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On a positive harmonic function in a half-plane.

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THEOREM 1. *Let $u(z)=u(x+iy)$ be harmonic and $u>0$ for $x>0$. Let C be a Jordan arc, contained in the half-plane $x>0$, which ends at $z=0$ and is contained in a Stolz domain, whose vertex is at $z=0$. If $u(z)$ is bounded on C, then $u(z)$ is bounded in a sector Δ: $|z|\leqq 1$, $|\arg z|\leqq\varphi_0<\frac{\pi}{2}$.*

PROOF. Since $u(z)>0$ for $x>0$, $u(z)$ can be expressed by

$$u(z)=\int_{-\infty}^{\infty}\frac{xd\chi(t)}{x^2+(y-t)^2}+cx, \qquad c\geqq 0, \tag{1}$$

where $\chi(t)$ is an increasing function of t, such that $\chi(0)=0$.[1]

From (1),

$$\int_{|t|\geqq 1}\frac{d\chi(t)}{t^2}<\infty. \tag{2}$$

Let $0<u(z)\leqq M$ on C and $z=x+iy$ lie on C, then $|y|\leqq k_0 x$ ($k_0=\text{const.}$), so that

$$M\geqq u(z)-cx\geqq\int_{-x}^{x}\frac{xd\chi(t)}{x^2+(|y|+|t|)^2}\geqq\int_{-x}^{x}\frac{d\chi(t)}{x(1+(k_0+1)^2)}$$

$$=\frac{\chi(x)-\chi(-x)}{x(1+(k_0+1)^2)},$$

hence

$$|\chi(t)|\leqq K|t|, \qquad |t|\leqq 1, \qquad K=(1+(k_0+1)^2)M. \tag{3}$$

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1) A. Dinghas: Über das Phragmén-Lindelöfsche Prinzip und den Julia-Carathéodoryschen Satz. Sitzungsber. Preuss. Akad. Wiss. (1938). M. Tsuji: On a positive harmonic function in a half-plane. Jap. Journ. Math. 15 (1939).

Let $z=x+iy\in\Delta$, then $y=kx$, $|k|\leqq\tan\varphi_0$,

$$u(z)-cx=\int_{|t|\leqq x}\frac{x\,d\chi(t)}{x^2+(kx-t)^2}+\int_{x\leqq|t|\leqq 1}\frac{x\,d\chi(t)}{x^2+(kx-t)^2}$$

$$+\int_{|t|\geqq 1}\frac{x\,d\chi(t)}{x^2+(kx-t)^2}=\mathrm{I}+\mathrm{II}+\mathrm{III}, \tag{4}$$

where

$$\mathrm{I}\leqq\frac{1}{x}\int_{-x}^{x}d\chi(t)\leqq 2K \qquad \text{by (3)},$$

$$\mathrm{II}=\int_x^1+\int_{-1}^{-x}=\mathrm{II}_1+\mathrm{II}_2,$$

$$\mathrm{II}_1\leqq\left[\frac{x\,\chi(t)}{x^2+(kx-t)^2}\right]_x^1+2\int_x^1\frac{x\,\chi(t)\,|t-kx|\,dt}{(x^2+(kx-t)^2)^2}\leqq\frac{x\,\chi(1)}{x^2+(kx-1)^2}$$

$$+2K\int_x^1\frac{x\,|t-kx|\,t\,dt}{(x^2+(kx-t)^2)^2}\leqq O(1)+2K\int_1^\infty\frac{|\tau-k|\,\tau\,d\tau}{(1+(\tau-k)^2)^2}=O(1),$$

$$t=\tau x.$$

Hence $\mathrm{II}_1=O(1)$. Similarly $\mathrm{II}_2=O(1)$, hence $\mathrm{II}=O(1)$.

$$\mathrm{III}\leqq\text{const.}\int_{|t|\geqq 1}\frac{d\chi(t)}{t^2}=O(1).$$

Hence $u(z)\leqq$ const. in Δ, q. e. d.

By Theorem 1, we can prove simply the following Loomis' theorem.[2)]

THEOREM 2. *Let $u(z)$ be harmonic and $u>0$ for $x>0$. If $\lim_{r\to 0}u(re^{i\alpha})=\lim_{r\to 0}u(re^{i\beta})=\omega$ $\left(\alpha<\beta, |\alpha|, |\beta|<\frac{\pi}{2}\right)$ exist, then $\lim_{r\to 0}u(re^{i\theta})=\omega$ uniformly for $|\theta|\leqq\varphi_0<\frac{\pi}{2}$.*

PROOF. By Theorem 1, $u(z)$ is bounded in Δ: $|z|\leqq 1$, $|\arg z|\leqq\varphi_0<\frac{\pi}{2}$, where we take $\varphi_0>|\alpha|, |\beta|$, so that, since $\lim_{r\to 0}u(re^{i\alpha})=\lim_{r\to 0}u(re^{i\beta})=\omega$, we have $\lim_{r\to 0}u(re^{i\theta})=\omega$ uniformly for $\alpha\leqq\theta\leqq\beta$.

2) L. H. Loomis: The converse of the Fatou theorem for positive harmonic functions. Trans. Amer. Math. Soc. 53 (1943).

Let D: $\frac{1}{2} \leqq |z| \leqq 1$, $|\arg z| \leqq \varphi_0 < \frac{\pi}{2}$ and consider a family of functions $u_\tau(z) = u(\tau z)$ $(0 < \tau \leqq 1)$ in D, then $u_\tau(z)$ are uniformly bounded in D, so that they form a normal family and $u_\tau(z) \to \infty$ $(\tau \to 0)$ in its partial domain: $\frac{1}{2} \leqq |z| \leqq 1$, $\alpha \leqq \arg z \leqq \beta$, so that $u_\tau(z) \to \infty$ in D, or $\lim_{r \to 0} u(re^{i\theta}) = \infty$ uniformly for $|\theta| \leqq \varphi_0 < \frac{\pi}{2}$.

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