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A remark on my former paper "Theory of Fuchsian groups".

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Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain. We denote its part, contained in $|z|<\rho<1$ by $D_0(\rho)$. We define the non-euclidean line- and surface-element by

$$ds=\frac{2|dz|}{1-|z|^2}, \quad d\sigma=\frac{4\,dxdy}{(1-|z|^2)^2}, \quad z=x+iy. \tag{1}$$

Let $a\in D_0$ and we denote its equivalents by a_ν $(\nu=0,1,2,\cdots, a_0=a)$ and $n(r,a)$ be the number of a_ν, contained in $|z|<r<1$ and put

$$N(r,a)=\int_{\frac{1}{2}}^{r}\frac{n(r,a)dr}{r}. \tag{2}$$

Let $a\in D_0, b\in D_0$ $(a\neq b)$, then there exists a potential function $u(z;a,b)$, which is invariant by G and is harmonic in $|z|<1$, except at a_ν, b_ν, where

$$u(z;a,b)-\log\frac{1}{|z-a_\nu|}, \quad u(z;a,b)+\log\frac{1}{|z-b_\nu|}$$

are harmonic.

Let $u^+=u$, if $u\geqq 0$, $u^+=0$, if $u\leqq 0$, and put for a fixed b

$$m(r,a)=\frac{1}{2\pi}\int_0^{2\pi}u^+(re^{i\theta};a,b)d\theta, \tag{3}$$

$$T(r,a)=m(r,a)+N(r,a). \tag{4}$$

Let $b\in D_0$ and $U: |z-b|<\delta$ and $a\in D_0(\rho)-U$. Then in my former paper,[1] I have proved that

$$T(r, a)=T_\rho(r)+O(1), \tag{5}$$

where $O(1)$ is uniformly bounded for $a\in D_0(\rho)-U$, $1/2\leqq r<1$ and

$$T_\rho(r)=\int_0^r \frac{S(r)dr}{r}, \quad S(r)=\frac{A(r)}{\sigma(D_0(\rho))}, \tag{6}$$

$$A(r)=\iint_{D_0(\rho)} n(r, a)d\sigma(a).$$

If $\sigma(D_0)<\infty$, $$T_\rho(r)=\frac{2\pi}{\sigma(D_0)}\log\frac{1}{1-r}+O(1), \tag{7}$$

If $\sigma(D_0)=\infty$, $$\lim_{r\to 1} T_\rho(r)\,/\log\frac{1}{1-r}=0.$$

It can be proved that $m(r, a)=O(1)$. Hence

THEOREM. $N(r, a)=T_\rho(r)+O(1)$, $a\in D_0(\rho)$, *where* $O(1)$ *is uniformly bounded for* $a\in D_0(\rho)$, $1/2\leqq r<1$.

Since $m(r, a)$ does not appear in the theorem, we need not prove (5). We shall prove the theorem directly, without any reference to the former paper, in the following lines. First we shall prove a lemma.

LEMMA. *If* $a\in D_0(\rho)$, *then*

$$n(r, a)\leqq\frac{K}{1-r},$$

where K *is a constant independent of* $a\in D_0(\rho)$.

PROOF. Let $a\in D_0(\rho)$ and $\varDelta: \left|\frac{z-a}{1-\bar{a}z}\right|<\delta$ be a disc. We take $\delta>0$ so small that $\varDelta$ is contained in $|z|<1$ and its equivalents $\varDelta_\nu$ do not overlap, for any $a\in D_0(\rho)$. Let a_ν, $\varDelta_\nu$ be equivalents of a, $\varDelta$, such that $a_\nu\in\varDelta_\nu$. If $|a_\nu|<r$, then $\varDelta_\nu$ is contained in $|z|<r'$ $(r<r')$, such that

1) M. Tsuji: Theory of Fuchsian groups. Jap. Journ. Match. 21 (1951).

$1-r \leqq \text{const.}\,(1-r')$, so that

$$n(r, a)\,\sigma(\Delta) \leqq 4 \iint_{|z|<r'} \frac{r\,dr\,d\theta}{(1-r^2)^2} \leqq \frac{\text{const.}}{1-r'} \leqq \frac{\text{const.}}{1-r}\,, \quad \text{hence } n(r, a) \leqq \frac{K}{1-r}\,.$$

PROOF OF THE THEOREM.

Let $b \in D_0(\rho)$, $b \neq 0$ and U: $|z-b|<\delta$ be contained in $D_0(\rho)$ and $a \in D_0(\rho)-U$. We assume that $a \neq 0$ and there are no a_ν, b_ν on $|z|=r$. Applying the Green's formula: $\int_C \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) ds = 0$ to $u=u(z\,;\,a, b)$, $v=\log\frac{r}{|z|}$ for the domain, bounded by $|z|=r$ and small circles about a_ν, b_ν in $|z|<r$ and a small circle about $z=0$ and then making the radii of these circles tend to zero, we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}\,;\,a, b)\,d\theta + \sum_{|a_\nu|<r} \log\frac{r}{|a_\nu|} - \sum_{|b_\nu|<r} \log\frac{r}{|b_\nu|} = u(0\,;\,a, b)\,,$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}\,;\,a, b)\,d\theta + \int_0^r \frac{n(r, a)\,dr}{r} - \int_0^r \frac{n(r, b)\,dr}{r} = u(0\,;\,a, b)\,.$$

Hence if there are no a_ν, b_ν on $|z|=1/2$, then

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{1}{2}e^{i\theta}\,;\,a, b\right) d\theta + \int_0^{\frac{1}{2}} \frac{n(r, a)\,dr}{r} - \int_0^{\frac{1}{2}} \frac{n(r, b)\,dr}{r} = u(0\,;\,a, b)\,,$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}\,;\,a, b)\,d\theta + N(r, a) - N(r, b) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{1}{2}e^{i\theta}\,;\,a, b\right) d\theta. \tag{1}$$

We see easily that (1) holds, if $a=0$, $b=0$ or there are a_ν, b_ν on $|z|=r$ and $|z|=1/2$, hence (1) holds in general.

In the following, const. denotes a constant, which is independent of $a \in D_0(\rho)-U$, $1/2 \leqq r < 1$, b being fixed. Since

$$\left|\int_0^{2\pi} u\left(\frac{1}{2}e^{i\theta}\,;\,a, b\right) d\theta\right| \leqq \text{const.}, \tag{2}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta};\ a,b)d\theta + N(r,a) = N(r,b) + O(1), \tag{3}$$

where $|O(1)| \leqq \text{const.}$.

Let $a \in D_0(\rho) - U$ and $\varDelta$: $|z-a| < \eta < \delta/2$, where η is taken so small that $\varDelta$ lies in $|z|<1$ and its equivalents $\varDelta_\nu$ do not overlap. Then $|a-b| > \delta/2$. Let U' : $|z-b| < \frac{\delta}{2}$ and U'_ν be its equivalents.

Then as was proved in the former paper,

$$|u(z;\ a,b)| \leqq \text{const., outside of } \sum_\nu \varDelta_\nu + \sum_\nu U'_\nu. \tag{4}$$

The part of $|z|=r$, which is contained in $\sum_\nu \varDelta_\nu$ consists of a finite number of arcs θ_i ($i=1,2,\cdots,N$) and that in $\sum_\nu U'_\nu$ consists of a finite number of arcs θ'_j ($j=1,2,\cdots,N'$). Hence by (4),

$$\int_0^{2\pi} |u|d\theta \leqq \sum_{i=1}^{N}\int_{\theta_i} |u|\, d\theta + \sum_{j=1}^{N'}\int_{\theta'_j} |u|\, d\theta + \text{const.}. \tag{5}$$

Now

$$\int_{\theta_i} |u|\, d\theta = \frac{1-r^2}{r}\int_{\theta_i} |u| \frac{|dz|}{1-|z|^2} \leqq 3(1-r)\int_{\theta_i} |u|\frac{|dz|}{1-|z|^2},\quad (1/2 \leqq r < 1).$$

Let $\tilde{\theta}_i$ be the equivalent of θ_i in D_0, then $\tilde{\theta}_i$ is a circular are in $\varDelta$ and since $|u|\frac{|dz|}{1-|z|^2}$ is invariant by G, we have

$$\int_{\theta_i} |u| \frac{|dz|}{1-|z|^2} = \int_{\tilde{\theta}_i} |u| \frac{|dz|}{1-|z|^2}.$$

In $\varDelta$, $|u| \leqq \log\frac{1}{|z-a|} + \text{const.}$, so that $\int_{\tilde{\theta}_i} |u| \frac{|dz|}{1-|z|^2} \leqq \text{const.}$,

hence

$$\sum_{i=1}^{N}\int_{\theta_i} |u|\, d\theta \leqq \text{const. } N(1-r).$$

Similarly

$$\sum_{j=1}^{N'}\int_{\theta'_j} |u|\, d\theta \leqq \text{const. } N'(1-r),$$

hence by (5),

$$\int_0^{2\pi}|u|d\theta \leqq \text{const. } N(1-r)+\text{const. } N'(1-r), \ +\text{const.}$$

If Δ_ν intersects with $|z|=r$, then $a_\nu \in \Delta_\nu$ lies in $|z|<r'$ $(r<r')$, such that $1-r \leqq$ const. $(1-r')$, so that by the lemma,

$$N \leqq n(r', a) \leqq \frac{\text{const.}}{1-r'} \leqq \frac{\text{const.}}{1-r},$$

hence

$$\int_0^{2\pi}|u|d\theta \leqq \text{const.} \tag{6}$$

Hence by (3),

$$N(r, a)=N(r, b)+O(1), \tag{7}$$

where $|O(1)| \leqq$ const., so that

$$\iint_{D_0(\rho)-U} N(r, a)d\sigma(a)=(\sigma(D_0(\rho)-\sigma(U))\,(N(r, b)+O(1))=$$
$$(\sigma(D_0(\rho)-\sigma(U))N(r, a)+O(1)\,. \tag{8}$$

Let U'': $|z-b|<2\delta$ and a be fixed in $D_0(\rho)-U''$ and b' vary in U: $|z-b|<\delta$, then (7) holds with b' instead of b, so that

$$\iint_U N(r, b')d\sigma(b')=\sigma(U)\, N(r, a)+O(1)\,. \tag{9}$$

If we add (8), (9), we have

$$N(r, a)=\frac{1}{\sigma(D_0(\rho))}\iint_{D_0(\rho)} N(r, a)d\sigma(a)+O(1)=$$
$$=T_\rho(r)+O(1). \tag{10}$$

We assumed that $a \in D_0(\rho)-U''$. If $a \in U''$, then we take b' outside of U'' and we see that (10) holds for $a \in U''$. Hence (10) holds for any $a \in D_0(\rho)$. If $\sigma(D_0)<\infty$, then we see that $O(1)$ in (8), (9), (10) is bounded for $\rho \to 1$, if $a \in D_0(\rho)$. Hence we have

$$N(r, a)=\frac{1}{\sigma(D_0)}\iint_{D_0} N(r, a)\, d\sigma(a)+O(1)$$
$$=\frac{1}{\sigma(D_0)}\int_0^r \frac{dr}{r}\iint_{D_0} n(r, a)d\sigma(a)+O(1)$$
$$=T_\rho(r)+O(1), \ a \in D_0(\rho)\,. \tag{11}$$

Since

$$\iint_{D_0} n(r, a)d\sigma(a) = 4\iint_{|z|<r} \frac{rdrd\theta}{(1-r^2)^2} = \frac{4\pi r^2}{1-r^2},$$

we have

$$T_\rho(r) = \frac{2\pi}{\sigma(D_0)} \log \frac{1}{1-r} + O(1). \tag{12}$$

If $\sigma(D_0) = \infty$, then

$$T_\rho(r) = \frac{1}{\sigma(D_0(\rho))} \int_0^r \frac{dr}{r} \iint_{D_0(\rho)} n(r, a)d\sigma(a) \leqq \frac{1}{\sigma(D_0(\rho))} \int_0^r \frac{dr}{r} \iint_{D_0} n(r, a)d\sigma(a)$$

$$= \frac{2\pi}{\sigma(D_0(\rho))} \log \frac{1}{1-r} + O(1).$$

Since $N(r, 0) = T_\rho(r) + O(1)$, we have $\overline{\lim\limits_{r\to1}}\, N(r, 0) / \log \frac{1}{1-r} \leqq \frac{2\pi}{\sigma(D_0(\rho))} \to 0$, $\rho \to 1$, so that $\lim\limits_{r\to1} N(r, 0) / \log \frac{1}{1-r} = 0$, hence

$$\lim_{r\to1} T_\rho(r) / \log \frac{1}{1-r} = 0\,. \tag{13}$$

Hence our theorem is proved.

REMARK. I have found that the proof of Theorem 8 of the former paper is false.

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