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On the singular point of integral equations of Volterra type

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I. Introduction.

1. The fundamental solutions of a system of ordinary linear differential equations are obtained, in the vicinity of its regular singular point, by a well-known method due to Frobenius. The purpose of the present paper is to give an analogous method for a system of linear integral equations of Volterra type.

Throughout this papar, we shall consider the following equations:

$$(1) \qquad x\,u_j(x)=\sum_{i=1}^{n}\int_0^x K_{ji}(x,t)\,u_i(t)\,dt \quad (j=1,2,\cdots,n),$$

where $K_{ji}(x,t)$ are regular with respect to x, t in a domain $|x|<r_1$, $|t|<r_2$ $(r_1, r_2>0)$. It is readily seen that the origin $(x=0)$ is a singular point of the system (1). We shall merely pay attention to the vicinity of this singular point.

2. The equations (1) are homogeneous, but non-homogeneous equatious such as

$$(2) \qquad x\,u_j(x)=f_j(x)+\int_0^x\Big(\sum_{i=1}^{n}A_{ji}(x,t)\,u_i(t)+B_j(x,t)\Big)\,dt \quad (j=1,2,\cdots,n)$$

can be reduced to the present case, if $f_j(x), A_{ji}(x,t)$ and $B_j(x,t)$ are regular for $|x|<r_1$ and $|t|<r_2$. In fact, substituting $x=0$ in (2), we see $f_j(0)=0$. Then, (2) can be reduced to

$$x\,u_j(x)=\int_0^x\Big(\sum_{i=1}^{n}A_{ji}(x,t)\,u_i(t)+B_j(x,t)+f'_j(t)\Big)\,dt.$$

Therefore, we may consider, in place of (2), the following homogeneous equations:

$$x\, u_j(x)=\int_0^x \Big(\sum_{i=1}^n A_{ji}(x,t)\, u_i(t)+(B_j(x,t)+f'_j(t))\, u_{n+1}(t)\Big)\, dt \quad (j=1,2,\cdots,n),$$

$$x\, u_{n+1}(x)=\int_0^x u_{n+1}(t)\, dt.$$

3. Let

$$K_{ji}(x,t)=\sum_{\alpha,\beta} a_{ji}^{\alpha\beta}\, x^\alpha\, t^\beta$$

be expansions of $K_{ji}(x,t)$, where $a_{ji}^{\alpha\beta}$ are all constants and α, β run over all non-negative integers. We consider the characteristic equation of the matrix (a_{ji}^{00}):

$$\text{(3)} \qquad \begin{vmatrix} a_{11}^{00}-\lambda & a_{12}^{00} & \cdots & a_{1n}^{00} \\ a_{21}^{00} & a_{22}^{00}-\lambda & \cdots & a_{2n}^{00} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{00} & a_{n2}^{00} & \cdots & a_{nn}^{00}-\lambda \end{vmatrix}=0\,.$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be n roots of (3). These roots are classified into two categories, according to the following conditions:

(i) *λ_k is a simple root of* (3) *and there exists no integer m, such that $\lambda_k+m=\lambda_j$, for any index j ($\neq k$)*;

(ii) *λ_k is a multiple root of* (3), *or, there exists at least an integer m, such that $\lambda_k+m=\lambda_j$ for some index j ($\neq k$).*

In accordance with the classification of λ_k, our results will be also stated in two parts, each dealing with the case (i) and the case (ii) which we call briefly the case of simple roots and the case of multiple roots.

II. The case of simple roots.

We shall treat, in this section, the case of simple roots. In order to have a quick start, we will demonstrate following lemmas.

LEMMA 1. *Suppose λ_k be a simple root of the equation* (3), *then there exists a formal solution of* (1),

(4) $$u_j(x)=cx^{\lambda_k-1}(b_{j0}+b_{j1}x+\cdots) \qquad (j=1,2,\cdots,n),$$

where c is an arbitrary constant, b_{jl} *are constants uniquely determined by* $a_{ij}^{\alpha\beta}$ $(\alpha+\beta\leqq l)$ *and at least one of* b_{j0} *is not zero.*

PROOF. Let

(5) $$u_j(x)=x^{\lambda_k-1}(c_{j0}+c_{j1}x+\cdots) \qquad (j=1,2,\cdots,n)$$

be a formal solution of (1). λ_k must be a root of (3) because, substituting (5) in (1), we obtain

$$\lambda_k c_{j0}=\sum_{i=1}^{n} a_{ji}^{00}c_{i0} \qquad (j=1,2,\cdots,n).$$

By a suitable linear transformation, we can take the matrix (a_{ji}^{00}) in a Jordan's canonical form

(6) $$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ \delta_1 & \lambda_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \delta_{n-1} & \lambda_n \end{pmatrix}$$

which satisfies the following conditions:

(iii) The multiple roots of (3) appear consecutively in the arrangement $\lambda_1, \lambda_2, \cdots, \lambda_n$.

(iv) $\delta_j=0$ or 1. If $\delta_j=1$, then $\lambda_{j+1}=\lambda_j$.

Substituting (5) in (1) and equating the coefficients of x^{λ_k} we obtain

$$\lambda_k c_{j0}=\delta_{j-1}c_{j-1\,0}+\lambda_j c_{j0}.$$

If $\delta_{k-1}=\delta_k=0$, $c_{k0}=c$ is an arbitrary constant and $c_{j0}=0$ for $j\neq k$. Equating the coefficients of x^{λ_k+m} $(m>0)$ we obtain

$$(\lambda_k+m)c_{jm}=\delta_{j-1}c_{j-1\,m}+\lambda_j c_{jm}+cd_{jm},$$

where d_{jm} are constants uniquely determined by $a_{ji}^{\alpha\beta}$ $(\alpha+\beta\leqq m)$. Therefore $b_{jm}=c_{jm}/c$ are uniquely determined by $a_{ji}^{\alpha\beta}$ $(\alpha+\beta\leqq m)$.

LEMMA 2. *Let* $\int_0^x t^{\lambda_k-1}\,dt$ *be convergent, then the series* $\sum_s b_{js}\,x^s$ *which appear in* (4), *have a radius of convergency which is not zero.*

PROOF. Put

$$P_{jN}(x)=\sum_{s<N} b_{js}\,x^s\,,$$

$$u_j(x)=cx^{\lambda_k-1}(P_{jN}(x)+w_j(x))$$

and substitute (4) in (1). Then we have

$$x^{\lambda_k}w_j(x)=f_j(x)+\sum_{i=1}^{n}\int_0^x K_{ji}(x,t)\,t^{\lambda_k-1}\,w_i(t)dt\,,$$

where

$$f_j(x)=-x^{\lambda_k}P_{jN}(x)+\sum_{i=1}^{n}\int_0^x K_{ji}(x,t)\,t^{\lambda_k-1}P_{jN}(t)\,dt\,.$$

Therefore $f_j(x)$ are bounded for $|x|<r_1'<r_1$. Since $x^{\lambda_k}w_j(x)$ and $\int_0^x K_{ji}(x,t)\,t^{\lambda_k-1}\,w_i(t)dt$ have no term whose order is lower than λ_k+N, $f_j(x)$ have no such term. Therefore, there exists a constant B_N such that

$$|x^{-\lambda_k}f_j(x)|\leqq B_N|x|^N \quad \text{for } |x|<r_1'.$$

Let

$$\mathfrak{F}=\{(w_1(x),w_2(x),\cdots,w_n(x))\}\,,$$

where $w_j(x)$ are regular functions for $|x|<r'$ such that $|w_j(x)|\leqq K|x|^N$. Then $\mathfrak{F}$ has following properties:

(i) $\mathfrak{F}\neq 0$.

(ii) $\mathfrak{F}$ is normal.

(iii) Let $\varphi_1 \in \mathfrak{F}$, $\varphi_2 \in \mathfrak{F}$, $\mu_1 > 0$, $\mu_2 > 0$, and $\mu_1 + \mu_2 = 1$, then $\mu_1\varphi_1 + \mu_2\varphi_2 \in \mathfrak{F}$.

(iv) $\lim\limits_{\varphi_i \in \mathfrak{F}} \varphi_i \in \mathfrak{F}$ if it exists.

Transform

$$\overline{w}_j(x) = x^{-\lambda_k} f_j(x) + x^{-\lambda_k} \sum_{i=1}^{n} \int_0^x K_{ji}(x,t) t^{\lambda_k - 1} w_i(t) dt. \tag{7}$$

Taking a suitable path of integration(1), we have

$$|\overline{w}_j(x)| \leqq \left(B_N + \frac{nAK|1+i\alpha|}{\mu - \nu\alpha + N}\right) |x|^N \quad \text{for } |x| < r_1',$$

where $A \geqq |K_{ji}(x,t)|$. Taking the value of K and N sufficiently large so that

$$\mu - \nu\alpha + N > |1+i\alpha| nA, \quad K \leqq B_N + \frac{nAK|1+i\alpha|}{\mu-\nu\alpha+N},$$

we see $|w_j(x) \leq K|x|^N$ and regular for $|x| < r_1'$. Therefore $(w_1(x), w_2(x), \cdots, w_n(x)) \in \mathfrak{F}$.

The transformation (7) implies that the mapping is continuous. Hence the lemma is proved by a theorem of fixed points.[1]

Now we can state the following theorem.

THEOREM 1. *Suppose* λ_k *be a simple root of* (3) *and* $\int_0^x t^{\lambda_k - 1} dt$ *be convergent, then* (4) *is a solution of* (1).

This follows immediately from lemmas 1, 2.

REMARK. *Let* λ_k *be neither negative real number nor zero. Then* $\int_0^x t^{\lambda_k - 1}\, dt$ *converges for a suitable path of integration.*[2]

PROOF. Put

$$x = \sigma e^{iw}, \quad \lambda_k = \mu + i\nu$$

(1) See the remark.

and let

$$(8) \qquad t=se^{i\theta}, \ \theta=\omega+\alpha\log(s/\sigma) \ (0<s\leq\sigma)$$

be the equations of path of integration. Then we have

$$|t^{\lambda_k}|=s^{\mu-\nu\alpha}\sigma^{\nu\alpha}e^{-\nu\omega}$$

In the first case that $\mu>0$, we take as $\alpha=0$. In the second case that $\mu\leqq 0$ and $\nu\neq 0$, we take as $\nu\alpha<\mu$. Then we see

$$|t^{\lambda_k}|\to 0 \text{ as } |t|\to 0.$$

III. The case of multiple roots.

In this section, we shall treat the case of multiple roots in the same manner as in the preceding section.

LEMMA 3. *Suppose* λ *be an* l*-ple root of* (3) *and the matrix* (6) *satisfy the following conditions:*

$$(\mathrm{i}) \qquad \lambda_1=\lambda_2=\cdots=\lambda_l=\lambda;$$

$$(\mathrm{ii}) \qquad \delta_{M_{i-1}+1}=\cdots=\delta_{M_{i-1}+m_i-1}=1, \ \delta_{M_i}=0,$$

$$\textit{where } M_i=\sum_{j=1}^{i}m_j, \ M_0=0 \textit{ and } M_k=l.$$

Under these assumptions there is a formal solution of (1),

$$(9) \qquad u_j(x)=x^{\lambda-1}\sum_{i=1}^{k}\sum_{p=1}^{m_i}c_{pi}\sum_{r=0}^{m_i-p}(\log x)^r\sum_s b_{jpirs}x^s \quad (j=1,2,\cdots,n),$$

where c_{pi} *are arbitrary constants and* b_{jpirs} *are constant uniquely determined by* $a_{ji}^{\alpha\beta}$ $(\alpha+\beta\leq s)$.

PROOF. Substituting

$$u_j(x)=x^{\lambda-1}\sum_{r=0}^{m-1}(\log x)^r\sum_s C_{jrs}x^s$$

in (1), where $m=\text{Max.}(m_1, m_2, \cdots, m_k)$, and equating the coefficients of $x^\lambda (\log x)^{m-1}$, we obtain

$$\lambda C_{j m-1\, 0}=\delta_{j-1} C_{j-1\, m-1\, 0}+\lambda_j C_{j\, m-1\, 0}.$$

Thence we see $C_{j\, m-1\, 0}=0$ for $j\neq M_i$ while $C_{j\, m-1\, 0}$ are undetermined for $j=M_i\ (i=1, 2, \cdots, k)$. Equating the coefficients of $x^\lambda (\log x)^{q-1}$ $(m>q>1)$, we obtain

$$\lambda C_{j\, q-1\, 0}=\delta_{j-1}(C_{j-1\, q-1\, 0}-\frac{q}{\lambda} C_{j-1\, q\, 0}+\cdots$$

$$+(-1)^{m-q}\frac{q(q+1)\cdots(m-1)}{\lambda^{m-q}} C_{j-1\, m-1\, 0})$$

$$+\lambda_j(C_{j\, q-1\, 0}-\frac{q}{\lambda} C_{j\, q\, 0}+\cdots+(-1)^{m-q}\frac{q(q+1)\cdots(m-1)}{\lambda^{m-q}} C_{j\, m-1\, 0}).$$

Therefore $C_{j\, q-1\, 0}$ are undetermined for $\lambda_j=\lambda$ while $C_{j\, q-1\, 0}=0$ for $\lambda_j\neq\lambda$ and $C_{j\, q\, 0}=0$ for $\delta_{j-1}=0$ while $C_{j\, q\, 0}=\sum_{h=M_{i-1}+1}^{j-1} C_{h\, q-1\, 0}\, b_{j\, h\, q}$ for $M_{i-1}+1<j\leqq M_i$, where $b_{j\, h\, q}$ are constants uniquely determined by λ. Thence we see that:

(i) $C_{j00}\ (j=1, 2, \cdots, l)$ are arbitrary constants.

(ii) $C_{M_{i-1}+d\, r\, 0}=\sum_{p=1}^{d-r} C_{M_{i-1}+p\, 0\, 0}\, b_{dpi\, r\, 0}$ for $0<r<d\leqq m_i$.

(iii) $C_{M_{i-1}+d\, r\, 0}=0$ for $r\geqq d$.

(iv) $C_{j\, r\, 0}=0$ for $\lambda_j\neq\lambda$.

(v) $C_{j\, r_s}=\sum_{i=1}^{k}\sum_{p=1}^{m_i-r} C_{M_{i-1}+p00}\, b_{j\, pir_s}$ for $s\geqq 1$.

where $b_{j\, pir_s}$ are constants uniquely determined by $a_{ji}^{\alpha\beta}$ $(\alpha+\beta\leqq s)$. Put $C_{M_{i-1}+p\, 0\, 0}=c_{pi}$, and we obtain (9).

LEMMA 4. *Let $\int_0^x t^{\lambda-1}dt$ be convergent, then the series $\sum_s b_{jpirs}\, x^s$ which appear in* (9), *have a radius of convergency which is not zero.*

PROOF. Put

$$P^N_{jpir}(x)=\sum_{s<N} b_{jpirs}\, x^s$$

$$\sum_s b_{jpirs}\, x^s = P^N_{jpir}(x)+w_{jpir}(x)$$

and substitute (9) in (1). Then we obtain

$$\sum_{r=0}^{m_i-p} x^\lambda (\log x)^r\, w_{jpir}(x)=\sum_{r=0}^{m_i-p} f_{jpir}(x)\,(\log x)^r$$

$$+\sum_{h=1}^{n}\int_0^x K_{jh}(x,t)\sum_{r=0}^{m_i-p} t^{\lambda-1}(\log t)^r\, w_{jpir}(t)dt$$

where

$$\sum_{r=0}^{m_i-p} f_{jpir}(x)\log x)^r=\sum_{h=1}^{n}\int_0^x K_{jh}(x,t)\sum_{r=0}^{m_i-p} t^{\lambda-1}(\log t)^r P^N_{jpir}(t)\,dt$$

$$-\sum_{r=0}^{m_i-p} x^\lambda (\log x)^r P^N_{jpir}(x)$$

because c_{pi} are arbitrary constants. Since $P^N_{jpir}(x)$ are polynomials and $K_{jh}(x,t)$ are analytic for $|x|<r_1$ and $|t|<r_2$, there exists a constant M such that

$$|K_{jh}(x,t)\, P^N_{jpir}(t)|<M \quad \text{for } |x|<r_1'<r_1,\ |t|<r_2'<r_2.$$

Moreover $\int_0^x |t^{\lambda-1}(\log t)^r|\,|dt|$ are convergent and there exists a constant B_σ such that

$$|x^\lambda (\log x)^r|\ B_\sigma \text{ for } |x|<r_1',\ |\arg x|<\sigma,$$

because $\int_0^x t^{\nu-1}\,dt$ is convergent. Therefore there exists a constant $\acute{M}'$ such that

$$|f_{jpir}(x)|<M' \quad \text{for } |x|<r_1''<\mathrm{Min}(r_1', 1).$$

Since $f_{jpir}(x)$ have no term whose order is lower than $\lambda+N$, there exists a constant B_N such that

$$|x^{-\lambda} f_{jpir}(x)|<B_N\,|x|^N \quad \text{for } |x|<r_1''.$$

Let

$$\mathfrak{F}=\{(w_{jpir}(x);\ j=1,2,\cdots,n\,;\ r=0,1,\cdots,m_i-p)\}$$

where $w_{jpir}(x)$ are regular functions for $|x|<r_1''$ such that

$$|w_{jpir}(x)|\leqq K|x|^N.$$

Then $\mathfrak{F}$ has same properties as lemma 2. Transform

(10)
$$x^\lambda \overline{w}_{jpir}(x)=f_{jpir}(x)+\sum_{h=1}^{n}\int_0^x K_{jh}(x,t)\,t^{\lambda-1}\,w_{jtir}(t)dt$$
$$-\sum_{h=1}^{n}(r+1)\int_0^x dt/t\int_0^t K_{jh}(x,t)t^{\lambda-1}w_{jpir+1}(t)dt+\cdots$$
$$+\sum_{h=1}^{n}(-1)^{m_i-r-1}(r+1)\cdots(m_i-1)\int_0^x dt/t\int_0^t dt/t\cdots$$
$$\int_0^t K_{jh}(x,t)\,t^{\lambda-1}\,w_{jpim_i-1}(t)dt$$

where the path of integration is (8). Then we have

$$\sum_{r=0}^{m_i-p} x^\lambda(\log x)^r\,\overline{w}_{jpir}(x)=\sum_{r=0}^{m_i-p} f_{jpir}(x)(\log x)^r$$

$$+\sum_{h=1}^{n}\int_0^x K_{jh}(x,t)\sum_{r=0}^{m_i-p} t^{\lambda-1}(\log t)^r w_{jpir}(t)dt$$

and

$$|\bar{w}_{jpir}(x)| \leqq (B_N + nAK|1+i\alpha|)/(\mu-\nu\alpha+N)$$

$$+(r+1)nAK|1+i\alpha|^2/(\mu-\nu\alpha+N)^2+\cdots$$

$$+(r+1)\cdots(m_i-1)nAK|1+i\alpha|^{m_i-r}/(\mu-\nu\alpha+N)^{m_i-r})|x|^N.$$

Take the value of K and N sufficiently large, and we see $|\bar{w}_{jpir}(x)| \leqq K$ and regular for $|x|<r_1''$, Therefore $\mathfrak{F} \ni (\bar{w}_{jpir}(x))$ and the mapping (10) is continuous.

THEOREM 2. *Suppose the equation* (3) *have roots whose differences be integer. Let the class* $(\lambda_1, \lambda_2, \cdots, \lambda_K)$ *satisfy the following conditions :*

(i) $\lambda_{L_{h-1}+1}=\lambda_{L_{h-1}+2}=\cdots=\lambda_{L_{h-1}+l_h}=\mu_h$

where $L_h=\sum_{j=1}^{h} l_j$, $L_0=0$ *and* $L_g=K$.

(ii) $\delta_{L_{h-1}+M_{h\,i-1}+1}=\cdots=\delta_{L_{h-1}+M_{h\,i-1}+m_{h\,i-1}}=1$

$\delta_{L_{h-1}+M_{hi}}=0$ *where* $M_{hi}=\sum_{j=1}^{i} m_{hj}$, $M_{h0}=0$ *and* $M_{hk_h}=l_h$.

(iii) $\mu_{h+1}=\mu_h+q_h$ *where* q_h *are positive integers.*

(iv) $\int_0^x t^{\mu_1-1}dt$ *is convergent.*

(v) *If there exists a root* λ_j *of* (3) *and a positive integer* q *such that* $\lambda_1+q=\lambda_j$, *then* $\lambda_j \in \{\lambda_1, \lambda_2, \cdots, \lambda_K\}$.

Under these assumptions

(11) $$u_j(x)=\sum_{p=1}^{K} c_p x^{\lambda_p-1}\sum_{r=0}^{R_p}(\log x)^r \sum_s b_{jprs}x^s$$

is a solution of (1), *where* c_p *are arbitrary constants,* b_{jpr_s} *are constants uniquely determined by* $a_{ji}^{\alpha\beta}$ $(\alpha+\beta\leqq s+\lambda_p-\lambda_1)$, *and*
$R_p=M_g-M_h+m_{hi}-d$ *for* $p=L_{h-1}+M_{h\,i-1}+d$, *where* $M_h=\sum_{j=1}^{h} m_j$ *and* $m_h=\text{Max}\ (m_{h1}, m_{h2},\cdots, m_{hk_h})$.

PROOF. We shall show (11) is a formal solution of (1). The convergency of the series $\sum_s b_{jprs}\, x^s$ can be proved by the same idea as lemma 4. Substitute

$$u_j(x)=\sum_{h=1}^{g} x^{\mu_h-1}\sum_{r=0}^{m_h-1}(\log x)^{M_{h-1}+r}\sum_s C_{jhr_s}x^s$$

in (1). Lemma 3 implies:

(i) $$C_{L_{h-1}+M_{h\,i-1}+dhoo}=c_{hido}\quad (i=1,2,\cdots,k_h;\ d=1,2,\cdots,m_{hi})$$

are undetermined.

(ii) $$C_{L_{h-1}+M_{h\,i-1}+dhro}=\sum_{p=1}^{d-r} c_{hip_o}b_{hip_o}\quad \text{for } 0<r<d\leqq m_{hi}.$$

(iii) $$C_{L_{h-1}+M_{h\,i-1}+dhro}=0\quad \text{for } r\geqq d.$$

(iv) $$C_{jhro}=0\ \text{for}\ \lambda_j\neq\mu_h.$$

(v) $$C_{jhrs}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}-r} c_{hip_o}b_{hip_s}\quad \text{for } 0<s<q_h.$$

where b_{hip_s} are constants uniquely determined by $a_{ji}^{\alpha\beta}(\alpha+\beta\leqq s)$. Equating the coefficients of $x^{\mu_{h+1}}(\log x)^{M_{h-1}+r}$ we obtain

$$\mu_{h+1}\,C_{jhrq_h}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}-r} c_{hip_o}\,b_{hipq_h}$$

$$+\delta_{j-1}\Big(\sum_{p=r}^{m_h-1}(-1)^{p-r}\ \frac{(M_{h-1}+p)\,!}{(M_{h-1}+r)\,!\ \mu_{h+1}^{p-r}}C_{j-1\,hpq_h}$$

$$+\sum_{p=0}^{m_{h+1}-1}(-1)^{m_h+p-r}\frac{(M_{k-1}+p)!}{(M_{h-1}+r)!\,\mu_{h+1}^{m_h+p-r}}C_{j-1\,h+1\,po})$$

$$+\lambda_j(\sum_{p=r}^{m_h-1}(-1)^{p-r}\;\frac{M_{h-1}+p)!}{(M_{h-1}+r)!\,\mu_{h+1}^{p-r}}C_{jhpq_h}$$

$$+\sum_{p=0}^{m_{h+1}-1}(-1)^{mh+p-r}\frac{M_h+p)!}{(M_{h-1}+r)!\,\mu_{h+1}^{mh+p-r}}C_{j\,h+1\,po}).$$

Therefore we see:

(vi) C_{jhrq_h} are undetermined for $\lambda_j=\mu_{h+1}$, while C_{jhrq_h} are uniquely determined by c_{hipo} and $a_{ji}^{\alpha\beta}\,(\alpha+\beta\leqq q_h)$ for $\lambda_j\neq\mu_{h+1}$.

(vii) $$C_{L_h+M_{h+1\,i-1}+d\,h+1\,co}=\sum_{m_{hi}=m_h}c_{hilo}\,b_{hilq_h}$$

$$+\sum_{p=1}^{d-1}C_{L_h+M_{h+1\,i-1}+phm_h-1\,q_h}\,b_{h+1\,i\,m_h-1\,o}\,.$$

(viii) $$C_{L_h+M_{h+1\,i-1}+d\,h\,r+1\,q_h}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}-r}c_{hipo}\,b_{hipp_h}$$

$$+\sum_{p=1}^{d-1}C_{L_h+M_{h+1\,i-1}+p\,h\,r\,q_h}b_{h+1\,i\,r\,o}.$$

Thence

(ix) $$C_{L_h+M_{h+1\,i-1}+d\,h\,o\,q_h}=c_{hidqh}\quad(i=1,2,\cdots,k_{h+1};\;d=1,2,\cdots,m_{h+1\,i})$$

are undetermined.

(x) $$C_{L_h+M_{h+1\,i-1}+d\,h\,r\,q_h}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}-r+d}c_{hipo}\,b_{hipq_h}\quad\text{for } d\leqq r.$$

$$C_{L_h+M_{h+1\,i-1}+d\,h\,r\,q_h}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}} c_{hipo}\, b_{hipq_h}+\sum_{p=1}^{d-r} c_{hipq_h}\, b_{hipo}$$

for $d>r$.

(xi) $$c_{h+1\,i\,d\,o}=\sum_{i=1}^{k_h}\sum_{p=1}^{d} c_{hipo}\, b_{hipq_h} \quad \text{for } d\leqq m_h.$$

$$c_{h+1\,i\,d\,o}=\sum_{i=1}^{k_h}\sum_{p=1}^{m_{hi}} c_{hipo}\, b_{hipq_h}+\sum_{p=1}^{d-m_h} c_{hipq_h}\, b_{hipo} \quad \text{for } d>m_h.$$

Successively, equating the coefficients of $x^{\mu_a}(\log x)^{M_h-1}\,(a=h+1, \cdots, g)$, we see:

(xii) $C_{j\,h\,o\,\lambda_j-\mu_h}$ are undetermined for $\lambda_j\geqq\mu_h$.

(xiii) $$C_{L_{a-1}+M_{a\,i-1}+d\,h+1\,r\,\mu_a-\mu_{h+1}}$$

$$=\sum_{e=h}^{\delta-1}\sum_{i=1}^{k_e}\sum_{p=1}^{m_{ei}} C_{L_{e-1}+M_{e\,i-1}+p\,h\,o\,\mu_e-\mu_h}\, b_{eipo\,\mu_a-\mu_e}$$

$$+\sum_{i=1}^{k_\delta}\sum_{p=1}^{\kappa} C_{L_{\delta-1}+M_{\delta\,i-1}+p\,h\,o\,\mu_\delta-\mu_h}\, b_{\delta ipo\,\mu_a-\mu_\delta}.$$

where $M_{\delta-1}-M_{h-1}\leqq M_{a-1}-M_h+d-r<M_\delta-M_{h-1}$ and $\kappa=M_{a-1}-M_{\delta-1}+d-m_h-r$. Put $C_{j1o\,\lambda_j-\lambda_1}=c_j$ and we obtain:

(xiv) c_p $(p=1, 2, \cdots, K)$ are arbitrary constants.

(xv) $$C_{jhr\,\mu_a-\mu_h+s}=\sum_{p=1}^{L_\delta} c_p\, b_{jhr\,\mu_a-\lambda_p+s}$$

$$+\sum_{i=1}^{k_{\delta+1}}\sum_{p=1}^{M_a-M_{h-1}-M_\delta-r} C_{L_\delta-M_{\delta+1\,i-1}+p1o\,\mu_{\delta+1}-\mu_1}\, b_{jhr\,\mu_a-\mu_{\delta+1}+s}$$

where $M_s \leqq M_a - M_{h-1} - r < M_{s+1}$, for $0 \leqq s < q_a$ and $a \geqq h$. This shows that (11) is a formal solution of (1).

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