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Canonical product for a meromorphic function in a unit circle.

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1. Canonical product.

Let $w(z)$ be a meromorphic function of finite order ρ in $|z|<1$, such that $\overline{\lim_{r\to 1}} \log T(w, r)/\log \frac{1}{1-r}=\rho$, where $T(w, r)$ is the characteristic function of $w(z)$. Let $a_n(n=1,2,\cdots)$ be zero points of $w(z)$ in $|z|<1$, which are different from 0, then

$$\sum_{n=1}^{\infty}(1-|a_n|)^{\rho+1+\varepsilon}<\infty \qquad \text{for any} \quad \varepsilon>0. \tag{1}$$

We shall define the canonical product $P(z)$, formed with a_n as follows.

If $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$, then

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)=\prod_{n=1}^{\infty}\frac{\bar{a}_n(a_n-z)}{1-\bar{a}_n z}, \tag{2}$$

which is regular and $|P(z)|<1$ in $|z|<1$ and $P(a_n)=0$ $(n=1,2,\cdots)$. We define the convergence exponent μ of a_n as $\mu=0$.

If $\sum_{n=1}^{\infty}(1-|a_n|)=\infty$, then let $\mu\geqq 0$ be such that

$$\sum_{n=1}^{\infty}(1-|a_n|)^{\mu+1-\varepsilon}=\infty, \quad \sum_{n=1}^{\infty}(1-|a_n|)^{\mu+1+\varepsilon}<\infty \quad \text{for any } \varepsilon>0. \tag{3}$$

We call μ the convergence exponent of a_n. By (1),

$$\mu\leqq\rho. \tag{4}$$

Let $p\geqq 1$ be a positive integer, such that

$$\sum_{n=1}^{\infty}(1-|a_n|)^{p}=\infty, \qquad \sum_{n=1}^{\infty}(1-|a_n|)^{p+1}<\infty. \tag{5}$$

If μ is not an integer, then $p=[\mu]+1$ and if μ is an integer, then $p=\mu$, or $p=\mu+1$, according as $\sum_{n=1}^{\infty}(1-|a_n|)^{\mu+1}<\infty$, or $\sum_{n=1}^{\infty}(1-|a_n|)^{\mu+1}=\infty$. Hence in any case, $p-1\leqq\mu$, so that by (4),

$$p-1\leqq\mu\leqq\rho\,. \tag{6}$$

We define $P(z)$ by

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)e^{\frac{1-|a_n|^2}{1-\bar{a}_n z}+\frac{1}{2}\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^2+\cdots+\frac{1}{p}\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^p}. \tag{7}$$

By (5), $P(z)$ is regular in $|z|<1$ and $P(a_n)=0$ $(n=1,2,\cdots)$.

THEOREM 1. *Suppose that* $\sum_{n=1}^{\infty}(1-|a_n|)=\infty$, *then*

$$\log^+|P(z)|\leqq 2^{p+1}\sum_{n=1}^{\infty}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon},$$

where $\varepsilon=0$, *if* $p=\mu$ *and* $0<\varepsilon\leqq 1-(\mu-[\mu])$, *if* $p\neq\mu$.

PROOF. We put

$$\Phi(z,a_n)=\log\left(1-\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)+\frac{1-|a_n|^2}{1-\bar{a}_n z}+\cdots+\frac{1}{p}\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^p, \tag{1}$$

then

$$\log^+|P(z)|\leqq\sum_{\frac{1-|a_n|^2}{|1-\bar{a}_n z|}\geqq\frac{1}{2}}{}_1\,\Re^+(\Phi(z,a_n))+\sum_{\frac{1-|a_n|^2}{|1-\bar{a}_n z|}<\frac{1}{2}}{}_2\,|\Phi(z,a_n)|\,. \tag{2}$$

In $\sum_2$,

$$\begin{aligned}|\Phi(z,a_n)|&\leqq\frac{1}{p+1}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+1}+\frac{1}{p+2}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+2}+\cdots\\&<\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+1}\left(1+\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|+\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^2+\cdots\right)\\&<\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+1}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots\right)=2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+1}.\end{aligned} \tag{3}$$

If $p=\mu$, then

$$|\Phi(z,a_n)|\leqq 2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1}. \tag{4}$$

If $p \neq \mu$, then $p-(\mu+\varepsilon) \geqq 0$, if $0<\varepsilon \leqq 1-(\mu-[\mu])$, so that

$$|\Phi(z, a_n)| \leqq 2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p+1} = 2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon} \cdot \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{p-(\mu+\varepsilon)}$$

$$\leqq 2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}\left(\frac{1}{2}\right)^{p-(\mu+\varepsilon)} \leqq 2\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}. \tag{5}$$

Hence

$$\sum\nolimits_2 |\Phi(z, a_n)| \leqq 2 \sum\nolimits_2 \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}, \tag{6}$$

where $\varepsilon=0$, if $p=\mu$ and $0<\varepsilon \leqq 1-(\mu-[\mu])$, if $p \neq \mu$.

Since $\log\left|1-\dfrac{1-|a_n|^2}{1-\bar{a}_n z}\right| = \log\left|\dfrac{\bar{a}_n(a_n-z)}{1-\bar{a}_n z}\right| \leqq 0$, we have in $\sum_1$

$$\Re^+(\Phi(z, a_n)) \leqq \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right| + \cdots + \frac{1}{p}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p$$

$$\leqq \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p \left(1+\left|\frac{1-\bar{a}_n z}{1-|a_n|^2}\right| + \cdots + \left|\frac{1-\bar{a}_n z}{1-|a_n|^2}\right|^{p-1}\right)$$

$$\leqq \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p (1+2+\cdots+2^{p-1}) < 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p. \tag{7}$$

If $p=\mu$, then since $\left|\dfrac{1-\bar{a}_n z}{1-|a_n|^2}\right| \leqq 2$ in $\sum_1$,

$$\Re^+(\Phi(z, a_n)) \leqq 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p$$

$$= 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1} \cdot \left|\frac{1-\bar{a}_n z}{1-|a_n|^2}\right| \leqq 2^{p+1} \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1}. \tag{8}$$

If $p \neq \mu$, then $0<\mu+1+\varepsilon-p \leqq 1$, if $0<\varepsilon \leqq 1-(\mu-[\mu])$, so that

$$\Re^+(\Phi(z, a_n)) \leqq 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^p = 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon} \cdot \left|\frac{1-\bar{a}_n z}{1-|a_n|^2}\right|^{\mu+1+\varepsilon-p}$$

$$\leqq 2^p \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon} 2^{\mu+1+\varepsilon-p} \leqq 2^{p+1} \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}. \tag{9}$$

Hence

$$\sum_1 \mathfrak{R}^+(\Phi(z, a_n)) \leqq 2^{p+1} \sum_1 \left| \frac{1-|a_n|^2}{1-\bar{a}_n z} \right|^{\mu+1+\varepsilon}, \tag{10}$$

where $\varepsilon=0$, if $p=\mu$ and $0<\varepsilon\leqq 1-(\mu-[\mu])$, if $p\neq\mu$.

Hence from (2), (6), (10), we have

$$\log^+ |P(z)| \leqq 2^{p+1} \sum_{n=1}^{\infty} \left| \frac{1-|a_n|^2}{1-\bar{a}_n z} \right|^{\mu+1+\varepsilon}.$$

LEMMA 1. *Let* $I=\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{\lambda+1}}$, $\lambda\geqq 0$, $0\leqq r<1$, *then*

$$I=O\left(\frac{1}{(1-r)^\lambda}\right), \quad \textit{if } \lambda>0, \qquad I=O\left(\log\frac{1}{1-r}\right), \quad \textit{if } \lambda=0.$$

PROOF.

$$I=\int_{|\theta|\leqq 1-r} \frac{d\theta}{|1-re^{i\theta}|^{\lambda+1}} + \int_{1-r\leqq|\theta|\leqq\pi} \frac{d\theta}{|1-re^{i\theta}|^{\lambda+1}} = I_1+I_2. \tag{1}$$

(i) If $\lambda>0$, then

$$I_1 \leqq \frac{1}{(1-r)^{\lambda+1}} \int_{|\theta|\leqq 1-r} d\theta = \frac{2}{(1-r)^\lambda}.$$

Since $|1-re^{i\theta}|^2=1-2r\cos\theta+r^2=(1-r)^2+4r\sin^2\frac{\theta}{2}\geqq(1-r)^2+a^2\theta^2$ ($a=$const.),

$$I_2=2\int_{1-r}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{\lambda+1}} \leqq 2\int_{1-r}^{\pi} \frac{d\theta}{((1-r)^2+a^2\theta^2)^{\frac{\lambda+1}{2}}}$$

$$\leqq \frac{2}{(1-r)^\lambda}\int_1^\infty \frac{dt}{(1+a^2t^2)^{\frac{\lambda+1}{2}}} = O\left(\frac{1}{1-r)^\lambda}\right), \quad \theta=(1-r)t.$$

Hence

$$I=O\left(\frac{1}{(1-r)^\lambda}\right). \tag{2}$$

(ii) If $\lambda=0$, then $I_1=O(1)$ and

$$I_2=2\int_{1-r}^{\pi}\frac{d\theta}{|1-re^{i\theta}|}\leqq 2\int_{1-r}^{\pi}\frac{d\theta}{\sqrt{(1-r)^2+a^2\theta^2}}$$

$$=2\int_{1}^{\frac{\pi}{1-r}}\frac{dt}{\sqrt{1+a^2t^2}}=O\left(\log\frac{1}{1-r}\right).$$

Hence

$$I=O\left(\log\frac{1}{1-r}\right). \tag{3}$$

THEOREM 2. *Suppose that* $\sum_{n=1}^{\infty}(1-|a_n|)=\infty$. *Let* ρ^* *be the order of* $P(z)$, *then*

$$\rho^*=\mu.$$

Since $\mu\leqq\rho$,

$$\rho^*\leqq\rho.$$

PROOF. Since by (4) of page 7 $\mu\leqq\rho^*$, we have only to prove that $\rho^*\leqq\mu$.

Let $T(r)$ be the characteristic function of $P(z)$, then since $P(z)$ is regular in $|z|<1$, we have by Theorem 1 and Lemma 1,

$$T(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|P(re^{i\theta})|\,d\theta+O(1)$$

$$\leqq \text{const.}\sum_{n=1}^{\infty}\int_0^{2\pi}\left|\frac{1-|a_n|^2}{1-\bar{a}_n re^{i\theta}}\right|^{\mu+1+\varepsilon}d\theta+O(1)=O\left(\frac{1}{(1-r)^{\mu+\varepsilon}}\right), \tag{1}$$

where $\varepsilon=0$, if $p=\mu$ and $\varepsilon>0$ is arbitrarily small, if $p\neq\mu$, so that $\rho^*\leqq\mu$, hence $\rho^*=\mu$.

As an application of Theorem 2, we shall prove

THEOREM 3. *Let* $w(z)$ *be a meromorphic function of finite order* ρ *in* $|z|<1$, *then* $w(z)$ *can be expressed in the form* $w(z)=\dfrac{w_1(z)}{w_2(z)}$, *where* $w_1(z)$, $w_2(z)$ *are regular and of order* $\leqq\rho$ *in* $|z|<1$.

PROOF. Let $P(z)$ be the canonical product, formed with poles $a_n\neq 0$ of $w(z)$ and $w(z)$ have a pole of the ν ($\geqq 0$)-th order at $z=0$, then if we put $w_2(z)=z^\nu P(z)$, $w_2(z)$ is regular and of order $\leqq\rho$ in $|z|<1$, so that $w_1(z)=w(z)\,w_2(z)$ is regular and of order $\leqq\rho$ in $|z|<1$. Hence $w(z)=\dfrac{w_1(z)}{w_2(z)}$ is the desired decomposition.

THEOREM 4. *Let $\mu \geqq 0$ be the convergence exponent of a_n and $C_n: |z-a_n| = (1-|a_n|^2)^{\mu+4}$ be a circle about a_n. If z lies outside of $C_n (n=1, 2, \cdots)$, then*

$$\log^+ \left|\frac{1}{P(z)}\right| \leqq \text{const. } \log \frac{1}{1-|z|} \cdot \sum_{n=1}^{\infty} \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}, \quad \frac{1}{2} \leqq |z| < 1,$$

where if $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$, then $\mu = 0$, $\varepsilon = 0$ and if $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$, then $\varepsilon = 0$, if $p = \mu$ and $0 < \varepsilon \leqq 1-(\mu-[\mu])$, if $p \neq \mu$.

PROOF. (i) Suppose that $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$, then

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{1-|a_n|^2}{1-\bar{a}_n z}\right) e^{\frac{1-|a_n|^2}{1-\bar{a}_n z} + \cdots + \frac{1}{p}\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^p}. \tag{1}$$

Hence if we put

$$\Psi(z, a_n) = \log \frac{1-\bar{a}_n z}{\bar{a}(a_n - z)} - \left(\frac{1-|a_n|^2}{1-\bar{a}_n z} + \cdots + \frac{1}{p}\left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right)^p\right), \tag{2}$$

then

$$\log^+ \left|\frac{1}{P(z)}\right| \leqq \sum_{\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right| \geqq \frac{1}{2}}{}_1 \Re^+ (\Psi(z, a_n)) + \sum_{\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right| < \frac{1}{2}}{}_2 |\Psi(z, a_n)|. \tag{3}$$

As the proof of theorem 1, we have

$$\sum\nolimits_2 |\Psi(z, a_n)| \leqq 2 \sum\nolimits_2 \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}, \tag{4}$$

where $\varepsilon = 0$, if $p = \mu$ and $0 < \varepsilon \leqq 1-(\mu-[\mu])$, if $p \neq \mu$.
In $\sum_1$,

$$\Re^+ (\Psi(z, a_n)) \leqq \log \left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n - z)}\right| + 2^{p+1} \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}. \tag{5}$$

If z lies outside of $C_n: |z-a_n| = (1-|a_n|^2)^{\mu+4}$, then in $\sum_1$,

$$\left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n - z)}\right| \leqq \frac{|1-\bar{a}_n z|}{|a_n|(1-|a_n|^2)^{\mu+4}} \leqq \frac{|1-\bar{a}_n z|}{|a_n|\left|\frac{1-\bar{a}_n z}{2}\right|^{\mu+4}}$$

$$\leqq \frac{\text{const.}}{|1-\bar{a}_n z|^{\mu+3}} \leqq \frac{\text{const.}}{(1-|z|)^{\mu+3}},$$

so that

$$\log\left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right| \leqq \text{const. } \log\frac{1}{1-|z|}, \quad \frac{1}{2}\leqq |z|<1.$$

Since in $\sum_1$, $\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right| \geqq \frac{1}{2}$, we have

$$\mathfrak{R}^+(\Psi(z, a_n)) \leqq \text{const. } \log\frac{1}{1-|z|}\cdot\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon},$$

so that

$$\sum_1 \mathfrak{R}^+(\Psi(z, a)) \leqq \text{const. } \log\frac{1}{1-|z|}\cdot\sum_1\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}, \quad \frac{1}{2}\leqq|z|<1. \tag{6}$$

Hence by (3), (4), (6), if z lies outside of C_n $(n=1, 2, \cdots)$,

$$\log^+\left|\frac{1}{P(z)}\right| \leqq \text{const. } \log\frac{1}{1-|z|}\cdot\sum_{n=1}^{\infty}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon}, \quad \frac{1}{2}\leqq|z|<1. \tag{7}$$

(ii) Next suppose that $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$, then $\mu=0$ and

$$P(z)=\prod_{n=1}^{\infty}\frac{\bar{a}_n(a_n-z)}{1-\bar{a}_n z}, \tag{1}$$

so that

$$\log\left|\frac{1}{P(z)}\right| \leqq \sum_{\frac{1-|a_n|^2}{|1-\bar{a}_n z|}\geqq\frac{1}{2}} \log\left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right| + \sum_{\frac{1-|a_n|^2}{|1-\bar{a}_n z|}<\frac{1}{2}}{}_2 \left|\log\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right|. \tag{2}$$

In $\sum_2$,

$$\left|\log\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right| \leqq \left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right| + \frac{1}{2}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^2 + \cdots \leqq 2\,\frac{1-|a_n|^2}{|1-\bar{a}_n z|},$$

so that

$$\sum_2\left|\log\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right| \leqq 2\sum_2\frac{1-|a_n|^2}{|1-\bar{a}_n z|}. \tag{3}$$

If z lies outside of $C_n : |z-a_n|=(1-|a_n|^2)^4$, then in $\sum_1$,

$$\log\left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right| \leqq \log\frac{|1-\bar{a}_n z|}{|a_n|(1-|a_n|^2)^4} \leqq \log\frac{|1-\bar{a}_n z|}{|a_n|\left|\frac{1-\bar{a}_n z}{2}\right|^4}$$

$$=\log\frac{16}{|a_n||1-\bar{a}_n z|^3}\leqq\text{const.}\log\frac{1}{1-|z|},\quad \frac{1}{2}\leqq|z|<1.$$

Since $\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|\geqq\frac{1}{2}$ in $\sum_1$,

$\log\left|\frac{1-\bar{a}_n z}{\bar{a}_n(a_n-z)}\right|\leqq\text{const.}\log\frac{1}{1-|z|}\cdot\frac{1-|a_n|^2}{|1-\bar{a}_n z|}$, so that

$$\sum_1\log\left|\frac{1-a_n z}{a_n(a_n-z)}\right|\leqq\text{const.}\log\frac{1}{1-|z|}\cdot\sum_1\frac{1-|a_n|^2}{|1-\bar{a}_n z|},$$
$$\frac{1}{2}\leqq|z|<1. \tag{4}$$

Hence by (2), (3), (4), if z lies outside of C_n $(n=1,2,\cdots)$,

$$\log\left|\frac{1}{P(z)}\right|\leqq\text{const.}\log\frac{1}{1-|z|}\cdot\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|1-\bar{a}_n z|},\quad\frac{1}{2}\leqq|z|<1. \tag{5}$$

2. Order of the derivative.

Let $w(z)$ be a meromorphic function of order ρ $(\leqq\infty)$ for $|z|<\infty$, then Whittaker and Valiron[1)] proved that $w'(z)$ is of the same order ρ. We shall prove the analogue for a meromorphic function in $|z|<1$. We shall use the following lemmas.

LEMMA 2. *Let $\mu\geqq 0$ be the convergence exponent of a_n. Then*

$$\sum_{r\leqq a_n<1}(1-|a_n|)^{\mu+1}=O((1-r)^2).$$

Hence let $C_n: |z-a_n|=(1-|a_n|^2)^{\mu+4}$ be a circle about a_n, then if $1-r_0$ is small, for any $r(r_0\leqq r<1)$, there exists $r'\left(r\leqq r'\leqq\frac{r+1}{2}\right)$, such that the circle $|z|=r'$ lies outside of C_n $(n=1,2,\cdots)$.

1) J. M. Whittaker: The order of the derivative of a meromorphic function. Journ. London Math. Soc. 11 (1936).

G. Valiron: Sur la distribution des valeurs des fonctions méromorphes. Acta Math. 47 (1926).

M. Tsuji: On the order of the derivative of a meromorphic function. Tohoku Math. Journ. 3 (1951).

Y. Komatu: The order of the derivative of a meromorphic function. Proc. Japan Acad. 27 (1951).

PROOF. Let $n(r)$ be the number of a_n, such that $|a_n|<r<1$, then since $\sum_{n=1}^{\infty}(1-|a_n|)^{\mu+1+\varepsilon}<\infty$, we have $n(r)=O\left(\frac{1}{(1-r)^{\mu+1+\varepsilon}}\right)$, $0<\varepsilon<1$. Let $0<r<\rho<1$, then

$$\sum_{r\leqq|a_n|<\rho}(1-|a_n|)^{\mu+4}=\int_r^\rho(1-t)^{\mu+4}\,dn(t)$$

$$\leqq(1-\rho)^{\mu+4}n(\rho)+(\mu+4)\int_r^\rho(1-t)^{\mu+3}n(t)\,dt\,.$$

Since $(1-\rho)^{\mu+4}n(\rho)=O((1-\rho)^{3-\varepsilon})\to 0$, as $\rho\to 1$, we have

$$\sum_{r\leqq|a_n|<1}(1-|a_n|)^{\mu+4}\leqq(\mu+4)\int_r^1(1-t)^{\mu+3}n(t)\,dt$$

$$=O\left(\int_r^1(1-t)^{2-\varepsilon}\,dt\right)=O((1-r)^{3-\varepsilon})=O((1-r)^2)\,.$$

LEMMA 3. (Hardy-Littlewood[2]). *Let $u(z)\geqq 0$ be a non-negative subharmonic function in $|z|\leqq 1$ and $0<\alpha<\frac{\pi}{2}$. Let $z=e^{i\theta}$ be any point of $|z|=1$ and $\omega(e^{i\theta},\alpha)$: $|\arg(1-ze^{-i\theta})|<\alpha$, $\frac{1}{2}\leqq|z|<1$ be a sector, whose vertex is at $e^{i\theta}$ and put*

$$M(\theta,\alpha)=\max_{z\in\omega(e^{i\theta},\alpha)}u(z)\,.$$

Then

$$\int_0^{2\pi}[M(\theta,\alpha)]^k\,d\theta\leqq A(k,\alpha)\int_0^{2\pi}[u(e^{i\theta})]^k\,d\theta\,,\qquad k>1\,,$$

where $A(k,\alpha)$ is a constant, which depends on k and α only.

Now we shall prove

THEOREM 5. *Let $w(z)$ be a meromorphic function of order ρ ($\leqq\infty$) in $|z|<1$ and ρ' be the order of $w'(z)$, then $\rho'=\rho$.*

PROOF. (i) First suppose that $w(z)$ is regular in $|z|<1$ and $\rho<\infty$ and let $T(w,r)$ be its characteristic function. By Nevanlinna's theorem[3],

2) G. H. Hardy and J. E. Littlewood: A maximal theorem with function-theoretic applications. Acta Math. 54 (1930).

3) R. Nevanlinna: Eindeutige analytische Funktionen. Berlin (1936) p. 240.

$$m\left(\frac{w'}{w}, r, \infty\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{w'(re^{i\theta})}{w(re^{i\theta})}\right| d\theta$$

$$= O\left(\log \frac{1}{1-r}\right) + O(\log T(r)) = O\left(\log \frac{1}{1-r}\right), \tag{1}$$

except in a set of intervals J_ν, such that $\sum_\nu \int_{J_\nu} d\left(\frac{1}{(1-r)^\lambda}\right) < \infty$, $(\lambda > 0)$. Hence

$$T\left(\frac{w'}{w}, r\right) = m\left(\frac{w'}{w}, r, \infty\right) + N\left(\frac{w'}{w}, r, \infty\right)$$

$$= m\left(\frac{w'}{w}, r, \infty\right) + N(w, r, 0)$$

$$\leqq m\left(\frac{w'}{w}, r, \infty\right) + T(w, r) = O\left(\frac{1}{(1-r)^{\rho+\varepsilon}}\right), \quad \varepsilon > 0, \tag{2}$$

except in J_ν. If we take $\lambda = \rho + \varepsilon$, then since $T\left(\frac{w'}{w}, r\right)$ is an increasing function of r, we see that (2) holds without exception, hence $\frac{w'(z)}{w(z)}$ is of order $\leqq \rho$, so that $w'(z)$ is of order $\leqq \rho$, hence

$$\rho' \leqq \rho. \tag{3}$$

Next we shall prove that $\rho \leqq \rho'$. Since $w'(z)$ is of order $\rho' \leqq \rho < \infty$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |w'(re^{i\theta})| d\theta = T(w', r) + O(1) = O\left(\frac{1}{(1-r)^{\rho'+\varepsilon}}\right), \quad \varepsilon > 0, \tag{4}$$

$$M(r) = \operatorname*{Max}_{|z|=r} \log^+ |w'(z)| = O\left(\frac{1}{(1-r)^{\rho'+1+\varepsilon}}\right), \quad \varepsilon > 0. \tag{5}$$

If we put

$$M(r, \theta) = \operatorname*{Max}_{0 \leqq t \leqq r} |w'(te^{i\theta})|, \tag{6}$$

then

$$|w(re^{i\theta})| \leqq |w(0)| + \int_0^r |w'(te^{i\theta})| dt \leqq |w(0)| + M(r, \theta),$$

so that

$$T(w, r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|w(re^{i\theta})|\,d\theta+O(1)\leqq\frac{1}{2\pi}\int_0^{2\pi}\log^+ M(r, \theta)\,d\theta+O(1)$$

$$\leqq\left(\frac{1}{2\pi}\int_0^{2\pi}(\log^+ M(r, \theta))^{1+\varepsilon}\,d\theta\right)^{\frac{1}{1+\varepsilon}}+O(1),\quad \varepsilon>0. \tag{7}$$

Since $\log^+ M(r, \theta)=\underset{0\leqq t\leqq r}{\operatorname{Max}}\log^+|w'(te^{i\theta})|$ and $\log^+|w'(z)|\geqq 0$ is a non-negative subharmonic function, we have by Lemma 3 and (4), (5),

$$\int_0^{2\pi}(\log^+ M(r, \theta))^{1+\varepsilon}d\theta\leqq\text{const.}\int_0^{2\pi}(\log^+|w'(re^{i\theta})|)^{1+\varepsilon}d\theta$$

$$=\text{const.}\int_0^{2\pi}\log^+|w'(re^{i\theta})|(\log^+(w'(re^{i\theta})|)^{\varepsilon}\,d\theta$$

$$\leqq\text{const.}\int_0^{2\pi}\log^+|w'(re^{i\theta})|(M(r))^{\varepsilon}\,d\theta$$

$$\leqq\frac{\text{const.}}{(1-r)^{\varepsilon(\rho'+1+\varepsilon)}}\int_0^{2\pi}\log^+|w'(re^{i\theta})|\,d\theta\leqq\frac{\text{const.}}{(1-r)^{\varepsilon(\rho'+1+\varepsilon)}}\cdot\frac{1}{(1-r)^{\rho'+\varepsilon}}. \tag{8}$$

Hence by taking $\varepsilon>0$ sufficiently small, we have by (7)

$$T(w, r)=O\left(\frac{1}{(1-r)^{\rho'+\delta}}\right)\quad\text{for any } \delta>0, \tag{9}$$

hence $\rho\leqq\rho'$, so that $\rho'=\rho$, if $\rho<\infty$. If $\rho'<\infty$, then from $\rho\leqq\rho'$, we have $\rho<\infty$, so that if $\rho=\infty$, then $\rho'=\infty$, hence $\rho'=\rho$ in general.
(ii) Next suppose that $w(z)$ has poles in $|z|<1$ and first suppose that $\rho<\infty$. Then by Theorem 3, $w(z)=\frac{w_1(z)}{w_2(z)}$, where $w_1(z)$, $w_2(z)$ are regular and of order $\leqq\rho$ in $|z|<1$. Since by (i), $w_1'(z)$, $w_2'(z)$ are of order $\leqq\rho$, we have from $w'(z)=\frac{w_1'(z)\,w_2(z)-w_1(z)\,w_2'(z)}{(w_2(z))^2}$,

$$\rho'\leqq\rho. \tag{1}$$

Next we shall prove that $\rho\leqq\rho'$. Let $a_n\neq 0$ be poles of $w'(z)$ and $P(z)$ be the canonical product, formed with a_n, then by Theorem 2, $P(z)$ is of order $\leqq\rho'$ and

$$w'(z)=\frac{Q(z)}{z^{\nu}P(z)},\qquad \nu\geqq 0, \tag{2}$$

where $Q(z)$ is regular and of order $\leqq\rho'$ in $|z|<1$.

Let $\mu \geqq 0$ be the convergence exponent of a_n and $C_n: |z-a_n| = (1-|a_n|^2)^{\mu+4}$ be a circle about a_n, then by Lemma 2, for any small $1-\tau$, there exists r_0 and $r(\tau \leqq r_0 < r < 1)$, such that the circles $|z|=r_0$ and $|z|=r$ lie outside of C_n $(n=1, 2, \cdots)$.

We take off the insides of C_n, which lie in $r_0<|z|<r$ from $r_0<|z|<r$ and let D be the remaining part. Then D consists of a finite number of connected domains. If $1-r_0$ is small, then there exists a connected one Δ among them, which contains the circles $|z|=r_0$ and $|z|=r$ on its boundary Γ. Let $L(r_0, r, \theta)$ be the segment $z=te^{i\theta}(r_0 \leqq t \leqq r)$. We modify $L(r_0, r, \theta)$ into $L^*(r_0, r, \theta)$ as follows.

If $L(r_0, r, \theta)$ does not meet $\{C_n\}$, then we put $L^*(r_0, r, \theta) = L(r_0, r, \theta)$. If $L(r_0, r, \theta)$ meets $\{C_n\}$, then the part of $L(r_0, r, \theta)$, which lies in Δ consists of an odd number of segments: $\overline{z_0 z_1}, \cdots, \overline{z_{2n} z_{2n+1}}$, where $z_\nu = t_\nu e^{i\theta}$ $(r_0=t_0<t_1<\cdots<t_{2n+1}=r)$. There exists an arc α_1 of Γ, which connects z_1 to z_2 and there exists an arc α_2 of Γ, which connects z_3 to z_4 and similarly we define $\alpha_3, \alpha_4, \cdots$, then

$$L^*(r_0, r, \theta) = \overline{z_0 z_1} + \alpha_1 + \overline{z_2 z_3} + \alpha_2 + \overline{z_4 z_5} + \cdots + \overline{z_{2n} z_{2n+1}}. \tag{3}$$

$L^*(r_0, r, \theta)$ lies outside of C_n $(n=1, 2, \cdots)$ and connects $z_0 = r_0 e^{i\theta}$ to $z = re^{i\theta}$, so that

$$\begin{aligned} |w(re^{i\theta})| &\leqq |w(r_0 e^{i\theta})| + \int_{L^*(r_0, r, \theta)} |w'(z)|\,|dz| \\ &= |w(r_0 e^{i\theta})| + \int_{L^*(r_0, r, \theta)} \left| \frac{Q(z)}{z^\nu P(z)} \right| |dz| \\ &\leqq |w(r_0 e^{i\theta}) + \sqrt{\int_{L^*(r_0, r, \theta)} |Q(z)|^2 |dz| \int_{L^*(r_0, r, \theta)} \left| \frac{1}{z^\nu P(z)} \right|^2 |dz|}, \end{aligned}$$

hence

$$\begin{aligned} \int_0^{2\pi} \log^+ |w(re^{i\theta})|\, d\theta &\leqq \int_0^{2\pi} \log^+ \left(\int_{L^*(r_0, r, \theta)} |Q(z)|^2 |dz| \right) d\theta \\ &+ \int_0^{2\pi} \log^+ \left(\int_{L^*(r_0, r, \theta)} \left| \frac{1}{z^\nu P(z)} \right|^2 |dz| \right) d\theta + O(1) = \mathrm{I} + \mathrm{II} + O(1). \end{aligned} \tag{4}$$

We shall prove that if $1-r_0$ is sufficiently small, then $L^*(r_0, r, \theta)$ is contained in a Stolz domain $\omega\left(e^{i\theta}, \dfrac{\pi}{6}\right)$: $|\arg(1-ze^{-i\theta})| < \dfrac{\pi}{6}$, whose vertex is at $e^{i\theta}$.

Let $\zeta \in L^*(r_0, r, \theta)$ and suppose that $\zeta \in \alpha_\nu$. Since $(1-|a_n|^2)^{\mu+4}$ is the radius of C_n and by Lemma 2, $\sum_{2|\zeta|-1 \leqq |a_n| < 1} (1-|a_n|)^{\mu+4} = O((1-|\zeta|)^2)$, we see that the diameter of α_ν is $O((1-|\zeta|)^2)$, so that if $1-r_0$ is small, ζ lies in $\omega\left(e^{i\theta}, \frac{\pi}{6}\right)$, hence $L^*(r_0, r, \theta)$ is contained in $\omega\left(e^{i\theta}, \frac{\pi}{6}\right)$.

Let $z' = r'e^{i\theta}$, $r' = \frac{r+1}{2}$. Since $L^*(r_0, r, \theta)$ is contained in $\omega\left(e^{i\theta}, \frac{\pi}{6}\right)$, we see easily that $L^*(r_0, r, \theta)$ is contained in a Stolz domain $\omega\left(z', \frac{\pi}{3}\right)$, which is bounded by two lines through z', making an angle $\frac{\pi}{3}$ with the radius of the circle $|z| = r'$, through z'.

Let $\omega_{r_0}\left(z', \frac{\pi}{3}\right)$ be the part of $\omega\left(z', \frac{\pi}{3}\right)$, which is contained in $r_0 \leqq |z| < r'$ and put

$$M(r', \theta) = \operatorname*{Max}_{z \in \omega_{r_0}\left(z', \frac{\pi}{3}\right)} |Q(z)|. \tag{5}$$

Then by Lemma 3,

$$\int_0^{2\pi} (\log^+ M(r', \theta))^{1+\varepsilon} d\theta \leqq \text{const.} \int_0^{2\pi} (\log^+ |Q(r'e^{i\theta})|)^{1+\varepsilon} d\theta, \quad \varepsilon > 0.$$

Hence

$$\begin{aligned} \mathrm{I} = \int_0^{2\pi} \log^+ \left(\int_{L^*(r_0, r, \theta)} |Q(z)|^2 \, |dz| \right) d\theta &\leqq \text{const.} \int_0^{2\pi} \log^+ M(r', \theta) \, d\theta \\ &\leqq \text{const.} \left(\int_0^{2\pi} (\log^+ M(r', \theta))^{1+\varepsilon} d\theta \right)^{\frac{1}{1+\varepsilon}} \\ &\leqq \text{const.} \left(\int_0^{2\pi} (\log^+ |Q(r'e^{i\theta})|)^{1+\varepsilon} d\theta \right)^{\frac{1}{1+\varepsilon}}. \end{aligned} \tag{6}$$

From this, we have as in (i),

$$\mathrm{I} = O\left(\frac{1}{(1-r')^{\rho'+\delta}} \right) = O\left(\frac{1}{(1-r)^{\rho'+\delta}} \right) \quad \text{for any } \delta > 0. \tag{7}$$

Next we shall evaluate II. Let z be any point on $L^*(r_0, r, \theta)$ and first suppose that $z\in\alpha_\nu$. Let z_ν be one of two end points of α_ν, which lies on $L(r_0, r, \theta)$, then as proved above, the diameter of α_ν is $O((1-|z_\nu|)^2)$, so that $|z-z_\nu|=O((1-|z_\nu|)^2)$, hence from $|\bar{a}_n z_\nu -1|-|z-z_\nu|\leqq|\bar{a}_n z-1|\leqq|\bar{a}_n z_\nu-1|+|z-z_\nu|$ and $|1-\bar{a}_n z_\nu|\geqq 1-|z_\nu|$, we have

$$|1-\bar{a}_n z_\nu|\,(1-O((1-|z_\nu|))\leqq|1-\bar{a}_n z|\leqq|1-\bar{a}_n z_\nu|\,(1+O((1-|z_\nu|)).$$

Hence if $1-|z|$ is small,

$$\frac{1}{|1-\bar{a}_n z|}\leqq\frac{2}{|1-\bar{a}_n z_\nu|}. \tag{8}$$

Since as easily be proved, $\dfrac{1}{|1-\bar{a}_n z_\nu|}\leqq\dfrac{2}{|1-\bar{a}_n re^{i\theta}|}$, we have

$$\frac{1}{|1-\bar{a}_n z|}\leqq\frac{4}{|1-\bar{a}_n re^{i\theta}|}. \tag{9}$$

If $z\in\overline{z_0 z_1}+\overline{z_2 z_3}+\cdots$, then $\dfrac{1}{|1-\bar{a}_n z|}\leqq\dfrac{2}{|1-\bar{a}_n re^{i\theta}|}$, so that (9) holds for any $z\in L^*(r_0, r, \theta)$.
Since by Theorem 4, if z lies outside of $C_n: |z-a_n|=(1-|a_n|^2)^{\mu+4}$ $(n=1, 2,\cdots)$,

$$\log^+\left|\frac{1}{P(z)}\right|\leqq\text{const.}\log\frac{1}{1-|z|}\cdot\sum_{n=1}^{\infty}\left|\frac{1-|a_n|^2}{1-\bar{a}_n z}\right|^{\mu+1+\varepsilon},\quad \frac{1}{2}\leqq|z|<1 \tag{10}$$

and $L^*(r_0, r, \theta)$ lies outside of C_n $(n=1, 2,\cdots)$, if we put

$$M(r,\theta)=\underset{z\in L^*(r_0,r,\theta)}{\mathrm{Max}}\log^+\left|\frac{1}{z^\nu P(z)}\right|, \tag{11}$$

then from (9), (10), we have

$$M(r,\theta)\leqq\text{const.}\log\frac{1}{1-r}\cdot\sum_{n=1}^{\infty}\left|\frac{1-|a_n|^2}{1-\bar{a}_n re^{i\theta}}\right|^{\mu+1+\varepsilon}, \tag{12}$$

so that by Lemma 1,

$$\mathrm{II}=\int_0^{2\pi}\log^+\left(\int_{L^*(r_0,r,\theta)}\left|\frac{1}{z^\nu P(z)}\right|^2|dz|\right)d\theta\leqq 2\int_0^{2\pi}M(r,\theta)\,d\theta$$

$$\leqq \begin{cases} \text{const.} \log \dfrac{1}{1-r} \cdot \dfrac{1}{(1-r)^{\mu+\varepsilon}}, & \text{if } \mu+\varepsilon>0, \\ \text{const.} \left(\log \dfrac{1}{1-r}\right)^2, & \text{if } \mu+\varepsilon=0. \end{cases} \tag{13}$$

Since $\mu \leqq \rho'$, we have by (4), (7), (13),

$$m(w, r, \infty) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w(re^{i\theta})|\, d\theta = O\left(\frac{1}{(1-r)^{\rho'+\delta}}\right) \text{ for any } \delta>0.$$

Hence if the circle $|z|=r$ lies outside of C_n $(n=1, 2, \cdots)$,

$$T(w, r) = m(w, r, \infty) + N(w, r, \infty) \leqq m(w, r, \infty) + N(w', r, \infty)$$

$$\leqq m(w, r, \infty) + T(w', r) = O\left(\frac{1}{(1-r)^{\rho'+\delta}}\right). \tag{14}$$

By Lemma 2, for any $r (r_0 \leqq r < 1)$, there exists r' $\left(r \leqq r' \leqq \dfrac{r+1}{2}\right)$, such that the circle $|z|=r'$ lies outside of C_n $(n=1, 2, \cdots)$, so that for any r $(0 \leqq r < 1)$,

$$T(w, r) \leqq T(w, r') = O\left(\frac{1}{(1-r')^{\rho'+\delta}}\right) = O\left(\frac{1}{(1-r)^{\rho'+\delta}}\right) \text{ for any } \delta>0.$$

Hence $\rho \leqq \rho'$, so that $\rho'=\rho$, if $\rho<\infty$. We can prove as in (i), that $\rho'=\infty$, if $\rho=\infty$, hence $\rho'=\rho$ in general.

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