries 3 and 3c.

COROLLARY 3d. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty}c_n z^n$ *be a function of the class* $S(p, 0)$, *then*

$$\frac{r^p}{(1+r)^{2p}}\left(\frac{1-r}{1+r}\right)^2 \leqq |f(re^{i\theta})| \leqq \frac{r^p}{(1-r)^{2p}}\left(\frac{1+r}{1-r}\right)^2, \quad 0\leqq r<1, \tag{4.9}$$

$$f(z) \ll \frac{z^p}{(1-z)^{2p}}\left(\frac{1+z}{1-z}\right)^2. \tag{4.10}$$

COROLLARY 3e. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty}c_n z^n$ *be starlike of order p in the direction of a ray containing a diametral segment, then*

$$\frac{r^p}{(1+r)^{2p}}\left(\frac{1-r^2}{1+r^2}\right)^2 \leqq |f(re^{i\theta})| \leqq \frac{r^p}{(1-r)^{2p}}, \quad 0\leqq r<1, \tag{4.11}$$

$$f(z) \ll \frac{z^p}{(1-z)^{2p}}. \tag{4.12}$$

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