Journal of the Mathematical Society of Japan Vol. 7, No. 1, January, 1955

On the kernel of semigroups.

By Hiroshi HASHIMOTO

(Received October 20, 1953)

The structure of the kernel of finite semigroups was studied by Suschkewitsch [1], and his study has been extended to bicompact semigroups by Numakura [2]. In the latter case, the set of idempotents plays an important rôle. In this note we shall define the kernel of semigroups which have minimal left and minimal right ideals, and investigate the relation between the kernel and minimal left (right) ideals. Thus we propose to extend the theory of bicompact semigroups to more general semigroups.

Let D_1 be a minimal left ideal, D_2 a minimal right ideal, and D the product of D_1 and D_2. Then, in order that a subset $L(R)$ of S be a minimal left (right) ideal of S, it is necessary and sufficient that $L(R)$ be represented in the following form:

$$L=D_1 a \qquad (R=aD_2),$$

where a is an element in $L(R)$. From this fact it follows that the product of any minimal left ideal and any minimal right ideal is always equal to D. Therefore D is determined uniquely irrespective of the selection of D_1 and D_2. D is a simple semigroup and is called the kernel of S. If we put $E=D_2D_1$, then E is a group contained in D_1 and D_2. Therefore every semigroup which has its kernel contains at least one idempotent. We obtain the following result, which shows the relation between the kernel and minimal left (right) ideals: the kernel D is decomposed into join of minimal left (right) ideals which have no element in common. On the other hand every minimal left (right) ideal can be divided into groups, which have no element in common. The structure of the kernel D is thus completely determined.

The author wishes to express here his hearty thanks to professor M. Moriya to whom he has been greatly indebted for his many valuable remarks and suggestions.

1. In this paper we limit ourselves to semigroups which have minimal left ideals and minimal right ideals.

Let D_1 be a minimal left ideal and D_2 a minimal right ideal of S. Then Sd_1 is a left ideal contained in D_1 for any element $d_1 \in D_1$, so that by definition

(1) $$Sd_1 = D_1 \quad \text{for every element } d_1 \text{ in } D_1,$$

and similarly

(2) $$d_2S = D_2 \quad \text{for every element } d_2 \text{ in } D_2.$$

Accordingly

(3) $$SD_1 = D_1,$$

(4) $$D_2S = D_2.$$

Now, if we put

(5) $$D = D_1D_2,$$

then by (3) and (4)

$$SD = SD_1D_2 = D_1D_2 = D,$$
$$DS = D_1D_2S = D_1D_2 = D,$$

namely

(6) $$SD = DS = D.$$

Thus D is an ideal in S and furthermore, by (1) and (2), D is a minimal ideal in S.

THEOREM 1 L. *Every minimal left ideal L of S can be represented in the form $L = Da$, where a is any element in L.*

PROOF. Since $SL \subset L$, we have $Da \subset L$ for any element a in L. However Da is a left ideal of S. Thus we have $L = Da$.

THEOREM 1 R. *Every minimal right ideal R of S can be represented in the form $R = aD$, where a is any element in R.*

By these theorems we have

(7) $$D_1 = Dd_1 \quad \text{for every element } d_1 \text{ in } D_1,$$

(8) $$D_2 = d_2D \quad \text{for every element } d_2 \text{ in } D_2,$$

and so

(9) $$DD_1 = D_1, \qquad D_2D = D_2.$$

By (6), (7) and (8)

(10) $$D_1 \subset D, \qquad D_2 \subset D.$$

On the other hand, we have by (9),

(11) $$D^2 = DD_1D_2 = D_1D_2 = D.$$

Therefore $$D = D_1D_2 \subset DD_2 \subset D^2 = D,$$

$$D = D_1D_2 \subset D_1D \subset D^2 = D,$$

hence

(12) $$D_1D = DD_2 = D.$$

THEOREM 2L. *Every minimal left ideal L of S can be represented in the form $L = D_1a$, where a is any element in L.*

PROOF. Since $SL = L$, $D_1a \subset L$ for any element a in L. However D_1a is a left ideal of S by (3) and hence $L = D_1a$ for every a in L.

THEOREM 2R. *Every minimal right ideal R of S can be represented in the form $R = aD_2$, where a is any element in R.*

D_1 is a minimal left ideal of S. Then we have

(13) $$D_1 = D_1d_1 \quad \text{for every element } d_1 \text{ in } D_1,$$

accordingly

(14) $$D_1^2 = D_1.$$

Similarly, we have

(15) $$D_2 = d_2D_2 \quad \text{for every element } d_2 \text{ in } D_2,$$

(16) $$D_2^2 = D_2.$$

From (6) and theorem 1 we can obtain the following theorems:

THEOREM 3. *Every minimal left (right) ideal is contained in D.*

THEOREM 4L. *Every minimal left ideal L of S can be represented in the form $L = D_1d$, where d is an element in D.*

THEOREM 4R. *Every minimal right ideal R of S can be represented in the form $R = dD_2$, where d is an element in D.*

2. E is defined by the product of D_2 and D_1. Then by (10) we see at once that

(17) $$E \subset D.$$

By (9) and (16) we can obtain the following results:

(18) $$D_1=DD_1=D_1D_2D_1=D_1E\,,$$

(19) $$DE=D_1D_2D_2D_1=D_1D_2D_1=DD_1=D_1\,.$$

Similarly

(20) $$D_2=D_2D=D_2D_1D_2=ED_2\,,$$

(21) $$ED=D_2\,.$$

(22) $$E^2=D_2D_1D_2D_1=D_2DD_1=D_2D_1=E\,.$$

By (17) (19) and (22) we have

(23) $$D_1=DE\supset E^2=E$$

and similarly

(24) $$D_2\supset E.$$

Finally by (14) and (16) we have

(25) $$ED_1=D_2E=E\,.$$

Here we remark that

(26) $$E=d_2D_1=D_2d_1\,,\quad \text{where}\quad d_1\in D_1\,,\ d_2\in D_2\,,$$

because from (7), (8) and (11) we obtain $E=D_2D_1=d_2DDd_1=d_2Dd_1=d_2D_1=D_2d_1$.

THEOREM 5. *E is a group.*

PROOF. Let e be any element in E. From (23) and (24) we see that e is an element in D_1 and D_2. Therefore by (13) and (15) we have

$$Ee=D_2(D_1e)=D_2D_1=E\,,$$

$$eE=(eD_2)D_1=D_2D_1=E$$

for every element e in E. Hence E is a group.

3. LEMMA 1. *Let p be any element in D_1d, where d is an element in D. Then D_1p is identical with D_1d.*

PROOF. Since p is an element in D_1d, p can be represented in the form d_1d, where d_1 is an element in D_1. By (13) we have $D_1p=D_1d_1d=D_1d$.

LEMMA 2. *If d and d' are elements in D, then either $D_1d=D_1d'$ or $D_1d\frown D_1d'=\phi$.*

PROOF. Let p be a common element in D_1d and D_1d'. By lemma 1 we have $D_1p=D_1d=D_1d'$.

LEMMA 3. *D is covered by the family $\{D_1d,\ d\in D\}$.*

PROOF. This is evident by (12).

LEMMA 4. *D_1d is a minimal left ideal of S for any element d in D.*

PROOF. It is easy to see that D_1d is a left ideal. If L is a left ideal of S in D_1d, then, for any element d_1d in D_1d, we have $L\supset SL\supset D_1d_1d=D_1d$ by theorem 2L. Hence D_1d is a minimal left ideal.

By lemma 4 and theorem 4 we have

THEOREM 6L. *In order that a subset L of S be a minimal left ideal of S, it is necessary and sufficient that L be represented in the following form:*

$$L=D_1d \qquad \text{where } d \text{ is an element in } D.$$

Similarly we obtain

THEOREM 6R. *In order that a subset R of S be a minimal right ideal of S, it is necessary and sufficient that R be represented in the following form:*

$$R=dD_2 \qquad \text{where } d \text{ is an element in } D.$$

THEOREM 7. *Let L be any minimal left ideal and R any minimal right ideal, then*

(i) $LR=D$,

(ii) *RL is a group.*

PROOF. By theorem 6, L and R can be represented as follows:

$$L=D_1d, \qquad R=d'D_2,$$

where d, d' are elements in D. Since $dd'\in D$ and D_1 is a minimal left ideal, we have $D_1=D_1dd'$ by lemma 4. Hence we obtain $LR=D$.

Next, we have $RL=d'D_2D_1d=d'Ed$. Any element p in $d'Ed$ can be represented in the form $p=d'ed$, where $e\in E$. Then $d'edd'Ed=d'ed''Ed$, where $d''=dd'$ is an element in D. Therefore d'' can be represented as $d''=d_1d_2$, where $d_1\in D_1$, $d_2\in D_2$; and by (26), $E=d_2'D_1$, $d_2'\in D_2$, therefore $ed''E=ed_1d_2d_2'D_1$. By (16) d_2d_2' is an element in D_2. If we put $d_2''=d_2d_2'$, we have $ed''E=ed_1d_2''D_1=ed'''D_1$, where $d'''=d_1d_2''$ is an element in D. Since $ed'''\in ED=D_2$ by (21), $d_2'''=ed'''$ is an element in D_2 and then

$ed''E = d_2'''D_1 = E$ by (26). Hence $d'edd'Ed = d'Ed$ for every element e in E. And also we have $d'Edd'ed = d'Ed$ for every element e in E. Therefore $d'Ed$ is a group.

Since, by theorem 7, D is equal to the product of any minimal left ideal and any minimal right ideal, we shall define D as the kernel of S.

THEOREM 8. *The kernel D is a simple semigroup.*

PROOF. Let d be any element in D, then $d = d_1 d_2$ where $d_1 \in D_1$ and $d_2 \in D_2$. Therefore we have $DdD = Dd_1 d_2 D = D_1 D_2 = D$ by (7) and (8). Hence D is a simple semigroup. [3]

From theorem 5 we have

THEOREM 9. *Every semigroup which has its kernel contains at least one idempotent.*

By lemmas 1–4 we have the following theorem which gives the relation between the kernel and minimal left (right) ideals:

THEOREM 10. *The kernel D is decomposed into join of minimal left (right) ideals which have no element in common.*

LEMMA 5. *Let p be any element in dE, where d is an element in D. Then $pE = dE$.*

PROOF. Since p is an element in dE, p can be represented in the form de, where e is an element in E. Then $pE = deE = dE$ by theorem 5.

LEMMA 6 L. *D_1 is decomposed into join of disjoint subsets dE with $d \in D$.*

PROOF. Let p be a common element in dE and $d'E$. Then by lemma 5 we have $pE = dE = d'E$. Thus if d and d' are two elements in D, either $dE = d'E$ or $dE \frown d'E = \phi$ holds. Since $DE = D_1$ by (19), D_1 is covered by the family $\{dE,\ d \in D\}$.

LEMMA 6 R. *D_2 is decomposed into join of disjoint subsets Ed with $d \in D$.*

By theorem 6, 10 and lemma 6, we can see that the kernel D is decomposed into join of subsets dEd', where d, d' are elements in D.

Now we have

LEMMA 7. *Let p be any element in dEd', where d and d' are elements in D. Then $pEd' = dEd'$.*

PROOF. Since p is an element in dEd', p can be represented in the form $p = ded'$ with $e \in E$. Then $pEd' = ded'Ed'$. But we have

$ed'E=E$ from the proof of theorem 7. Hence $pEd'=dEd'$. Therefore, if there is a common element p in $d'Ed$ and $d''Ed$, where d' and d'' are elements in D, we have by lemma 7 $pEd=d'Ed=d''Ed$. So, if d' and d'' are two elements in D, then either $aEd=bEd$ or $aEd\frown bEd=\phi$ holds. Thus we have the following results:

THEOREM 11 L. *A minimal left ideal D_1d with $d\in D$ is decomposed into join of disjoint subsets $d'Ed$, where d' belongs to D.*

Similarly we have

THEOREM 11 R. *A minimal right ideal dD_2 with $d\in D$ is decomposed into join of disjoint subsets dEd', where d' belongs to D.*

By theorem 6, 10 and 11 we have

THEOREM 12. *The kernel D is decomposed into join of disjoint subsets dEd', where d and d' are elements in D.*

If we put $d=d_1d_2$ and $d'=d_1'd_2'$, where d_1, d_1' are elements in D_1 and d_2, d_2' elements in D_2, then by (13) and (15) we have

$$dEd'=d_1(d_2D_2)(D_1d_1')d_2'=d_1D_2D_1d_2'=d_1Ed_2'.$$

LEMMA 8. *If $d_1\in D_1$ and $d_2\in D_2$, then d_1Ed_2 is a group.*

PROOF. Let p be any element in d_1Ed_2. Then $p=d_1ed_2$, $e\in E$ and $d_1ed_2d_1Ed_2=d_1Ed_2d_1ed_2=d_1Ed_2$ for every element e in E. Thus d_1Ed_2 is a group.

Now, theorem 11 and 12 can be expressed as follows:

THEOREM 13. *Every minimal left (right) ideal is decomposed into join of groups which have no element in common.*

THEOREM 14. *The kernel D is decomposed into join of groups d_1Ed_2, which have no element in common, where $d_1\in D_1$, and $d_2\in D_2$. In this case $d_1Ed_2d_1'Ed_2'=d_1Ed_2'$ holds true.*

Let p be any element in d_1E where $d_1\varepsilon D_1$. Then $p=d_1e$, $e\in E$ and $pE=d_1eE=d_1E$. Therefore if d_1 and d_1' are two elements in D_1, then we can see that $d_1E=d_1'E$ or $d_1E\frown d_1'E=\phi$. Similarly, if d_2 and d_2' are two elements in D_2, then $Ed_2=Ed_2'$ or $Ed_2\frown Ed_2'=\phi$. We shall remark here that the following relations hold:

$$d_1Ed_1'E=d_1E\ , \qquad Ed_2Ed_2'=Ed_2'.$$

LEMMA 9. *For an element d_1 in D_1 every element d_1' in D_1 which satisfies $d_1E=d_1'E$ is contained in d_1E.*

PROOF. Let d_1' be an element in D_1 which satisfies $d_1E=d_1'E$.

Since $D_1E=D_1$ by (18), D_1 is covered by the family $\{d_1E,\ d_1\in D_1\}$, so that there exists in D_1 an element d_1'' such that $d_1'\in d_1''E$. Therefore we have $d_1E=d_1'E=d_1''E\ni d_1'$ by lemma 5.

Let p be any element in $D_1\frown D_2$. Then p is an element in D_2 and by (15), $pE=(pD_2)D_1=D_2D_1=E$. Now p is an element in D_1, therefore, by lemma 9, p is an element in E and we have $D_1\frown D_2\subset E$. And $D_1\frown D_2\supset E$ holds by (23) and (24). Hence we have the following results:

THEOREM 15. $E=D_1\frown D_2$.

THEOREM 16. *The intersection of a minimal left ideal and a minimal right ideal is a group.*

Department of Mathematics,
Yamanashi University.

References

[1] A. Suschkewitsch, Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umgekehrbarkeit, Math. Ann., 99, (1928).

[2] K. Numakura, On bicompact semigroups, Math. Jour. Okayama Univ., 1, (1952).

[3] D. Rees, On semigroups, Proc. Cambridge Philos. Soc., 36, (1940).