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A characterization theorem for lattices with Hausdorff interval topology.

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1. Introduction. The problem of finding necessary and sufficicnt conditions that determine Hausdorff interval topologies in lattices was posed by Birkhoff [1][1]. It has been solved in the particular case of Boolean algebras[2] by Katetov [2] and by Northam [3]. The latter has found a necessary condition that a lattice be Hausdorff in the interval topology, the condition being that *every closed interval in the lattice has a finite separating set*[3]. In this note, we shall show that the notion of a certain type of separating set for the lattice is strong enough to yield a characterization of lattices with Hausdorff topology. We obtain this result from consideration of the relationship between a sub-basis for the closed sets and the Hausdorff separation principle[4].

We here recollect some standard terms and introduce a definition of comparison for subsets of a partially ordered set. Let P be a set of points, written $a, b, \cdots, x, y$. P is *partially ordered* if it is subject to a binary relation $\leqq$ which is reflexive, antisymmetric, and transitive. P is a *lattice* if it contains with every pair of elements their least upper bound and greatest lower bound. In P, if neither $x \leqq y$ nor $y \leqq x$, then x and y are said to be *incomparable* and this is denoted

1) Numbers in brackets represent references listed at the end of the paper.

2) If B is a Boolean algebra, then B has a Hausdorff interval topology if and only if, for every non-zero x in B, there exists some atom e such that $e \leqq x$. (An ***atom*** is a non-zero element e such that $0 < y \leqq e$ implies that $y = e$.)

3) Northam defines a separating set for closed intervals in the following way. Let x and y be two elements in a partially ordered set, with $x < y$. A set of elements (a_i) is called a *separating set* for the closed interval $[x, y]$ if $x < a_i < y$, all i, and every element in $[x, y]$ is comparable with at least one a_i. This requires that intervals containing less then three elements are said to be separated by the empty set.

4) I am indebted to L. Gillman for several suggestions for notation which I have used below.

by $x \# y$. If X and Y are nonempty subsets of P, we define $X < Y$ to mean that $x \in X$, $y \in Y$ implies that either $x < y$ or $x \# y$. Similarly $X \leqq Y$ means that either $x \leqq y$ or $x \# y$ whenever $x \in X$, $y \in Y$. (We shall take the liberty of writing $a \leqq Y$ when X reduces to a set consisting of the single element a.) The *interval topology* for P is defined by taking as a sub-basis for the closed sets the class $\mathfrak{F}$ of all sets (half intervals) of the form $[x\colon x \leqq a]$ and $[x\colon a \leqq x]$. It is convenient to introduce the notation $\hat{a}$ and $\check{a}$ to denote, respectively, the preceding half intervals. By a *covering* of an arbitrary set M we mean a collection of subsets of M whose union is M. We let E' denote the complement of a set E.

2. The Hausdorff interval topology.

LEMMA. *Let $(W_\lambda)_{\lambda \in \Gamma}$ be an indexed class of sets which is a covering for a space X. If $(\Gamma_\alpha)_{\alpha \in A}$ is in turn a covering of Γ, then*

$$\bigcap_{\alpha \in A} [(\bigcup_{\lambda \in \Gamma_\alpha} W_\lambda)'] = 0 .$$

PROOF. Take the dual of $\bigcup_{\alpha \in A} \bigcup_{\lambda \in \Gamma_\alpha} W_\lambda = X$.

THEOREM. *A necessary and sufficient condition that the interval topology of a lattice L be Hausdorff is that, for every pair of elements a, b in L with $a < b$, there exist finite nonempty subsets A and B (depending on a, b) in L such that both of the following conditions are satisfied.*

(i) $a < A \leqq b$, $a \leqq B < b$;

(ii) $(\check{x})_{x \in A}$, $(\hat{y})_{y \in B}$, *is a covering of L.*

PROOF. We shall show first that (i) and (ii) are necessary in any partially ordered set P that has a Hausdorff interval topology. Let a, b be two elements in P such that $a < b$. If P is Hausdorff, then a and b may be separated by two basic open sets V_a, V_b. That is, there exist disjoint open sets V_a, V_b such that $a \in V_a$, $b \in V_b$, and V_a and V_b each has a complement consisting of a union of a finite number of sets in the sub-basis $\mathfrak{F}$. Hence there are four finite subsets A_1, A_2, B_1, B_2 in P such that

$$V_a' = [\bigcup_{x \in A_1} \hat{x}] \cup [\bigcup_{x \in A_2} \check{x}] ,$$
$$V_b' = [\bigcup_{y \in B_1} \hat{y}] \cup [\bigcup_{x \in B_2} \check{y}] .$$

We assert that (i) and (ii) are satisfied with finite sets A and B defined by

$$A=[x: x \text{ minimal in } A_2 \cup B_2],$$

$$B=[y: y \text{ maximal in } A_1 \cup B_1].$$

Since the sets V_a and V_b are disjoint, their complements form covering of P, and we see that the class of sets $(\check{x})_{x\in A_2 \smile B_2}$, $(\hat{y})_{y\in A_1 \smile B_1}$ (together) form a covering of P. The restriction of the index sets to those x which are minimal in $A_2 \cup B_2$ and to those y which are maximal in $A_1 \cup B_1$ evidently gives a subcovering of P. Hence we obtain (ii). Now, if $x \in A_2$, clearly either $a<x$ or $a \# x$. On the other hand, if $x \in B_2$ and $x \leqq a$, then $x \leqq b$ (since $a<b$), which is impossible. Hence we may conclude that $a \prec A$. Now, since b lies in the complement of the open set V_a, $x \leqq b$ for at least one x in A, and the minimality condition on A therefore precludes $b<x$ for any x in A. Hence, A is a finite set of pairwise incomparable elements and satisfies condition (i). The remainder of (i) is obtained by the dual argument.

We now consider sufficiency, and show first that if P is any partially ordered set in which (i) and (ii) hold, then any pair of elements a, b, for which $a<b$ holds, may be separated by disjoint open sets. For in this case, suppose that A and B are nonempty finite sets in P which satisfy conditions (i) and (ii) with respect to the comparable pair a, b. Define two sets U_a, U_b by their complements,

$$U'_a = \bigcup_{x\in A} \check{x},$$

$$U'_b = \bigcup_{y\in B} \hat{y}.$$

Since their complements are finite unions of closed sets, U_a and U_b are open. By (i), a is in U_a, and b is in U_b. By (ii), and the preceding lemma, U_a and U_b are disjoint. Finally we consider the case of two incomparable elements p, q in a lattice L such that L satisfies (i) and (ii). Let a and b, respectively, be the greatest lower bound and least upper bound of the pair p, q. Let A and B be two sets specified by (i) and (ii) with respect to a and b. We shall add the element p to the set B (if B does not already contain it), and call the resulting set B^*. (So, B^* may be B.) Similarly, we shall add the element q to the set A and call the resulting set A^*. Evidently the sets A^* and B^*

also satisfy conditions (i) and (ii), with respect to a and b. Now we first define, for any z in L,

$$A_z=[x:\ x\in A^*,\ x\not\leq z],$$

$$B_z=[y:\ y\in B^*,\ z\not\leq y].$$

In terms of these sets, we define open sets U_p and U_q by their complements,

$$U'_p=[\bigcup_{x\in A_p}\check{x}]\cup[\bigcup_{y\in B_p}\hat{y}],$$

$$U'_q=[\bigcup_{x\in A_q}\check{x}]\cup[\bigcup_{y\in B_q}\hat{y}].$$

Evidently U_p contains p, and U_q contains q. If we show that $A_p\cup A_q=A^*$ and $B_p\cup B_q=B^*$, then we may conclude, by the preceding lemma, that U_p and U_q are disjoint. So let x be any element in A^*-A_p. Then $x\leq p$. But we cannot also have $x\leq q$, because this would imply that $x\leq a$, which contradicts $a<A^*$. We conclude that this x lies in A_q. The dual argument gives the corresponding result for B_p and B_q, and this completes the proof.

3. An example. We here give an example of a lattice L_0 in which a pair of comparable points cannot always be separated by disjoint open sets, but in which every closed interval (set of the form $[x:\ a\leq x\leq b]$) has a finite separating set. Let L_0 be the union of an infinite set of chains (C^α), $\alpha=0,1,2,\cdots$, each C^α being of the form

$$x_1^\alpha<x_2^\alpha<\cdots<x_{N_\alpha}^\alpha \qquad (2<N_\alpha<\infty,\ \text{all } \alpha).$$

The comparability relations in L_0 are specified in the following way. If $\alpha'\neq\alpha''$ and $1<n<N_{\alpha'}$, $1<m<N_{\alpha''}$, then $x_n^{\alpha'}\# x_m^{\alpha''}$. Otherwise, $x_{N_0}^0<\cdots<x_{N_3}^3<x_{N_1}^1$, and $x_1^0=x_1^\alpha$, all $\alpha=1,2,\cdots$.

First observe that every closed interval is either a chain or is of the form $[x:\ x_1^0\leq x\leq x_{N_\alpha}^\alpha]$ for some $\alpha=1,2,\cdots$. In the latter case, an obvious finite separating set is the pair of elements $x_{N_\alpha-1}^\alpha$, $x_{N_{\alpha+1}}^{\alpha+1}$. Let us agree to call the set consisting of the elements of C^α minus the two end elements of C^α the *interior* of C^α. Now suppose that L_0 were Hausdorff in the interval topology. Then, applying the theorem above to the pair of (comparable) elements x_1^0 and $x_{N_0}^0$, we should be able to separate this pair of elements with disjoint open sets such that each of these sets has a complement consisting of a finite union of similarly oriented half intervals. It is readily verified, however, that any such open set necessarily contains the interiors of all but a finite number of the chains C^{α}. Hence the open sets are not disjoint, L_0 is not Hausdorff.

Finally, we note that, although (in the theorem) the set $A \smile B$ is a separating set for L, the statement of the theorem could not be weakened to require only that there exists a finite set D such that $a \leqq D \leqq b$ and D separates L. A simple counter-example is the lattice with a maximal chain $a < b < c < d$, and an infinite set (x_i) of pairwise incomparable elements such that $a < x_i < c$, all i, and an infinite set (y_j) of pairwise incomparable elements such that $b < y_j < d$, all j, and $x_i \# y_j$, all i, j. Let D be the set consisting of the two elements b and c. Then $b \leqq D \leqq c$, and D separates L, but this lattice is easily verified to be not Hausdorff.

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