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On the structure of complete local rings.

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I. S. Cohen [1] proved a series of theorems which clearify the structure of complete local rings. These theorems were generalized by M. Nagata [2], [3] in the following sense. By a *local ring* one means usually a ring, say $\mathfrak{o}$, satisfying the following conditions: (i) $\mathfrak{o}$ is commutative and has a unity 1. (ii) The non-unit elements of $\mathfrak{o}$ form a maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. (iii) $\mathfrak{o}$ is Noetherian. Now Nagata generalizes this concept in calling a *local ring* a ring satisfying (i), (ii) and the following conditions (iii′) instead of (iii).

(iii′) $$\bigcap_{i=1}^{\infty} \mathfrak{m}^i = (0)\,.$$

It is clear that one can topologize as usual local rings in this sense and speak of *complete* local rings. Structure-theorems corresponding to Cohen's were given in [2], [3] for complete local rings in this generalized sense[1].

The purpose of this paper is to prove the theorems of Nagata in a simplified manner. By local rings we mean always those in Nagata's sense. The above introduced notations $\mathfrak{o}$ and $\mathfrak{m}$, meaning a local ring and its maximal ideal, will be used throughout the paper. Moreover we shall always denote with F the residue field $\mathfrak{o}/\mathfrak{m}$, and with φ the canonical mapping of $\mathfrak{o}$ onto F.

§ 1.

THEOREM 1. *Let $\mathfrak{o}$ be a complete local ring having the same characteristic as the residue field F, then $\mathfrak{o}$ contains a subfield K which is isomorphic to F, such that $\varphi(K)=F$*[2].

1) An incompleteness of a proof in [2] was corrected in [3], but it is to be noticed that it was not sufficiently amended.

2) A. Geddes [4] also proved this theorem by an analogons method as ours.

In case where the characteristic of $\mathfrak{o}$ and F are both zero, this theorem may be easily proved by using Hensel's lemma. (cf. [1] or [2]: [1] treats only the case $\mathfrak{o}$ is a local ring in the ordinary sense, but it is easy to modify the proof so as to adapt also to our case as was done in [2]).

So we shall treat in the following, only the case where F has a characteristic $p(\neq 0)$.

We shall call a local ring $\mathfrak{o}$ is a *primary local ring,* if some power of $\mathfrak{m}$ is a zero ideal of $\mathfrak{o}$. A primary local ring in this sense is complete because any Cauchy sequence in such a ring has only a finite number of elements mutually djfferent from each other.

If $\mathfrak{o}$ is primary, then there exists some positive integer n such that $\mathfrak{m}^{p^n}=(0)$. $F^{p^2},\cdots,F^{p^n}$ are subfields of F, and we have $F\supset F^p\supset F^{p^2}\supset\cdots\supset F^{p^n}$.

LEMMA 1. *Let $\mathfrak{o}$ be a primary local ring having the same characteristic p as the residue field F, then $\mathfrak{o}$ contains a subfield K', which is isomorphic to F^{p^n} such that $\varphi(K')=F^{p^n}$.*

PROOF. An element of F^{p^n} may be written in form α^{p^n}, $\alpha\in F$. Let a be an elements of $\varphi^{-1}(\alpha)$, then a^{p^n} is uniquely determined by α^{p^n}, not depending on the choice of a from $\varphi^{-1}(\alpha)$. For if b is another element of $\varphi^{-1}(\alpha)$, then we have $a-b\in\mathfrak{m}$, and therefore $a^{p^n}-b^{p^n}\in\mathfrak{m}^{p^n}=(0)$. Thus we have a correspondence ψ' of F^{p^n} into $\mathfrak{o}$, such that $\psi'(\alpha^{p^n})=a^{p^n}$. It is evident that ψ' preserves addition and multiplication. Hence $\psi'(F^{p^n})$ is clcarly a required subfield of $\mathfrak{o}$.

We shall call a *p-basis* of F a system $\{\gamma_\tau\}_{\tau\in\Gamma}$ of elements of F, satisfying the following conditions:

(i) $[F^p(\gamma_{\tau_1},\gamma_{\tau_2},\cdots,\gamma_{\tau_\gamma}):F^p]=p^\gamma$ for any γ elements $\tau_1,\tau_2,\cdots,\tau_\gamma$ of Γ.

(ii) $F^p(\gamma_\tau)_{\tau\in\Gamma}=F$.

If F is not perfect, i. e. $F\neq F^p$ then a p-basis of F is not a null set, and existence of such a p-basis is easily shown by using Zorn's lemma.

LEMMA 2. *Let $\mathfrak{o}$ be a primary local ring, then $\mathfrak{o}$ contains a subfield K, which is isomorphic to F such that $\varphi(K)=F$.*

PROOF. In case $F=F^{p^n}$, this lemma is already proved by Lemma 1. Now, we shall assume $F\neq F^{p^n}$, i. e. $F\neq F^p$. Let us choose arbitrarily an element c_τ from $\varphi^{-1}(\gamma_\tau)$ once for all $\tau\in\Gamma$. Any element of F may be written uniquely in form of a polynomial $\bar{P}(\gamma_{\tau_1}, \gamma_{\tau_2},\cdots, \gamma_{\tau_k})$ of γ_τ, $\tau\in\Gamma$, with coefficients in F^{p^n}, of a degree less than p^n in each γ_τ.

Now by Lemma 1, $\mathfrak{o}$ contains a subfield $\psi'(F^{p^n})$ isomorphic to F^{p^n}- we shall denote with ψ' this isomorphism. The polynomial obtained from $\bar{P}(X_1, X_2,\cdots, X_k)\in F^{p^n}[X_1, X_2,\cdots, X_k]$ by replacing the coefficient $f\in F^{p^n}$ by $\psi'(f)\in\psi'(F^{p^n})$ will be denoted with $P(X_1, X_2, \cdots, X_k)$.

Then, the correspondence ψ: $\bar{P}(\gamma_{\tau_1}, \gamma_{\tau_2},\cdots, \gamma_{\tau_k})\rightarrow P(c_{\tau_1}, c_{\tau_2}, \cdots, c_{\tau_k})$ of F into $\mathfrak{o}$ preserves addition and multiplication, and ψ is an extension of ψ'. Hence, $\psi(F)$ is a required subfield of $\mathfrak{o}$.

PROOF OF THEOREM 1. Now, let $\mathfrak{o}$ be a complete local ring having the characteristic $p(\neq 0)$, then the local rings $\mathfrak{o}_1=\mathfrak{o}/\mathfrak{m}$, $\mathfrak{o}_2=\mathfrak{o}/\mathfrak{m}^2$, $\mathfrak{o}_3=\mathfrak{o}/\mathfrak{m}^3,\cdots$ are primary in the sense above-mentioned, and every $\mathfrak{o}_i$ has a residue field isomorphic to F. Let $\theta_1, \theta_2, \theta_3,\cdots$ be the canonical homomorphisms of $\mathfrak{o}$ onto $\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3,\cdots$, $\varphi_{i,i+1}$ the canonical homomorphism of $\mathfrak{o}_{i+1}$ onto $\mathfrak{o}_i$, and φ_i the canonical homomorphism of $\mathfrak{o}_i$ onto F respectively. By Lemma 2, these rings $\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3,\cdots$ contain subfields $K_1, K_2, K_3,\cdots$ respectively, each of which is isomordhic to F. We can assume that $\varphi_{i,i+1}(K_{i+1})=K_i$. Indeed it is possible from the proof of Lemma 2 to construct K_{i+1} in such a way that $\varphi_{i,i+1}(\varphi_{i+1}^{-1}(\gamma_\tau)\cap K_{i+1})=\varphi_i^{-1}(\gamma_\tau)\cap K_i$, $\tau\in\Gamma$ where $\{\gamma_\tau\}_{\tau\in\Gamma}$ is a p-basis of F, then it is evident that $\varphi_{i,i+1}(K_{i+1})=K_i$.

Now, let $\alpha\in F$ and α_i be an element of $\varphi_i^{-1}(\alpha)\cap K_i$, $i=1, 2, 3,\cdots$ and let a_i be an element of $\theta_i^{-1}(\alpha_i)$, then we have a sequence $a_1, a_2, a_3,\cdots$ in $\mathfrak{o}$. This sequence is a Cauchy sequence in $\mathfrak{o}$, because $\theta_i(a_i)=\theta_i(a_{i+1})$ namely $a_i-a_{i+1}\in\mathfrak{m}^i$. Since $\mathfrak{o}$ is complete, this sequence has a limit a in $\mathfrak{o}$. It is evident that a is uniquely determined by α, not depending on the choice of a_i from $\theta_i^{-1}(\alpha_i)$, $i=1, 2, 3, \cdots$. This correspondence $\alpha \to a$ of into $\mathfrak{o}$ preserves evidently addition and multiplication. Hence, the image of F under this correspondence is a required subfield of $\mathfrak{o}$.

§ 2

Now let us suppose that $\mathfrak{o}$ has a characteristic different from the characteristic of the residue field F.

Then, as is well known, either the characteristic of $\mathfrak{o}$ is zero and $\mathfrak{m} \ni p(\neq 0)$, or the characteristic of $\mathfrak{o}$ is $p^r(\gamma > 1)$ and $\mathfrak{m} \ni p$. Then we have the following theorem, the generalization of the theorem due to Cohen.

THEOREM 2. *A complete local ring $\mathfrak{o}$ having a characteristic different from the characteristic $p(\neq 0)$ of the residue field F contains a complete subring R with the maximal ideal (p), such that $\varphi(R)=F$.*

To prove this theorem, we begin by a special case, i. e. by the following lemma.

LEMMA 3. *If $\mathfrak{o}$ is primary (in the sense mentioned in § 1), and has a characteristic different from the characteristic p of F, then $\mathfrak{o}$ contains a subring R with the maximal ideal (p), such that $\varphi(R)=F$.*

PROOF. Let n be a positive integer such that $\mathfrak{m}^n=(0)$. (It is to be noticed that here the meaning of n is different from that in § 1). Now let us consider the following sequence of subfields of F: F^p, $F^{p^2}, F^{p^3}, \cdots, F^{p^n}, \cdots, F^{p^{2n}}$. Let us denote with ψ_0 the mapping of F^{p^n} into $\mathfrak{o}$ defined as follows. Every element of F^{p^n} is of the form α^{p^n}, $\alpha \in F$. Let a be an element of $\mathfrak{o}$ such that $\varphi(a)=\alpha$. Then we put $\psi_0(\alpha^{p^n})=a^{p^n}$. It can be shown namely that a^{p^n} is independent of the choice of a in $\varphi^{-1}(\alpha)$. In fact, let $a+u$, $u \in \mathfrak{m}$ be another element of $\varphi^{-1}(\alpha)$. Then we have

$$(a+u)^{p^n} = a^{p^n} \sum_{i=1}^{p^n} \binom{p^n}{i} a^{p^n-i} u^i .$$

But it is easy to see that $p^{n-j} | \binom{p^n}{i}$ if $i=sp^j$, $(s, p)=1$, so that $\binom{p^n}{i} u^i \in \mathfrak{m}^n=(0)$ for all $i=1, 2, \cdots, p^n$. Then we have $(a+u)^{p^n}=a^{p^n}$.

Using this mapning ψ_0 of F^{p^n} into $\mathfrak{o}$, write A_{p^n}, $A_{p^{n+1}}$, $A_{p^{n+2}}$,

$\cdots, A_{p^{2n}}$ for $\psi_0\ (F^{p^n}), \psi_0(F^{p^{n+1}}), \psi_0(F^{p^{n+2}}), \cdots, \psi_0(F^{p^{2n}})$ respectively. Clearly we have $A_{p^n} \supset A_{p^{n+1}} \supset \cdots \supset A_{p^{2n}}$. It is easy to see that every A_{p^i}, $i=n, n+1, \cdots, 2n$ is closed under multiplication.

Put now $A = A_{p^{2n}} + pA_{p^{2n-1}} + p^2 A_{p^{2n-2}} + \cdots + p^{n-1} A_{p^{n+1}}$. We shall show that this subset A of $\mathfrak{o}$ is closed under addition and multiplication.

Let us begin by showing that $p^{n-1}A_{p^{n+1}}$ is closed under addition. This result follows immediately from the result $a^{p^{n+1}} + b^{p^{n+1}} \equiv (a+b)^{p^{n+1}}$ (mod. p). Now, assuming $p^{k+1}A_{p^{2n-k-1}} + p^{k+2}A_{p^{2n-k-2}} + \cdots + p^{n-1}A_{p^{n+1}}$ as closed under addition, we can prove that $p^k A_{p^{2n-k}} + p^{k+1}A_{p^{2n-k-1}} + p^{k+2}A_{p^{2n-k-2}} + \cdots + p^{n-1}A_{p^{n+1}}$ is also closed under addition. For, if we take two elements $p^k a^{p^{2n-k}}$ and $p^k b^{p^{2n-k}}$ of $p^k A_{p^{2n-k}}$, the sum of these two may be written in the form:

$$p^k a^{p^{2n-k}} + p^k b^{p^{2n-k}} = p^k (a+b)^{p^{2n-k}} - \sum_{i=1}^{p^{2n-k}-1} \binom{p^{2n-k}}{i} a^{p^{2n-k}-i} b^i .$$

But $p^k\binom{p^{2n-k}}{i}$ is zero unless $p^{n+1} | i$, because if $p^{n+1} \nmid i$, then $p^{n-k} | \binom{p^{2n-k}}{i}$. Hence, the second term of right hand side may be written in the form:

$$\sum c\, p^{2n-j} a^{p^{2n-k-j}-sp^j} b^{sp^j}, \qquad 2n-k>j>n, \quad (s,p)=1$$

where c and s are rational integers. By assumption of the induction, this term belongs to the set $p^{k+1}A_{p^{2n-k-1}} + p^{k+2}A_{p^{2n-k-2}} + \cdots + p^{n-1} A_{p^{n+1}}$. Hence we see that A is closed under addition. Then A is also closed under multiplication because $p^i A_{p^{2n-i}} \cdot p^j A_{p^{2n-j}} \subset p^{i+j} A_{p^{2n-i-j}}$.

If F is perfect, it is clear that A is a local ring with the maximal ideal (p), such that $\varphi(A) = F$. So we shall assum that F is not perfect, i. e. $F \neq F^p$.

Now, let us choose arbitrarily an element c_τ once for all from $\varphi^{-1}(\gamma_\tau)$, $\tau \in \Gamma$ as in the proof of lemma 2, where $\{\gamma_\tau\}_{\tau\in\Gamma}$ is a p-basis of F. Then the mapping $\bar{P}(\gamma_{\tau_1}, \gamma_{\tau_2}, \cdots, \gamma_{\tau_k}) \to P(c_{\tau_1}, c_{\tau_2}, \cdots, c_{\tau_k})$ of F into $\mathfrak{o}$ as used in the proof of lemma 2 may be considered as an extension of ψ_0, where $\bar{P}(\gamma_{\tau_1}, \gamma_{\tau_2}, \cdots, \gamma_{\tau_k})$ is a polynominal with coefficients in

F^{p^n}. Let us denote this mapping with ψ and put $\psi(F)=S$, S has clearly the property: $S\ni a$ implies $S\ni a^{-1}$; $a, a^{-1}\in\mathfrak{o}$. It is easy to see that we obtain the same subset S if we consider the polynomials $\bar{P}(\gamma_{\tau_1}, \gamma_{\tau_2},\cdots, \gamma_{\tau_k})$ with the coefficients in $F^{p^{n+i}}$ instead of in F^{p^n} where i is one of the numbers $1, 2,\cdots, n$.

On the other hand, let R be the ring generated by A and $\{c_\tau\}_{\tau\in\Gamma}$. Then we have $R=S++pS+p^2S+\cdots+p^{n-1}S$ by the construction of S and the relation $A=A_{p^{2n}}+pA_{p^{2n-1}}+p^2A_{p^{2n-2}}+\cdots+p^{n-1}A_{p^{n+1}}$.

It is evident that R is a local ring with the maximal ideal (p), such that $\varphi(R)=F$. This completes the proof of lemma 3.

PROOF OF THEOREM 2. Let k be a positive integer such that $\mathfrak{m}^k\not\ni p$. Then $\mathfrak{o}/\mathfrak{m}^k, \mathfrak{o}/\mathfrak{m}^{k+1}, \mathfrak{o}/\mathfrak{m}^{k+2}\cdots$, are primary rings. We see by Lemma 3 that every $\mathfrak{o}/\mathfrak{m}^i$, $i=k, k+1, k+2,\cdots$ includes the subring R_i with the maximal ideal (p), which is mapped on F by the canonical mapping $\mathfrak{o}/\mathfrak{m}^i\rightarrow F$. Let $\theta_k, \theta_{k+1}, \theta_{k+2},\cdots$ be canonical mappings of $\mathfrak{o}$ onto $\mathfrak{o}/\mathfrak{m}^k, \mathfrak{o}/\mathfrak{m}^{k+2}, \mathfrak{o}/\mathfrak{m}^{k+2}\cdots$ respectively, $\varphi_{i,i+1}$ the canonical mappfng of $\mathfrak{o}/\mathfrak{m}^{i+1}$ onto $\mathfrak{o}/\mathfrak{m}^i$, and φ_i be a canonical mapping of $\mathfrak{o}/\mathfrak{m}^i$ onto F. We can assume that $\varphi_{i,i+1}(R_{i+1})=R_i$. Indeed it is clearly possible by the proof of lemma 3 to construct R_{i+1} in such a way that

$$\varphi_{i,t+1}(\varphi_{i+1}^{-1}(\gamma_\tau)\cap R_{i+1})=\varphi_i^{-1}(\gamma_\tau)\cap R_i\,,\quad \tau\in\Gamma$$

where $\{\gamma_\tau\}_{\tau\in\Gamma}$ is a p-basis of F, then it is evident that

$$\varphi_{i,i+1}(R_{i+1})=R_i\,.$$

Now let us call a *fundamental sequence* a sequence $\alpha_k, \alpha_{k+1}, \alpha_{k+2}, \cdots(\alpha_i\in R_i)$ satisfying $\varphi_{i,i+1}(\alpha_{i+1})=\alpha_i$, $i=k, k+1, k+2,\cdots$. If $\alpha_k, \alpha_{k+1}, \alpha_{k+2},\cdots$ and $\beta_k, \beta_{k+1}, \beta_{k+2},\cdots$ are fundamental seqnences, then $\alpha_k+\beta_k$, $\alpha_{k+1}+\beta_{k+1}$, $\alpha_{k+2}+\beta_{k+2},\cdots$ and $\alpha_k\cdot\beta_k, \alpha_{k+1}\cdot\beta_{k+1}, \alpha_{k+2}\cdot\beta_{k+2},\cdots$ are obviously also fundamental sequences. In this sense, fundamental sequences form a ring $\mathfrak{R}$. We shall write fundamental sequences with Greek capitals

$$\boldsymbol{A}=(\alpha_k, \alpha_{k+1}, \alpha_{k+2},\cdots),\ \boldsymbol{B}=(\beta_k, \beta_{k+1}, \beta_{k+2},\ldots)$$

and with

$$\boldsymbol{A}+\boldsymbol{B}=(\alpha_k+\beta_k, \alpha_{k+1}+\beta_{k+1}, \alpha_{k+2}+\beta_{k+2},\cdots)\,,$$

$$\boldsymbol{A}\cdot\boldsymbol{B}=(\alpha_k\beta_k, \alpha_{k+1}\beta_{k+1}, \alpha_{k+2}\beta_{k+2},\cdots).$$

Now we can map the ring $\mathfrak{R}\ni\boldsymbol{A},\boldsymbol{B},\cdots$ onto a ring R of $\mathfrak{o}$ in the following way. Let $\boldsymbol{A}=(\alpha_k, \alpha_{k+1}, \alpha_{k+2},\cdots)\in\mathfrak{R}$. Choose an element a_i of $\theta_i^{-1}(\alpha_i)$ arbitrarily and consider the sequence $a_k, a_{k+1}, a_{k+2},\cdots$. This sequence $a_k, a_{k+1}, a_{k+2},\cdots$ is a Cauchy sequence in $\mathfrak{o}$ because $\theta_i(a_{i+1}-a_i)=0$ i.e. $a_{i+1}-a_i\in\mathfrak{m}^i$, so it has a limit a in $\mathfrak{o}$. Moreover, it is evident that this limit a is uniquely determined by $\boldsymbol{A}$, not depending on the choice of a_i in $\theta_i^{-1}(\alpha_i)$.

We shall write $\Phi(\boldsymbol{A})=a$. This mapping Φ of $\mathfrak{R}$ into $\mathfrak{o}$ is clearly a homomorphism. So $\Phi(\mathfrak{R})=R$ is a subring of $\mathfrak{o}$, and it is clear that $\varphi(R)=F$.

Now we shall show that R has the maximal ideal (p). Remark first that $R_i\subset\mathfrak{o}/\mathfrak{m}^i$, $i=k, k+1, k+2,\cdots$ has the maximal ideal (p), and $\varphi_{i,i+1}$ maps units and non-units of R_{i+1} to units and non-units of R_i respectively. Thus $\boldsymbol{A}=(\alpha_k, \alpha_{k+1}, \alpha_{k+2},\cdots)$ is a unit or a non-unit of $\mathfrak{R}$ according as α_k is a unit or a non-unit of R_k, i.e. $\alpha_k\notin(p)$ or $\in(p)$. Therefore $\Phi(\boldsymbol{A})=a$ is a unit or a non-unit of R, according as $a\notin(p)$ or $\in(p)$.

It remains to show the completeness of R. To the purpose let us considor a Cauchy sequence $a_1, a_2, a_3,\cdots$ in R. Such a sequence is clearly a Cauchy sequence in $\mathfrak{o}$, so has a limit a in $\mathfrak{o}$. On the other hand, the sequence $\theta_i(a_1), \theta_i(a_2), \theta_i(a_3)\cdots$ is a Cauchy sequence in R_i, so has a limit α_i in R_i since R_i is primary, manely complete. It is evident that $\theta_i(a)=\alpha_i$, $(\alpha_k, \alpha_{k+1}, \alpha_{k+2},\cdots)=\boldsymbol{A}\in\mathfrak{R}$ and $\Phi(\boldsymbol{A})=a$, and this means that $a\in R$. Hence R is complete.

To go further, we need the following lemma.

LEMMA 4. *Let $\mathfrak{o}$ be a complete local ring with the maximal ideal (p), then any complete subring $\mathfrak{o}'$ of $\mathfrak{o}$ with a maximal ideal (p), such that $\mathfrak{o}'/(p)=F=\mathfrak{o}/(p)$ coincides with $\mathfrak{o}$ itself.*

PROOF. Any element a of $\mathfrak{o}$ may be expressed in the form $a_0'+a_1p$; $a_0'\in\mathfrak{o}'$, $a_1\in\mathfrak{o}$ as $\varphi(\mathfrak{o})=\varphi(\mathfrak{o}')$. In particular, we have a relation $a_1=a_1'+a_2p$; $a_1'\in\mathfrak{o}'$, $a_2\in\mathfrak{o}$. Hence we have $a=a_0'+a_1'p+a_2p^2$: a_0', $a_1'\in\mathfrak{o}'$, $a_2\in\mathfrak{o}$. Generally, we have a relation $a=a_0'+a_1'p+a_2'p^2+\cdots a_{i-1}'p^{i-1}+a_ip^i$; $a_0', a_1', a_2',\cdots, a_{i-1}'\in\mathfrak{o}'$, $a_i\in\mathfrak{o}$. The sequence $a_0', a_0'+a_1'p, a_0'+a_1'p+a_2'p^2,\cdots$ has a limit a' in $\mathfrak{o}'$ since $\mathfrak{o}'$ is complete by assumption. On the other hand, $a-(a'_0+a'_1p+a'_2p^2+\cdots+a'_{i-1}p^{i-1})=a_ip^i\in(p^i)$, and this converges to zero when $i\rightarrow\infty$. This means $a-a'=0$, therefore $a\in\mathfrak{o}'$.

Combining this lemma with Theorem 2, we obtain the following theorem.

THEOREM 3. *Let $\mathfrak{o}$ be a complete local ring having a characteristic different from the characteristic p of the residue field F. If F is perfect, then there exists only one complete subring R of $\mathfrak{o}$ with the maximal ideal (p), such that $\varphi(R)=F$. In the other case, complete subrings $R, R',\cdots$ of $\mathfrak{o}$ with the maximal ideal (p), such that $\varphi(R)=F$, $\varphi(R')=F,\cdots$, are mutually isomorphic to each other.*

PROOF. We shall first show that any R can be regarded as a subring of $\mathfrak{o}$ constructed by the method used in the proof of Theorem 2. In considering

$$\mathfrak{R}_0=\{\boldsymbol{A}\,;\,\boldsymbol{A}=(\alpha_k,\alpha_{k+1},\alpha_{k+2},\cdots),\ \alpha_i\in R/\mathfrak{m}^i\subset\mathfrak{o}/\mathfrak{m}^i,\ \varphi_{i,i+1}(\alpha_{i+1})=\alpha_i\}\ ,$$

we obtain a subring $\Phi(\mathfrak{R}_0)=R_0$ of R, but Lemma 4 asserts $R_0=R$. In fact, this construction with R_0 is the one of the constructions in the proof of theorem 2.

So the first part of this theorem follows from the proof of Lemma 3 and Theorem 2.

For the second part of this theorem, we may regard all $R, R',\cdots$ as constructed in the mentioned way, using eventually different c_τ's in $\varphi^{-1}(\gamma_\tau)$, $\tau\in\Gamma$ in the proof of Lemma 3, Then the isomorphism of $R, R',\cdots$ is evident from the proof of Lemma 3 and Theorem 2.

Now we can easily prove thefollowing result. (cf. [1] and [2]).

THEOREM 4. *Let $\mathfrak{o}$ and $\mathfrak{o}'$ be complete local rings with a maximal ideal (p), having the residue field isomorphic to each other. If $\mathfrak{o}$ has the same characteristic as $\mathfrak{o}'$. then $\mathfrak{o}$ and $\mathfrak{o}'$ are mutually isomorphic to each other. If the characteristic of $\mathfrak{o}$ is p^γ $(\neq 0)$, and the characteristic of $\mathfrak{o}'$ is zero or $p^{\gamma'}$ $(\gamma'>\gamma)$, then there exists a homomorphism of $\mathfrak{o}'$ onto $\mathfrak{o}$.*

PROOF. The first part of this theorem is evident by the proof of Theorem 3. The second part of the theorem can be proved immediately from the first part in considering the factor ring $\mathfrak{o}'/(p^\gamma)$.

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