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On the radial order of a certain regular function in a unit circle.

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1. Seidel and Walsh[1] proved the following theorem.

THEOREM 1. *Let $w=f(z)$ be regular and univalent in $|z|<1$. Then there exists a null set E on $|z|=1$, such that if $e^{i\theta}$ does not belong to E, then*

$$f'(z)=o\left(\frac{1}{\sqrt{|z-e^{i\theta}|}}\right) \quad \textit{uniformly for a fixed } \theta,$$

when $z\to e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$.

We shall give a simple proof as follows.

PROOF. Let D be the image of $|z|<1$ on the w-plane. Then since by an elementary transformation, we can map D on a finite domain, we may assume that D is a finite domain, so that

$$\int_0^{2\pi}\int_0^1 |f'(re^{i\theta})|^2 r dr d\theta < \infty ,$$

hence for almost all θ in $[0, 2\pi]$,

$$\int_0^1 |f'(re^{i\theta})|^2 dr < \infty . \tag{1}$$

Let (1) hold for $\theta=0$ and we shall prove that

$$f'(z)=o\left(\frac{1}{\sqrt{|z-1|}}\right) \quad \text{uniformly,} \tag{2}$$

1) W. Seidel and J. L. Walsh: On the derivatives of functions analytic in the unit circle and their raddii of univalence and of p-valence. Trans. Amer. Math. Soc. 52 (1942). F. Ferrand: C. R. Acad. des Sci. du 10 novembre 1941 and Thèse du 12 janvier 1942. J. Wolf: Inégalités remplies par derivées des fonctions holomorphes, univalentes et bornées dans un demi-plan. Commentarii Math. Helvetici. 45 (1952-53).

when $z\to 1$ from the inside of a Stolz domain, whose vertex at $z=1$. For any $\varepsilon>0$, let the set of $r\,(0<r<1)$, such that

$$|f'(r)|>\frac{\varepsilon}{\sqrt{1-r}} \tag{3}$$

consist of open intervals $I_\nu=(r_\nu, r'_\nu)\;(\nu=1,2,\cdots)$, where

$$0<r_1<r'_1<r_2<r'_2<\cdots<r_\nu<r'_\nu<1, \tag{4}$$

$$|f'(r_\nu)|=\frac{\varepsilon}{\sqrt{1-r_\nu}}, \quad |f'(r'_\nu)|=\frac{\varepsilon}{\sqrt{1-r'_\nu}}.$$

Then

$$\int_{r_\nu}^{r'_\nu}|f'(r)|^2dr>\varepsilon^2\int_{r_\nu}^{r'_\nu}\frac{dr}{1-r}=\varepsilon^2\log\frac{1-r_\nu}{1-r'_\nu}. \tag{5}$$

Since $\int_0^1|f'(r)|^2dr<\infty$, we take ν_0 so large that $\int_{r_{\nu_0}}^1|f'(r)|^2dr<\varepsilon^3$, then by (5), $\log\frac{1-r_\nu}{1-r'_\nu}<\varepsilon$, or

$$0<\frac{r'_\nu-r_\nu}{1-r_\nu}<1-e^{-\varepsilon}<\varepsilon \qquad (\nu\geqq\nu_0). \tag{6}$$

If we apply Koebe's distortion theorem for

$$F(\zeta)=\frac{f(z)-f(r_\nu)}{(1-r_\nu)f'(r_\nu)}\,\zeta+\cdots,\quad \zeta=\frac{z-r_\nu}{1-r_\nu},\quad |\zeta|<1,$$

then we have

$$|f'(z)|\leqq\frac{1+|\zeta|}{(1-|\zeta|)^3}|f'(r_\nu)|.$$

If $r_\nu\leqq r\leqq r'_\nu$, then $|\zeta|=\left|\frac{r-r_\nu}{1-r_\nu}\right|<\varepsilon$ by (6), so that if ε is small,

$$|f'(r)|\leqq 2|f'(r_\nu)|=\frac{2\varepsilon}{\sqrt{1-r_\nu}}\leqq\frac{2\varepsilon}{\sqrt{1-r}}.$$

Hence

$$|f'(r)|\leqq\frac{2\varepsilon}{\sqrt{1-r}} \qquad (r_{\nu_0}\leqq r<1). \tag{7}$$

Let

$$\Delta: \quad |z|<1, \quad |\arg(z-1)|\leqq\varphi_0<\frac{\pi}{2} \tag{8}$$

be a Stolz domain, whose vertex is at $z=1$ and z be any point of Δ and suppose that $\Im z\geqq 0$. Through z we draw a perpendicular L to the part of the boundary of Δ, which lies in the upper half-plane and let $z=r$ be the intersection of L with the real axis, then

$$|z-r|\leqq(1-r)\sin\varphi_0, \qquad |z-1|\leqq 1-r. \tag{9}$$

If we apply Koebe's distortion theorem for the disc: $|\zeta-r|<1-r$, then by (7), (9),

$$|f'(z)|\leqq K|f'(r)|\leqq\frac{2K\varepsilon}{\sqrt{1-r}}\leqq\frac{2K\varepsilon}{\sqrt{|z-1|}}, \quad K=\frac{1+\sin\varphi_0}{(1-\sin\varphi_0)^2}.$$

Since $\varepsilon>0$ is arbitrary, we have

$$f'(z)=o\left(\frac{1}{\sqrt{|z-1|}}\right) \quad \text{uniformly,} \tag{10}$$

when $z\to 1$ from the inside of Δ.

2. We shall prove the following theorem, which is related to Theorem 1.

THEOREM 2. *Let $w=f(z)$ be regular in $|z|<1$ and*

$$\iint_{|z|<1}|f(z)|^p dxdy<\infty, \quad p>0, \qquad z=x+iy.$$

Then there exists a null set E on $|z|=1$, such that if $e^{i\theta}$ does not belong to E, then

(i) *if p is a positive integer,*

$$f(z)=O\left(\frac{1}{|z-e^{i\theta}|^{\frac{1}{p}}}\right),$$

(ii) *if p is not a positive integer, for any $\delta>0$,*

$$f(z)=O\left(\frac{1}{|z-e^{i\theta}|^{\frac{1+\delta}{p}}}\right)$$

uniformly, when $z \to e^{i\theta}$ *from the inside of any Stolz domain, whose vertex is at* $z=e^{i\theta}$.

First we shall prove a lemma.

LEMMA. *Let* $w=f(z)$ *be regular in* $|z|<1$ *and*

$$\iint_{|z|<1} |f(z)|^p dxdy < \infty, \qquad p>0.$$

We put

$$A(r,\theta)=\iint_{|z-re^{i\theta}|<1-r} |f(z)|^p dxdy.$$

Then there exists a null set E *on* $|z|=1$, *such that if* $e^{i\theta}$ *does not belong to* E, *then for any* $\delta>0$,

$$A(r,\theta)=O((1-r)^{1-\delta}), \qquad r\to 1.$$

PROOF. We put

$$B(r,\theta)=\int_0^{1-r}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |f(re^{i\theta}+\rho e^{i(\theta+\varphi)})|^p \rho d\rho d\varphi. \tag{1}$$

Then since $\rho \leqq 1-r$,

$$\int_0^{2\pi} B(r,\theta)d\theta \leqq (1-r)\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{1-r}\int_0^{2\pi} |f(re^{i\theta}+\rho e^{i(\theta+\varphi)})|^p d\rho d\theta. \tag{2}$$

If we put $re^{i\theta}+\rho e^{i(\theta+\varphi)}=Re^{i\Theta}$, then

$$R=\sqrt{r^2+\rho^2+2r\rho\cos\varphi}, \quad \Theta=\theta+\tan^{-1}\frac{\rho\sin\varphi}{r+\rho\cos\varphi}. \tag{3}$$

We change variables from (ρ,θ) to (R,Θ) in (2), then since

$$dR\,d\Theta=\frac{\partial(R,\Theta)}{\partial(\rho,\theta)}d\rho d\theta=\frac{r\cos\varphi+\rho}{\sqrt{r^2+\rho^2+2r\rho\cos\varphi}}d\rho d\theta$$

$$\geqq \frac{r\cos\varphi+\rho}{r+\rho}d\rho d\theta \geqq \cos\varphi d\rho d\theta$$

$$\geqq \cos\frac{\pi}{4}d\rho d\theta=\frac{1}{\sqrt{2}}d\rho d\theta,$$

we have

$$\int_0^{2\pi} B(r,\theta)d\theta \leqq \sqrt{2}\,(1-r)\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_r^1\int_0^{2\pi} |f(Re^{i\Theta})|^p dR d\Theta$$

$$= \frac{\pi}{\sqrt{2}}\,(1-r)\int_r^1\int_0^{2\pi} |f(Re^{i\Theta})|^p dR d\Theta$$

$$\leqq K(1-r)\,, \tag{4}$$

where

$$K = \frac{\pi}{\sqrt{2}}\int_0^1\int_0^{2\pi} |f(Re^{i\Theta})|^p dR d\Theta\,. \tag{5}$$

Let $r_\nu = 1 - \frac{1}{2^\nu}$ $(\nu = 1, 2, \cdots)$, then

$$\int_0^{2\pi} B(r_\nu,\theta)d\theta \leqq K(1-r_\nu)\,. \tag{6}$$

Let $\delta > 0$ and e_ν be the set of θ, such that

$$B(r_\nu,\theta) > (1-r_\nu)^{1-\delta}\,, \tag{7}$$

then by (6),

$$me_\nu < K(1-r_\nu)^\delta = \frac{K}{2^{\nu\delta}}\,.$$

Hence if we put $E_\nu = e_\nu + e_{\nu+1} + \cdots$, $E = \lim\limits_{\nu} E_\nu$, then

$$mE = 0\,. \tag{8}$$

It is sufficient to prove that if $z=1$ does not belong to E, then

$$A(r,0) = O((1-r)^{1-\delta})\,, \qquad r \to 1\,. \tag{9}$$

Since $z=1$ does not belong to E, $z=1$ does not belong to a certain E_{ν_0}, so that by (7),

$$\int_0^{1-r_\nu}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |f(r_\nu + \rho e^{i\varphi})|^p \rho d\rho d\varphi = B(r_\nu, 0) \leqq (1-r_\nu)^{1-\delta} \quad (\nu \geqq \nu_0)\,. \tag{10}$$

Let

$$D_\nu: \qquad |z - r_\nu| < 1 - r_\nu\,, \qquad |\arg(z - r_\nu)| < \frac{\pi}{4} \tag{11}$$

and

$$\Delta_\nu: \qquad |z - \rho_\nu| < 1 - \rho_\nu \qquad (\rho_\nu > 0) \tag{12}$$

be a circular disc, which is contained in D_ν and touches the boundary of D_ν, then by a simple calculation, we have

$$1-\rho_\nu=\frac{1-r_\nu}{1+\sqrt{2}}, \tag{13}$$

so that

$$A(\rho_\nu,0)=\iint_{\Delta_\nu}|f(z)|^p dxdy\leqq\iint_{D_\nu}|f(z)|^p dxdy=B(r_\nu,0)\leqq(1-r_\nu)^{1-\delta}$$

$$=K_1(1-\rho_\nu)^{1-\delta},\quad K_1=(1+\sqrt{2})^{1-\delta}\qquad(\nu\geqq\nu_0). \tag{14}$$

Let $\rho_\nu\leqq\rho\leqq\rho_{\nu+1}$, then $1-\rho_{\nu+1}=\frac{1}{2}(1-\rho_\nu)\leqq1-\rho\leqq1-\rho_\nu$, so that

$$A(\rho,0)\leqq A(\rho_\nu,0)\leqq K_1(1-\rho_\nu)^{1-\delta}=K_2(1-\rho)^{1-\delta},\quad K_2=2^{1-\delta}K_1, \tag{15}$$

or

$$A(\rho,0)=O((1-\rho)^{1-\delta}),\qquad \rho\to1.$$

Hence (9) is proved.

$E=E(\delta)$ depends on $\delta>0$. If we take $\delta_1>\delta_2>\cdots>\delta_\nu\to0$ and put $E=\sum_{\nu=1}^{\infty}E(\delta_\nu)$, then E satisfies the condition of the lemma.

3. Proof of Theorem 2.

Since the first part (i) can be proved similarly as Seidel and Walsh, we assume that p is not a positive integer and we shall prove (ii). Let E be the null set on $|z|=1$, which satisfies the condition of the lemma. It is sufficient to prove that if $z=1$ does not belong to E, then

$$f(z)=O\left(\frac{1}{|z-1|^{\frac{1+\delta}{p}}}\right)\quad\text{uniformly}, \tag{1}$$

when $z\to1$ from the inside of a Stolz domain Δ, whose vertex is at $z=1$.

Since $z=1$ does not belong to E, for any $\delta>0$,

$$A(r,0)\leqq K(1-r)^{1-\delta},\qquad r_0\leqq r<1. \tag{2}$$

Let Δ be defined by (8) of the proof of Theorem 1 and z be any point of Δ and suppose that $\Im z\geqq0$. Through z we draw a perpendicular

L to the part of the boundary of Δ, which lies in the upper half-plane and let $z=r$ be the intersection of L with the real axis, then since $|z-r| \leqq (1-r)\sin\varphi_0$, we have

$$1-r-|z-r| \geqq (1-\sin\varphi_0)(1-r)=\rho,$$

so that a disc: $|\zeta-z|<\rho$ is contained in a disc: $|\zeta-r|<1-r$. Since $|f(z)|^p$ $(p>0)$ is subharmonic,

$$\pi\rho^2|f(z)|^p \leqq \iint_{|\zeta-z|<\rho} |f(\zeta)|^p dxdy \leqq \iint_{|\zeta-z|<1-r} |f(\zeta)|^p dxdy = A(r,0) \leqq K(1-r)^{1-\delta},$$

or

$$|f(z)| \leqq \frac{K_1}{(1-r)^{\frac{1+\delta}{p}}} \leqq \frac{K_1}{|z-1|^{\frac{1+\delta}{p}}}, \qquad K_1=\left(\frac{K}{\pi(1-\sin\varphi_0)^2}\right)^{1/p}. \tag{3}$$

Hence

$$f(z)=O\left(\frac{1}{|z-1|^{\frac{1+\delta}{p}}}\right) \quad \text{uniformly}, \tag{4}$$

when $z\to 1$ from the inside of Δ, q. e. d.

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