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On complex analytic vector bundles.

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The theory of complex line bundles, which was developed by A. Weil and K. Kodaira, D. C. Spencer, has proved to be of importance in the theory of algebraic varieties. (Cf. [12], [6]-[11].)[1] As a natural generalization, one thinks of complex analytic vector bundles at once, which are also of significance in view of Chern's theory of characteristic classes. (Cf. [2], [4] and [5].)

In this paper, we shall consider complex analytic vector bundles over an algebraic variety by Kodaira's method [8], and prove an analytical imbedding theorem for these bundles, analogous to Chern's topological imbedding theorem. As a consequence of this theorem, we shall find that the characteristic classes of complex analytic vector bundles are of analogous form to that of Todd canonical systems. This result generalizes a result of Chern [4].

The author communicated his results to Prof. Kodaira and learned that the same results had already been obtained by J. P. Serre by a different method. The author is very grateful to Profs. Akizuki and Kodaira for their kind encouragement.

§1. Complex vector bundles (topological considerations).

In this section we summarize the theory of characteristic classes in a form suitable for later use.

We denote by E^r an r-dimensional vector space over the field of complex numbers, and by $GL(r)$ the general linear group operating on E^r. A fiber bundle whose fibers are E^r's and whose structure group is $GL(r)$, shall be called a *complex vector bundle* or an *E^r-bundle.*

If $\mathfrak{F}$ is an E^r-bundle over the base space V, then $\mathfrak{F}$ can be described in terms of an open covering $\{U_\alpha\}$ and a system of transition

1) Numbers in [ ] refer to the bibliography at the end of the paper.

functions $\{g_{\alpha\beta}\}$, where $g_{\alpha\beta}$ is a mapping from $U_\alpha \frown U_\beta$ into $GL(r)$, satisfying the condition

$$(1.1) \qquad g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{in} \quad U_\alpha \frown U_\beta \frown U_\gamma .$$

If we denote by π the projection from $\mathfrak{F}$ to V, then a point $\mathfrak{p}$ of $\pi^{-1}(U_\alpha)$ can be described by coordinates (z, ζ_α) with $z = \pi(\mathfrak{p}) \in U_\alpha$ and $\zeta_\alpha \in E^r$, and if $\mathfrak{p} \in \pi^{-1}(U_\alpha \frown U_\beta)$, the coordinates (z, ζ_α) and (z, ζ_β) of $\mathfrak{p}$ with respect to $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ respectively are combined by the relation

$$(1.2) \qquad \zeta_\alpha = g_{\alpha\beta} \zeta_\beta .$$

Two systems of transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same bundle if and only if

$$(1.3) \qquad g'_{\alpha\beta} = f_\alpha^{-1} g_{\alpha\beta} f_\beta ,$$

where each f_α is a mapping from U_α into $GL(r)$.

In this paper we shall consider the case where V is a compact complex analytic manifold and shall assume that all mappings and differential forms which appear are infinitely many times differentiable.

In particular, if $\mathfrak{F}$ is defined by a system of complex analytic transition functions, then $\mathfrak{F}$ has a structure of complex analytic manifold in which π is a regular analytic mapping. Then we can speak of a complex analytic vector bundle. In this case, equivalence of transition functions is defined by (1.3) with analytic f_α.

We shall denote by $H(r, N)$ the Grassmann variety which consists of the vector subspaces E^r of a fixed E^N, and by $\mathfrak{R}(r, N)$ the space of all pairs (E^r, y), where E^r is a point of $H(r, N)$, and y is a vector in E^r. $\mathfrak{R}(r, N)$ is a complex analytic manifold and has a structure of an analytic E^r-bundle over $H(r, N)$. In fact, if $(p_{i_1 \cdots i_r})$ is the Plücker coordinate of E^r, then the bundle structure of $\mathfrak{R}/H$ is defined by the covering $\{U_\alpha\}$, $U_\alpha = H - (p_\alpha)_0$, and the transition functions

$$(1.4) \qquad g_{\alpha\beta}(p) = \frac{1}{p_{i_1 \cdots i_r}} \begin{pmatrix} p_{j_1 i_2 \cdots i_r} & p_{i_1 j_2 \cdots i_r} & \cdots & p_{i_1 \cdots i_{r-1} j_1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{j_r i_2 \cdots i_r} & p_{i_1 j_r \cdots i_r} & \cdots & p_{i_1 \cdots i_{r-1} j_r} \end{pmatrix} .$$

(Here we write α, β, instead of $(i_1 \cdots i_r)$, $(j_1, \cdots, j_r)$ for brevity.)

Now Chern's imbedding theorem ([2], ths. 1 and 2) asserts that *If V is a compact C^∞ manifold of topological dimension d, and $\mathfrak{F}$*

an E^r-bundle over V, then there is a mapping $f: V \to H(r, N)$ such that $\mathfrak{F}$ is equivalent to $f^{-1}(\mathfrak{R})$, the bundle induced by f and $\mathfrak{R}/H$, provided $2(N-r) \geqq d$. Moreover, if $\mathfrak{F}$ and $\mathfrak{F}'$ are two E^r-bundles over V and f, f' corresponding mappings (into the same H), then $\mathfrak{F}$ and $\mathfrak{F}'$ are equivalent if and only if f and f' are homotopic.

Because of this fact, we call $\mathfrak{R}$ the *universal bundle* over H.

Next, for a sequence of integers $0 \leqq a_1 \leqq \cdots \leqq a_r \leqq N-r$, we have a Schubert variety $\Omega_{a_1 \cdots a_r}$, and $\Omega_{a_1 \cdots a_r}$'s generate the homology group of $H(r, N)$. In particular $\Omega_{(p)} = \Omega_{N-r-1, \cdots, N-r-1, N-r, \cdots, N-r}$ (p of $(N-r-1)$'s and $r-p$ of $(N-r)$'s; $p=1, \cdots, r$) generate, together with $\Omega_{(0)} = H$ itself, the homology ring of H.

If we denote by $V(r, p)$ the variety of (not necessarily ortho-normal) p-frames in E^r, and by $\mathfrak{R}(p, r, N)$ the $V(r, p)$-bundle associated to $\mathfrak{R}(r, N)/H(r, N)$, then

$\Omega_{(p)}$ is dual to the cohomology class of the primary obstructions of $\mathfrak{R}(r-p+1, r, N)$.

For an E^r-bundle over V, we take $f: V \to H(r, N)$ as in the imbedding theorem, then the image $f^*(\Omega_{(p)})$ of $\Omega_{(p)}$ (considered as a cohomology class) is the class of the primary obstructions of the $V(r, r-p+1)$-bundle associated to $\mathfrak{F}/V$. This class is the p-th basic characteristic class of $\mathfrak{F}$.

We shall make use of Chern's theory of connections in a vector bundle (Cf. [3], Chap. 3).

Let $\mathfrak{F}$ be an E^r-bundle over V defined by a covering $\{U_\alpha\}$ and a system of transition functions $\{g_{\alpha\beta}\}$. Then a system $\theta = \{\theta_\alpha\}$ of square matrices θ_α of degree r, whose elements $\theta_{\alpha ij}$ $(i, j = 1, 2, \cdots, r)$ are linear differential forms in U_α, is said to define a connection in $\mathfrak{F}$, provided that the relation

$$\theta_\alpha = g_{\alpha\beta}(\theta_\beta - \omega_{\alpha\beta}) g_{\alpha\beta}^{-1} \tag{1.5}$$

holds in $U_\alpha \frown U_\beta$, where $\omega_{\alpha\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta}$.

A system $\varphi = \{\varphi_\alpha\}$ of vectors $\varphi_\alpha = {}^t(\varphi_{\alpha 1}, \cdots, \varphi_{\alpha r})$, whose components are differential forms of degree d in U_α, is called an $\mathfrak{F}$-vectorial differential form of degree d, if

$$\varphi_\alpha = g_{\alpha\beta} \varphi_\beta \quad \text{in} \quad U_\alpha \frown U_\beta, \tag{1.6}$$

and a similar system $\psi = \{\psi_\alpha\}$ of matrices $\psi_\alpha = (\psi_{\alpha, ij})$ is called an $\mathfrak{F}$-

tensorial differential form of adjoint kind[2] if

$$\psi_\alpha = g_{\alpha\beta}\psi_\beta g_{\alpha\beta}^{-1} \quad \text{in} \quad U_\alpha \frown U_\beta . \tag{1.6}$$

In this connection a usual differential form σ on V is called a scalar differential form.

By the aid of a connection, we can define covariant differentiation of these forms as follows:

$$\begin{cases} D\sigma = d\sigma \\ D\varphi = d\varphi_\alpha + \theta_\alpha \wedge \varphi_\alpha \\ D\psi = d\psi_\alpha + \theta_\alpha \wedge \psi_\alpha + (-1)^{d+1}\psi_\alpha \wedge \theta_\alpha \end{cases} \quad \text{in} \quad U_\alpha \frown U_\beta . \tag{1.7}$$

D obeys the usual law of covariant differentiation, especially

$$D^2\varphi_\alpha = \Theta_\alpha \wedge \varphi_\alpha , \tag{1.8}$$

where

$$\Theta_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha . \tag{1.9}$$

$\Theta = \{\Theta_\alpha\}$ is the curvature form of the connection. We put

$$\chi_\alpha = (1/2\pi\sqrt{-1})\Theta_\alpha . \tag{1.10}$$

Now we have a theorem proved by A. Weil:

Let $P(Y_1, \cdots, Y_r)$ be a symmetric multilinear polynomial in Y's, which are tensorial forms of adjoint kind. If P is invariant by the adjoint group, then $P(\Theta, \cdots, \Theta)$ determines a cohomology class of V, which is independent of the choice of the connection.

As a special case of this theorem, we see that

$$det(\lambda I_r + \chi) = \sum c_p \lambda^{r-p} \qquad (I_r \doteq \text{unit matix of degree } r) \tag{1.11}$$

defines cohomology classes c_p of V, independently of the choice of θ.

As to the connections of the bundle $\mathfrak{R}(r, N)/H(r, N)$, we find one as follows: We restrict the structure group $GL(r)$ to the unitary group $U(r)$, and consider the bundle $\mathfrak{R}'/H$ of ortho-normal r-frames over H. Then

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2) Here we alter Chern's terminology "type" into "kind", since "type" is used in another meaning.

$$H(r, N) \cong U(N)/(U(r) \times U(N-r)),$$

$$\mathfrak{R}' \cong U(N)/U(N-r),$$

and the bundle structure of $\mathfrak{R}'/H$ is given by the natural mapping

$$U(N)/U(N-r) \rightarrow U(N)/(U(r) \times U(N-r)).$$

We put

$$u^{-1}du = (\theta_{\lambda\mu}), \qquad \lambda, \mu = 1, \cdots, N,$$

where u is a variable point of $U(N)$. Then it is seen that (θ_{ij}), $(i, j = 1, \cdots, r)$, which are differentials defined locally on H, determine a connection of $\mathfrak{R}'/H$ and hence that of $\mathfrak{R}/H$. Its curvature form is given by

$$\Theta_{ij} = -\sum_{s=r+1}^{N} \theta_{is} \wedge \theta_{sj}.$$

Then by Chern's theorem ([2], th. 5), we see that *the c_p given by* (1.11) *is the p-th basic characteristic class of the bundle $\mathfrak{F}/V$.* (First this holds for $\mathfrak{R}/H$, and then by the covariant character of characteristic classes and connections, it holds for any $\mathfrak{F}/V$ because of the imbedding theorem.)

By the way, we remark that the duality theorem for complex vector bundles can very simply be deduced from the above. In fact if $\mathfrak{F}$ and $\mathfrak{F}'$ are E^r- and E^s-bundles over V, defined by the systems of transitions functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ respectively, then the Whitney product fo $\mathfrak{F}$ and $\mathfrak{F}'$ is defined by $\{g_{\alpha\beta} \dotplus g'_{\alpha\beta}\}$ (the direct sum of matrices), and hence the matrices of its connections and curvature forms are also the direct sums of those of $\mathfrak{F}$ and $\mathfrak{F}'$.

If $\mathfrak{F}$ and $\mathfrak{F}'$ are as above, we can define another vector bundle $\mathfrak{F} \otimes \mathfrak{F}'$ by $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$, where $\otimes$ denotes the Kronecker product of matrices. Later we shall use the case where $\mathfrak{F}$ is an E^1-bundle $\mathfrak{B}$ and hence $g_{\alpha\beta}$ are simply functions. In this case we shall write $\mathfrak{B}\mathfrak{F}'$ instead of $\mathfrak{B} \otimes \mathfrak{F}'$. If the characteristic polynomials of $\mathfrak{B}$ and $\mathfrak{F}'$ are $\lambda + X$ and $F(\lambda)$ respectively, then $F(\lambda + X)$ is the characteristic polynomial of $\mathfrak{B}\mathfrak{F}'$.

§ 2. Kodaira-Spencer's lemma.

Let $\mathfrak{F}$ be an analytic E^r-bundle over a compact analytic manifold

V, defined by an open covering $\{U_\alpha\}$ and a system of transition functions $\{g_{\alpha\beta}\}$. As we have already referred, the structure group can be reduced, from the topological point of view, to $U(r)$, that is to say, there exist C^∞-mappings

$$\sigma_\alpha : U_\alpha \to GL(r),$$
$$u_{\alpha\beta} : U_\alpha \frown U_\beta \to U(r)$$

such that

(2.1) $$g_{\alpha\beta} = \sigma_\alpha^{-1} u_{\alpha\beta} \sigma_\beta \quad \text{in} \quad U_\alpha \frown U_\beta .$$

If we put

(2.2) $$h_\alpha = {}^t\bar{\sigma}_\alpha \cdot \sigma_\alpha ,$$

then each h_α is a positive definite Hermitean matrix and it is easy to see that

(2.3) $$\theta_\alpha = h_\alpha^{-1} d' h_\alpha \text{ [3]}$$

defines a connection in $\mathfrak{F}$. Its curvature form is

$$\Theta_\alpha = d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha ,$$

i.e. simply

(2.4) $$\Theta_\alpha = d''\theta_\alpha .$$

Put $\chi_\alpha = (1/2\pi\sqrt{-1})\Theta_\alpha$, then

(2.5) $${}^t\bar{\chi}_\alpha = h_\alpha \chi_\alpha h_\alpha^{-1} .$$

From this, we conclude that the p-th basic characteristic class of $\mathfrak{F}$ contains a real closed differential form of type (p, p).

From now on, we assume that V is a compact Kähler variety. Then to a differential form φ defined on an open set of V, the adjoint form $*\varphi$ is associated.

Let $\Phi_{\mathfrak{F}}$ be the linear space of all the $\mathfrak{F}$-vectorial defferential forms on V, and let $\varphi = \{\varphi_\alpha\}$, $\psi = \{\psi_\alpha\}$ be elements of $\Phi_{\mathfrak{F}}$. Then we define the inner product of φ and ψ by

(2.6) $$(\varphi, \psi) = \int_V \sum_{i,j} (h_\alpha)_{ji} \varphi_{\alpha i} \wedge \overline{*\psi_{\alpha j}} .$$

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3) If z's are complex parameters of V, then z's and $\bar{z}$'s can be considered as real parameters of V. d' denotes exterior differentiation with respect to z's and d'' one with respect to $\bar{z}$'s.

The definition is legitimate since by (1.6) and (2.1) the integrand on the right hand side is independent of α. By (2.2) this is a positive definite Hermitean inner product.

The covariant differentiation defined by (1.7) (with respect to the connection (2.3)) is an endomorphism of $\Phi_{\mathfrak{F}}$. It is divided into two parts

$$D=\partial+d'', \tag{2.7}$$

where ∂ is defined by

$$\partial\varphi_\alpha=d'\varphi_\alpha+\mathrm{e}(\theta_\alpha)\varphi_\alpha\text{[4]},$$

and is an operator of type (1,0), while d'' is obviously of type (0,1).

The adjoint operator of ∂ and d'' exist and are given by

$$\begin{cases} \partial \longleftrightarrow \delta', & \\ d'' \longleftrightarrow \vartheta, & \vartheta\varphi_\alpha=\delta''\varphi_\alpha-*\mathrm{e}(\theta_\alpha)*\varphi_\alpha, \end{cases}$$

where $\delta'=-*d''*$ and $\delta''=-*d'*$ are usual operators in the theory of harmonic integrals.

From (1.8), we have

$$(d''\partial+\partial d'')\varphi_\alpha=\mathrm{e}(\Theta_\alpha)\varphi_\alpha. \tag{2.8}$$

As in [6], we define the Laplace-Beltrami's operator $\square$ by

$$\square=d''\vartheta+\vartheta d'',$$

and we say that φ is harmonic if $\square\varphi=0$. Then the theory of harmonic integrals can be applied and we have

$$\varphi=H\varphi+d''(\vartheta G\varphi)+\vartheta Gd''\varphi,$$

for every $\varphi\in\Phi_{\mathfrak{F}}$.

LEMMA 1. *A harmonic $\mathfrak{F}$-vectorial form φ satisfies*

$$(\mathrm{e}(\chi)\Lambda\varphi,\varphi)\geqq 0.$$

PROOF. $\square\varphi=0$ is equivalent to $d''\varphi=0$ and $\vartheta\varphi=0$. From the formula

$$d''\Lambda-\Lambda d''=\sqrt{-1}\,\delta'$$

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4) $\mathrm{e}(\theta_\alpha)$ denotes the exterior multiplication of θ_α, considered as a local operator to vectorial differential forms.

we have

$$\sqrt{-1}(\delta'\varphi, \delta'\varphi) = ((d''\Lambda - \Lambda d'')\varphi, \delta'\varphi) = (d''\Lambda\varphi, \delta'\varphi)$$
$$= (\Lambda\varphi, (\vartheta\delta' + \delta'\vartheta)\varphi) = ((d''\partial + \partial d'')\Lambda\varphi, \varphi).$$

Since $d''\partial + \partial d'' = \mathrm{e}(\Theta) = 2\pi\sqrt{-1}\,\mathrm{e}(\chi)$, the lemma is proved.[5]

We shall calculate the integrand of $(\mathrm{e}(\chi)\Lambda\varphi, \varphi)$, in the case where φ is of type $(n, 1)$. (n is the complex dimension of V.) Take a point $P \in V$ and express ds^2 as

$$ds^2 = 2\sum_{\lambda=1}^{n} \omega_\lambda \cdot \bar{\omega}_\lambda$$

in a neighbourhood of P. (We fix a U_α such that $P \in U_\alpha$. As all quantites refer to U_α, we omit the suffix α for a while.) Express $\chi = (\chi_{ij})$ and $\varphi = (\varphi_j)$ as

$$\chi_{ij} = \sqrt{-1}\sum_{\lambda,\mu} \chi_{ij;\lambda\mu}\omega_\lambda \wedge \bar{\omega}_\mu,$$

$$\varphi_j = \sum_\lambda \varphi_{j\lambda}\omega_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_\lambda,$$

then we have

$$\sum_{i,k} h_{ki}(\mathrm{e}(\chi)\Lambda\varphi)_i \wedge (*\bar{\varphi})_k = \sum_{j,\lambda;k,\mu} H_{j\lambda;k\mu}\varphi_{j\lambda}\bar{\varphi}_{k\mu}dv \tag{2.9}$$

with

$$H_{j\lambda;k\mu} = \sum_i h_{ki}\chi_{ij;\lambda\mu}. \tag{2.10}$$

By (2.5), $(H_{j\lambda;k\mu})$ is an Hermitean matrix. Combining this with the previous lemma, we obtain

LEMMA 2. *If $H = (H_{j\lambda;k\mu})$ is negative definite at every point of V, then there is no harmonic form of type* $(n, 1)$ *other than* 0.

Next we define the canonical complex line bundle $\mathfrak{K}$ over V as in [7], then we have the following theorem.

THEOREM 1. (*Kodaira-Spencer's lemma*).[6] *Let $\mathfrak{F}$ be an analytic E^r-bundle over V. If the matrix H defined by* (2.10) *for the bundle $\mathfrak{K}^{-1}\mathfrak{F}$ is everywhere negative definite, then we have*

$$H^1(V; \Omega(\mathfrak{F})) = 0,$$

5) Note that our characteristic class of a complex line bundle differs by sign from that in [1] and [8].

6) Kodaira proved this theorem for complex line bundles ([7]), and Spencer generalized it to vector bundles. Our proof is slightly different and goes along the line of [1].

where $\Omega(\mathfrak{F})$ denotes the sheaf of germs of holomorphic cross sections of $\mathfrak{F}$.

PROOF. As in [6], we see that $H^q(V; \Omega^p(\mathfrak{F}))$ is isomorphic with the module of harmonic $\mathfrak{F}$-vectorial forms of type (p, q), where $\Omega^p(\mathfrak{F})$ denotes the sheaf of germs of holomorphic F-vectorial forms of degree p. On the other hand, we have $\Omega(\mathfrak{F}) \cong \Omega^0(\mathfrak{F})$ and $\Omega^0(\mathfrak{F}) \cong \Omega^n(\mathfrak{K}^{-1}\mathfrak{F})$. (Cf. [7].) Hence our theorem follows from Lemma 2.

§ 3. Analytical imbedding theorem.

Following Kodaira, we consider the quadratic transform $\tilde{V}$ of V, with the center $M \in V$. $\tilde{V}$ is a compact Kähler variety of dimension n, and there is an everywhere regular analytic mapping P_M from $\tilde{V}$ onto V. P_M is biregular except at the center M and $P_M^{-1}(M) = S$ is an $(n-1)$-dimensional subvariety of $\tilde{V}$, analytically homeomorphic with a projective space.

For an E^r-bundle $\mathfrak{F}$ over V defined as usual, a bundle $\tilde{\mathfrak{F}} = P_M^{-1}(\mathfrak{F})$ on $\tilde{V}$ is determined. Clearly $\tilde{\mathfrak{F}}$ induces the trivial bundle 0 on S. Also there is a complex line bundle $\{S\}$ on $\tilde{V}$, defined by the $\tilde{V}$-divisor S. Then we have the exact sequence

$$0 \to \Omega(\{S\}^{-1}\tilde{\mathfrak{F}}) \xrightarrow{i} \Omega(\tilde{\mathfrak{F}}) \xrightarrow{r} \Omega_S(0) \to 0\,, \tag{3.1}$$

where Ω is as in th. 1, r is the restriction mapping and i means the multiplication by local equations of S. (Cf. [8].)

Suppose that the bundle $\{S\}^{-1}F$ satisfies the condition of th. 1. Then we see that

$$r^* : H^0(\tilde{V}; \Omega(\tilde{\mathfrak{F}})) \to H^0(S; \Omega_S(0)) \tag{3.2}$$

is an onto mapping.

Now $H^0(\tilde{V}; \Omega(\tilde{\mathfrak{F}})$ and $H^0(V; \Omega(\mathfrak{F}))$ are isomorphic by the adjoint homomorphism P_M^* of P_M. If we take a base $\zeta^{(1)}, \cdots, \zeta^{(M)}$ of $H^0(V; \Omega(\mathfrak{F}))$, then the corresponding cross sections $\tilde{\zeta}^{(1)}, \cdots, \tilde{\zeta}^{(M)}$ form a base of $H^0(\tilde{V}; \Omega(\tilde{\mathfrak{F}}))$, and we conclude from (3.2) that the matrix of components of $\tilde{\zeta}$'s

$$\begin{pmatrix} \tilde{\zeta}_1^{(1)} \cdots\cdots \tilde{\zeta}_1^{(M)} \\ \cdots\cdots\cdots \\ \tilde{\zeta}_r^{(1)} \cdots\cdots \tilde{\zeta}_r^{(M)} \end{pmatrix}$$

has the rank r on S, because $H^0(S;\Omega_S(0))$ has the base ${}^t(1,0,\cdots,0)$, $\cdots$, ${}^t(0,\cdots,0,1)$ which makes up a matrix of rank r. Hence the matrix

$$\zeta_\alpha=\begin{pmatrix}\zeta_{\alpha 1}^{(1)} \cdots\cdots \zeta_{\alpha 1}^{(M)}\\ \cdots\cdots\cdots\cdots \\ \zeta_{\alpha r}^{(1)} \cdots\cdots \zeta_{\alpha r}^{(M)}\end{pmatrix} \tag{3.3}$$

of functions on V (or rather on U_α which contains M) has the rank r at M. Thus we obtain

THEOREM 2. *If $\mathfrak{F}$ is an E^r-bundle over V such that $\{S\}^{-1}\tilde{\mathfrak{F}}$ satisfies the condition of th. 1. for every quadratic transform $\tilde{V}$ of V, then the matrix* (3.3) *composed of components of a base of $H^0(V;\Omega(\mathfrak{F}))$ has the rank r at every point of V.*

Now let ζ_α be as above, and consider $(\zeta_{\alpha i}^{(1)},\cdots,\zeta_{\alpha i}^{(M)})$ as the components of a vector $X_{\alpha i}$ in E^M. Then, under the condition of th. 2,

$$\Phi_\alpha:\ U_\alpha\ni P\to E^r=X_{\alpha 1}\smile\cdots\smile X_{\alpha r}\in H(r,M) \tag{3.4}$$

defines a regular analytic mapping from U_α into $H(r,M)$. Since $\zeta_\alpha=g_{\alpha\beta}\zeta_\beta$ in $U_\alpha\frown U_\beta$. we see readily that $\Phi_\alpha=\Phi_\beta$ on $U_\alpha\frown U_\beta$. Hence by (3.4), there is defined an everywhere regular analytic mapping Φ from V into $H(r,M)$. The relation between the bundles $\mathfrak{N}/H$ and $\mathfrak{F}/V$ is as follows:

We denote the sets of indices such as $(i_1,\cdots,i_r)$, $(j,\cdots,j_r)$, $\cdots$ by $\lambda,\mu,\cdots$, and put

$$V_\lambda=\{P\,|\,P\in V,\ det(\zeta_v^{iu})\neq 0 \text{ at } P\}\,,$$

$$W_{\alpha\lambda}=U_\alpha\frown V_\lambda\,.$$

Then $\{W_{\alpha\lambda}\}$ form an open covering of V and in $W_{\alpha\lambda}\frown W_{\beta\mu}$ we have

$$\begin{pmatrix}\zeta_{\alpha 1}^{(i_1)}\cdots\zeta_{\alpha r}^{(i_1)}\\ \cdots\cdots\cdots\\ \zeta_{\alpha 1}^{(i_r)}\cdots\zeta_{\alpha r}^{(i_r)}\end{pmatrix}=(s_{\lambda\mu}\circ\Phi)\begin{pmatrix}\zeta_{\beta 1}^{(j_1)}\cdots\zeta_{\beta r}^{(j_1)}\\ \cdots\cdots\cdots\cdots\\ \zeta_{\beta 1}^{(j_r)}\cdots\zeta_{\beta r}^{(j_r)}\end{pmatrix}{}^t g_{\alpha\beta}\,,$$

where $g_{\alpha\beta}$ and $s_{\lambda\mu}$ denote the transition functions of $\mathfrak{F}/V$ and $\mathfrak{N}/H$ respectively. Since $f_{\alpha\lambda}=(\zeta_{\alpha v}^{(iu)})$ is a regular analytic mapping from $W_{\alpha\lambda}$ into $GL(r)$, the above relation shows that $\Phi^{-1}(\mathfrak{N})$ is defined by the system $\{{}^t g_{\alpha\beta}^{-1}\}$. Hence we have

THEOREM 3. *If th. 2. holds for an E^r-bundle over V, then there is an everywhere regular analytic mapping Φ from V into a suitable*

$H(r, M)$, and $\mathfrak{F}$ is defined by the system of transition functions $\{{}^t h_{\alpha\beta}^{-1}\}$, where $\{h_{\alpha\beta}\}$ is a system of transition functions for $\Phi^{-1}(\mathfrak{R})$.

Finally we assume that V is a non-singular algebraic variety in a projective space. Then the complex line bundle $\mathfrak{B}$ defined by a generic hyperplane section X of V has the characteristic homology class $-X$,[7] and this is dual to the cohomology class $-(1/2\pi)\Omega$, where Ω denotes the fundamental form of the standard Kähler metric of V.

By making use of lemma 1 and formula (3) in [8], and of the fact that the curvature forms Θ_α, Θ'_α for $\mathfrak{F}$ and $\mathfrak{B}\mathfrak{F}$ are related by

$$\Theta'_\alpha = \Theta_\alpha - \sqrt{-1}\,\Omega \otimes I_r,$$

we can easily conclude

THEOREM 4. *For every analytic E^r-bundle $\mathfrak{F}$ over a non-singular algebraic variety V, there is a natural number m such that ths. 2. and 3. hold for $\mathfrak{B}^m\mathfrak{F}$.*

Inspection on curvature forms (2.4) shows that the p-th basic characteristic classes of two E^r-bundles, which are defined by $\{g_{\alpha\beta}\}$ and $\{{}^t g_{\alpha\beta}^{-1}\}$ respectively, differ only by a factor $(-1)^p$. From this we have the following theorem, which generalizes Chern's theorem, [4], th. 7.

THEOREM 5. *For an analytic E^r-bundle $\mathfrak{F}$ over a non-singular algebraic variety V, take m as in th. 4. and form the mapping $\Phi: V \to H(r, M)$ as in th. 3. Then the characteristic polynomial of $\mathfrak{F}$ is given by $F(\lambda - mX)$, where*

$$F(\lambda) = \sum_{p=0}^{r} \Phi^{-1}(\Omega_{(p)}) \cdot \lambda^{r-p}.$$

REMARK. It is evident that not all analytic E^r-bundles are associated to regular analytic mappings. If we consider a Jacobian variety J of a curve and its divisor X, which correspond to the principal matrix of Pfaffian 1, then $l(X)=1$ and $\{X\}$ cannot be associated to a regular analytic mapping into a Grassmann variety, in spite that the characteristic class of $\{X\}$ contains an everywhere negative definite form.

(Added in proof) After this paper was written, the author proved

7) See footnote 5).

that Chern classes of the tangential vector bundle of an algebraic variety are identical with Todd canonical systems, which is a supplementary result to this paper.

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