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On geometry of numbers.

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1. Introduction. In a recent paper Tsuji [3] has given some theorems which may be considered as a generalization to Fuchsian groups of the classical theorems of Blichfeldt [1] and Minkowski in the geometry of numbers. In both cases, however, the generalization is restricted to circular domains.

The object of this paper is to carry this generalization further. First, by considering more general groups than the Fuchsian groups; second, by considering a kind of domains more general than the circular domains.

2. A preliminary integral formula. Let S be a space of points in which operates a transitive group of transformations G. For a point $P \in S$ and an element $s \in G$ we denote with sP the transform of P by s. The elements s of G with $sP_0=P_0$ for a given point P_0 of S form a subgroup G_0 of G. Let G_1 be a simply-transitive subgroup of G. Then we have obviously $G_1G_0=G$, $G_1 \frown G_0=\{e\}$ (unit group), and we can identify G_1 with the homogeneous space G/G_0 or with the space S by assigning $x \in G_1$ to $xG_0 \in G/G_0$ or to $xP_0 \in S$. G may be then considered as operating on G_1, as well as on S, in the following manner. Let s, $t \in G$, $x \in G_1$, then sxt, properly an element of G, is identified with the element of G_1 corresponding to $sxtG_0$ in G/G_0. We assume now G_1 to be locally compact and provide G/G_0 resp. S with the same topology as that of G_1. We assume further that G_1 is unimodular, i. e. that the left invariant measure of G_1 is also right invariant; and that this measure is also invariant with respect to G, so that we have $m(H)=m(sHt)$ for a point-set H in G_1 and any $s, t \in G$. This may be symbolically written as follows ([4, 34]):

(2.1) $dx=dx^{-1}$, $dx=d(sxt)$ for any $s, t \in G$,

(2.2) $m(H)=\int_{G_1}\varphi(x)dx$,

where dx denotes the element of volume of G_1 and $\varphi(x)$ the characteristic function of H. By the above identification of G_1 with S, (2.2) gives also the measure of a set H in S if we set $\varphi(x)=1$ if $xP_0\in H$ and $\varphi(x)=0$ otherwise. Throughout the paper we shall only consider measurable sets.

Let K_0 be a set of N fixed points P_i $(i=1,2,3,\dots,N)$ of S. If we set now $\varphi_i(x)=1$ if $xP_i\in H$ and $\varphi_i(x)=0$ otherwise, according to (2.2) and (2.1) we have

$$m(H)=\int_{G_1}\varphi_i(x)dx=\int_{G_1}\varphi_i(x^{-1})dx \tag{2.3}$$

where $\varphi_i(x^{-1})=1$ if $P_i\in xH$ and $\varphi_i(x^{-1})=0$ otherwise.

Let $f(P_i)$ be a function defined on the points P_i. We have

$$\begin{aligned} I&=\int_{G_1}\sum_{P_i\in xH}f(P_i)dx=\sum_1^N\int_{G_1}f(P_i)\varphi_i(x^{-1})dx \\ &=\sum_1^N f(P_i)\int_{G_1}\varphi_i(x^{-1})dx=m(H)\sum_1^N f(P_i)\,. \end{aligned} \tag{2.4}$$

In particular, if $f(P_i)=1$ and $\nu(K_0\cap xH)$ denotes the number of points P_i which belong to xH, we have, [2],

$$\int_{G_1}\nu(K_0\cap xH)dx=Nm(H)\,. \tag{2.5}$$

3. An analogue to Blichfeldt's theorem. Let us suppose that there exists a partition of S into fundamental domains $D_h(h=0,1,2,\dots)$ and a discrete subgroup F of G (*) such that:

a) Each D_h is the transform of D_0 by the transformation x_h of F (x_0=unit element of F). That is

$$D_h=x_hD_0\,,\qquad x_h\in F\,. \tag{3.1}$$

b) Each $x_h\in F$ transforms a fundamental domain D_k into a fundamental domain D_l, with $l\neq k$ if $h\neq 0$.

c) The fundamental domains are measurable and $0<m(D_h)=m(D_0)<\infty$.

Let K_0 be a set of N points P_i $(i=1,2,\dots,N)$ contained in D_0 and let $f(P_i)$ be a function defined on the points P_i such that $f(x_hP_i)=f(P_i)$.

(*) Note that F is not necessarily a subgroup of G_1.

Since S is the sum of the sets D_h and the space of the group G_1 coincides with S, the integral (2.4) can be written

$$I=\int_{G_1}\sum_{P_i\in xH} f(P_i)dx=\sum_h\int_{D_h}\sum_{P_i\in xH} f(P_i)dx\,. \tag{3.2}$$

By the change of variables $x'=x_h^{-1}x$, having into account the invariance (2.1), we have

$$I=\sum_h\int_{D_0}\sum_{P_i\in x_h xH} f(P_i)dx=\sum_h\int_{D_0}\sum_{x_h^{-1}P_i\in xH} f(P_i)dx \tag{3.3}$$

and, by (2.4),

$$\int_{D_0}\sum_h\sum_{x_h^{-1}P_i\in xH} f(P_i)dx=m(H)\sum_1^N f(P_i)\,. \tag{3.4}$$

This means that if we draw on S the lattice formed by all points $x_hK_0(h=0,1,2,\ldots)$ and for each position xH of H with $x\in D_0$, we carry out the addition $\sum f(P_i)$ over the lattice points contained in xH (with the assumption $f(x_hP_i)=f(P_i)$), the integral formula (3.4) holds.

Therefore, the mean value of the sum $\sum f(P_i)$ $(P_i\in xH)$ is equal to

$$m.\,v.\Big(\sum f(P_i)\Big)=\frac{m(H)}{m(D_0)}\sum_1^N f(P_i) \tag{3.5}$$

and we have

THEOREM 1. *Let S, G, G_1, F be the space and the groups already specified. Given N fixed points P_i inside the fundamental domain D_0, we consider the lattice of all points x_hP_i $(x_h\in F)$ and a function $f(P_i)$ defined on the points P_i such that $f(x_hP_i)=f(P_i)$. Then, for every measurable set H of S there are transforms $x'H$ $(x'\in G_1)$ for which the sum $\sum f(P_i)$ extended over the lattice points contained in $x'H$ is not less than the right hand side of* (3.5) *and transforms $x''H(x''\in G_1)$ for which that sum is not greater than the right hand side of* (3.5).

If $f(P_i)=1$, (3.5) gives that the mean value of the number of lattice points contained in H is $Nm(H)/m(D_0)$ and the Theorem gives bounds for the number of lattice points that suitable transforms of H can contain.

If S is the euclidean space, G the group of motions, G_1 the group of translations and F the subgroup of translations which leaves unchanged the lattice of points with integral coordinates, then Theorem 1 coincides with Blichfeldt's theorem.

If S is the unit circle $|z|<1$ of the complex plane, G the group of non-euclidean motions $(z'=(bz+a)/(\bar{a}z+\bar{b}))$, G_1 the group of non-euclidean translations $(z'=(z+a)/(1+\bar{a}z), |a|<1)$ and F is a Fuchsian group, then Theorem 1 gives the Theorem 5 of Tsuji [3] generalized to domains not necessarily convex.

4. A lemma. With the same notations of n°3, let H be a set of points such that $m(H)>m(D_0)$. Let $D_i(i=1, 2,\dots)$ be the fundamental domains which have common point with H and let us consider the set of points $\sum_i(D_0\frown x_i^{-1}H)$. Since the measure is preserved by x_i, having into account (3.1), we have

$$m(D_0\frown x_i^{-1}H)=m(x_i(D_0\frown x_i^{-1}H))=m(x_iD_0\frown H)=m(D_i\frown H)$$

and therefore

$$m\Big(\sum_i(D_0\frown x_i^{-1}H)\Big)=\sum_i m(D_0\frown x_i^{-1}H)=\sum_i m(D_i\frown H)=m(H)$$

As a consequence, since $m(H)>m(D_0)$, the sets $D_0\frown x_h^{-1}H$ overlap, that is, there is a point P which belong to two sets $D_0\frown x_i^{-1}H$, say $D_0\frown x_1^{-1}H$, $D_0\frown x_2^{-1}H$. Therefore P belongs to $x_1^{-1}H$ and to $x_2^{-1}H$ and since x_1^{-1}, x_2^{-1} belong to F, we have the following

LEMMA. *If $m(H)>m(D_0)$, then the equivalents of H with respect to F overlap.*

For the particular case of Fuchsian groups this lemma coincides with Theorem 1 of Tsuji [3].

5. An analogue to Minkowski's theorem. With the same nomenclature as in n°3, let us now consider the case $N=1$. That is, given a fixed point P_0 in D_0, we consider the point lattice $x_hP_0(x_h\in F)$.

Given a domain H which contains P_0, we shall say that a domain H^* contained in H is an m-domain of H (with respect to the group G) if, for $x\in G$, $y\in G$, the following two conditions are satisfied:

a) If $xP_0\in H^*$, then $x^{-1}P_0\in H^*$.

b) If $xP_0\in H^*$ and $yP_0\in H^*$, then $xyP_0\in H$.

Let us assume that H admits an m-domain H^* such that $m(H^*)>m(D_0)$. Then, by the lemma of n°4, the equivalent sets of H^* with respect to F overlap. This means that there exists a point $P=xP_0$ which belongs to H^* and to a transform, say x_1H^*, of H^*. That is, we have $xP_0\in H^*$, $xP_0\in x_1H^*$, or

$$xP_0\in H^*\,, \qquad x_1^{-1}xP_0\in H^*\,.$$

Since H^* is an m-domain of H, we have by a)

$$xP_0\in H^*\,,\quad (x_1^{-1}x)^{-1}P_0=x^{-1}x_1P_0\in H^*$$

and by b)

$$xx^{-1}x_1P_0=x_1P_0\in H\,.$$

Since x_1P_0 is a lattice point, we have the following

THEOREM 2. *If the domain H which contains the point P_0 possesses an m-domain H^* such that $m(H^*)>m(D_0)$, then H contains a lattice point distinct from P_0.*

EXAMPLES. 1. Let S be the n-dimensional euclidean space, and let now $G=G_1$ be the group of translations in it; let F be the subgroup which leaves unchanged the lattice of points with integral coordinates. Let P_0 be the origin of coordinates and let H be a convex domain with center of symmetry at P_0. Then it is easy to see that the homothetic domain of H with center P_0 and ratio $\frac{1}{2}$ is an m-domain of H. Moreover, if V denotes the volume of H, we have $m(H^*)=V/2^n$ and $m(D_0)=1$. Therefore, if $V>2^n$, then H contains a lattice point distinct from the origin and we have the classical theorem of Minkowski.

2. Let S be the unit circle $|z|<1$ of the complex plane, G the group of non-euclidean (n. e.) motions $(z'=(bz+a)/(\bar{a}z+\bar{b}))$ and F a Fuchsian group. Let P_0 be the origin $z=0$. Let H be the disc $|z|\leq\rho<1$ and let H^* be the disc $|z|\leq\rho_0$, where ρ_0 is related to ρ by the equation

$$\rho=\frac{2\rho_0}{1+\rho_0^2}\,.$$

According to the n. e. metric $ds=2|dz|/(1-|z|^2)$, the n. e. radius of H^* is one half of that of H. We want to prove that H^* is an m-domain of H, that is, that the conditions a), b) are satisfied.

a) Let x be the n. e. motion $z'=(bz+a)/(\bar{a}z+\bar{b})$. Since P_0 is the origin $z=0$, the point xP_0 is the point $z'=a/\bar{b}$. The point $x^{-1}P_0$ is $z''=-a/b$ and therefore if we assume $|z'|=|a/b|\leq\rho_0$, we have $|z|=|a/b|=|z''|\leq\rho_0$.

b) Let r be the n. e. length of ρ and σ the n. e. length of the segment which unites $P_0(z=0)$ with the point yP_0; if we assume $yP_0\in H^*$, then we have $\sigma\leq r/2$. Since the n. e. motions preserve the n. e. lengths, the n. e. length of the segment (arc of geodesic) which unites the points xP_0 and xyP_0 is equal to σ. Therefore, from $xP_0\in H^*$ and $\sigma\leq r/2$ we deduce $xyP_0\in H$.

Consequently H^* is an m-domain of H and Theorem 2 can be applied. Having into account that, in this particular case, we have $m(H)=4m(H^*)+\pi^{-1}m^2(H^*)$, if $m(H)>4m(D_0)+\pi^{-1}m^2(D_0)$, then H contains a lattice point distinct from $z=0$, and we have the theorem 2 of Tsuji [3].

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