Journal of the Mathematical Society of Japan Vol. 6, No. 2, September, 1954

Construction of the set theory from the theory of ordinal numbers.

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(Received May 26, 1953)

The object of this paper is to show that we can construct—in a sense explained below—the set theory of Fraenkel-v. Neumann from the theory of ordinal numbers. This latter theory means the system based on the axioms 1.1–1.17 listed below, and the set theory of Fraenkel-v. Neumann means the one based on the axioms A, B, C, D and E given in Gödel [1]. Gödel [1] constructs also a model Δ of the set theory on the theory of ordinal numbers. We follow him in modifying his method, but we pursue here another purpose as him. We restrict namely our system of logic to the one called 'G^1LC without bound functions' in our former paper [2], which is obtained from Gentzen's system LK (Logistischer klassischer-Kalkül, [3]) by introducing the 'variable of the height 0 or 1'. Some acquaintance with [1], [2] is assumed in this paper.

Thus the result of this paper shows that, if the theory of ordinal numbers is consistent, so also is the set theory of Fraenkel-v. Neumann in the sense of G^1LC without bound functions.

The author wishes to express his thanks to Dr. T. Iwamura who has given him valuable remarks in the course of this work.

First we shall list the axioms of the theory of ordinal numbers, 1.1–1.17.

1.1 $\forall x(x=x)$

1.2 $0<\omega$

1.3 $\forall x\forall y\ (x<y \vee x=y \vee y<x)$

1.4 $\forall x\forall y \neg(x=y \wedge x<y)$

1.5 $\forall x\forall y \neg(x<y \wedge y<x)$

1.6 $\forall x\forall y\forall z(x<y \wedge y<z \vdash x<z)$

1.7 $\forall x(0<x \vee 0=x)$

1.8 $0'=1$

1.9 $\forall x(x<y \vdash x'=y \vee x'<y)$

1.10 $\forall x(x<x')$

1.11 $\forall x \forall y(x'=y' \vdash x=y)$

1.12 $\forall x(x<\omega \vdash x'<\omega)$

1.13 $\forall \varphi \forall x \forall y\{x=y \vdash (\varphi[x] \vdash\dashv \varphi[y])\}$

1.14 $\forall \varphi \forall x\{\varphi[0] \wedge \forall y(\varphi[y] \vdash \varphi[y']) \wedge x<\omega \vdash \varphi[x]\}$

1.15 $\forall \varphi[\{\forall x \neg \varphi[x] \vdash \mathrm{Min}(z)\varphi[z]=0\} \wedge \{\mathrm{E}x\varphi[x] \vdash \varphi[\mathrm{Min}(z)\varphi[z]]\}$
$\wedge \forall x\{\varphi[x] \vdash x \geqq \mathrm{Min}(z)\varphi[z]\}]$

1.16 $\forall \varphi_2 \forall u[\forall x \forall y \forall z(\varphi_2[x, z] \wedge \varphi_2[y, z] \vdash x=y)$
$\vdash \mathrm{E}x \forall y\{\mathrm{E}z(\varphi_2[y, z] \wedge z<u) \vdash y<x\}]$

1.17 $\forall u \mathrm{E}v \forall \varphi_2[\forall x \forall y \forall z(\varphi_2[x, z] \wedge \varphi_2[y, z] \vdash x=y)$
$\vdash \mathrm{E}x\{x<v \wedge \forall y \neg(\varphi_2[x, y] \wedge y<u)\}]$

We call 1.14 the axiom of mathematical induction, 1.15 the axiom of minimum, 1.16 the axiom of substitution and 1.17 the axiom of cardinal numbers. The system of all these axioms 1.1–1.17 will be denoted by Γ_0.

In this paper we shall use the following abbreviated wording. When Γ_0, $\Delta \rightarrow A$ is provable, we shall say briefly that $\Delta \rightarrow A$ is provable, or that we have $\Delta \rightarrow A$ as it is usual in the mathematics; if, moreover, Δ is void, we say that A is provable or that we have A.

From 1.15 we have easily for any variety of type 1 $\{x\}\mathfrak{A}(x)$ and a term t,

2. $\forall x(x<t \vdash (\forall y(y<x \vdash \mathfrak{A}(y)) \vdash \mathfrak{A}(x))) \rightarrow \forall x(x<t \vdash \mathfrak{A}(x))$, which is called the principle of transfinite induction and will be useful in several proofs.

By 1.15 we have easily the following lemma.

3. If an axiom $\forall x_1 \cdots \forall x_n \mathrm{E}y\mathfrak{A}(y, x_1, \cdots, x_n)$ is provable, we can introduce a new function satisfying
$\forall x_1, \cdots \forall x_n \mathfrak{A}(f(x_1, \cdots, x_n), x_1, \cdots, x_n)$

For the proof of consistency, in fact, we have only to replace $f(x_1, \cdots, x_n)$ with $\mathrm{Min}\,(y)\mathfrak{A}(y, x_1, \cdots, x_n)$.

From this lemma it follows easily that we can introduce new functions $\max(*_1, *_2)$ and $\min(*_1, *_2)$ satisfying

4.1 $\forall x \forall y(x \leqq y \vdash\!\dashv \max(x, y)=y)$

4.2 $\forall x \forall y(y \leqq x \vdash\!\dashv \max(x, y)=x)$

4.3 $\forall x \forall y(\max(x, y)=\max(y, x))$

4.4 $\forall x \forall y(x \leqq y \vdash\!\dashv \min(x, y)=x)$

4.5 $\forall x \forall y(y \leqq x \vdash\!\dashv \min(x, y)=y)$

4.6 $\forall x \forall y(\min(x, y)=\min(y, x))$

5. Lemma on the recursive predicate (first form)

Let $\mathfrak{F}(\alpha)$ be a formula with full indication for α (i. e. such that $\mathfrak{F}(\beta)$ has no α) satisfying (i. e. such that the following formula is provable)

5.1 $\forall \varphi \forall \psi\{\forall x(\varphi[x] \vdash\!\dashv \psi[x]) \vdash\!-(\mathfrak{F}(\{x\}\varphi[x]) \vdash\!\dashv \mathfrak{F}(\{x\}\psi[x]))\}$

and let $\mathfrak{A}(a)$ be

$\forall \varphi\{\forall u(u<a \vdash\!-(\varphi[u] \vdash\!\dashv \mathfrak{F}(\{x\}(\varphi[x] \wedge x<u)))) \vdash\!-\mathfrak{F}(\{x\}(\varphi[x] \wedge x<a))\}$

Then we have

$$\forall x(\mathfrak{A}(x) \vdash\!\dashv \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y<x))), \text{ and for any } \{x\}\mathfrak{C}(x) \text{ and } \{x\}\mathfrak{B}(x),$$

$$\forall x\{\mathfrak{C}(x) \vdash\!\dashv \mathfrak{F}(\{y\}(\mathfrak{C}(y) \wedge y<x))\} \wedge \forall x\{\mathfrak{B}(x) \vdash\!\dashv \mathfrak{F}(\{y\}(\mathfrak{B}(y) \wedge y<x))\}$$
$$\rightarrow \forall x(\mathfrak{C}(x) \vdash\!\dashv \mathfrak{B}(x))$$

is provable.

PROOF. The second part of the lemma is a simple application of the transfinite induction making use of 5.1. As for the first part, we have only to prove

$\forall x[x<a \vdash\!-\{\mathfrak{A}(x) \vdash\!\dashv \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y<x))\}] \rightarrow \mathfrak{A}(a) \vdash\!\dashv \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y<a))$.

Let $\mathfrak{B}$ be the left side of this sequence. As $\mathfrak{A}(a), \mathfrak{B} \rightarrow \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y<a))$ follows immediately from the definition of $\mathfrak{A}(a)$, we have only to prove $\neg \mathfrak{A}(a), \mathfrak{B} \rightarrow \neg \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y<a))$.

For this purpose let $\mathfrak{C}$ denote

$\forall x[x<a \vdash\!-\{\alpha[x] \vdash\!\dashv \mathfrak{F}(\{y\}(\alpha[y] \wedge y<x))\}] \wedge \neg \mathfrak{F}(\{y\}(\alpha[y] \wedge y<a))$

Then by the transfinite induction and 5.1 we have successively the sequences

$\mathfrak{C}, \mathfrak{B} \rightarrow \forall x(x<a \vdash\!-(\alpha[x] \vdash\!\dashv \mathfrak{A}(x)))$

$\mathfrak{C}, \mathfrak{B} \rightarrow \mathfrak{F}(\{y\}(\alpha[y] \wedge y < a)) \vdash\!\dashv \mathfrak{F}(\{y\}\mathfrak{A}(y) \wedge y < a))$

$\mathfrak{C}, \mathfrak{B} \rightarrow \neg \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y < a))$

$\neg \mathfrak{A}(a), \mathfrak{B} \rightarrow \neg \mathfrak{F}(\{y\}(\mathfrak{A}(y) \wedge y < a))$ q. e. d.

In the same way we can easily prove

6. Lemma on the recursive predicate (second form)

If $\forall y \forall \varphi \forall \psi \{\forall x_1 \forall x_2(\varphi[x_1, x_2] \vdash\!\dashv \psi[x_1, x_2]) \vdash (\mathfrak{F}(y, \{x_1, x_2\}\varphi[x_1, x_2]) \vdash\!\dashv \mathfrak{F}(y, \{x_1, x_2\}\psi[x_1, x_2])\}$ is provable and if we write $\mathfrak{A}_2(b, a)$ for $\forall \varphi_2\{\forall u [u < a \vdash \forall v\{\varphi_2[v, u] \vdash\!\dashv \mathfrak{F}(v, \{x, y\}(\varphi_2[x, y] \wedge y < u))\}]$
$\vdash \mathfrak{F}(b, \{x, y\}(\varphi_2[x, y] \wedge y < a))\}$
then we have
$\forall x \forall y\{\mathfrak{A}_2(y, x) \vdash\!\dashv \mathfrak{F}(y, \{u, v\}(\mathfrak{A}_2(u, v) \wedge v < x))\}$.

7. Lemma on the recursive function

For any function $g(*_1, *_2)$ and any variety of type (1,1) $\{x, y\}$ $\mathfrak{B}(x, y)$ satisfying the axiom $\forall x \forall y \mathrm{E} z(\mathfrak{B}(z, y) \wedge x < z)$, we can introduce a new function $f(*_1, *_2)$ satisfying

7.1 $\forall y \forall x\{\forall z(z < x \vdash g(f(z, y), y) \leqq f(x, y)) \wedge \mathfrak{B}(f(x, y), y)\}$

7.2 $\forall y \forall x \forall u\{\forall z(z < x \vdash g(f(z, y), y) \leqq u) \wedge \mathfrak{B}(u, y) \vdash f(x, y) \leqq u\}$

7.3 $\forall y \forall x \forall z\{x < z \vdash f(x, y) \leqq f(z, y)\}$

If moreover $g(*_1, *_2)$ satisfies $\forall y \forall x \forall z\{x < z \vdash g(x, y) \leqq g(z, y)\}$ and $\mathfrak{B}(x, y)$ is of the form $h(y) \leqq x$, then we have

7.4 $\forall y \forall x f(x', y) = g(f(x, y), y)$

PROOF. Let $\mathfrak{S}(b, \alpha_2)$ be $\forall v(\mathrm{E} u \alpha_2[v, u] \vdash g(v, c) \leqq b) \wedge \mathfrak{B}(b, c)$, $\mathfrak{F}(b, \alpha_2)$ be $b = \mathrm{Min}(w)\mathfrak{S}(w, \alpha_2)$, $\mathfrak{A}_2(b, a)$ be the formula obtained from $\mathfrak{F}(b, \alpha_2)$ as before and let $f(a, c)$ mean $\mathrm{Min}(w)\mathfrak{S}(w, \{x, y\}(\mathfrak{A}_2(x, y) \wedge y < a))$. Then we can prove 7.1 and 7.2 as follows.

First we obtain easily $\mathfrak{F}(b_1, \alpha_2), \mathfrak{F}(b_2, \alpha_2) \rightarrow b_1 = b_2$ and so
$\mathfrak{A}_2(b_1, a), \mathfrak{A}_2(b_2, a) \rightarrow b_1 = b_2$

Hence by the axiom of substitution we have easily

$\mathrm{E} w \forall v(\mathfrak{A}_2(v, u) \wedge u < a \vdash g(v, c) \leqq w)$

and so $\mathrm{E} w \mathfrak{S}(w, \{x, y\}(\mathfrak{A}_2(x, y) \wedge y < a))$ by the axiom which $\mathfrak{B}(*)$ satisfies. From this follows immediately $\mathfrak{F}(f(a, c), \{x, y\}(\mathfrak{A}_2(x, y) \wedge y < a))$. So we have $\mathfrak{A}(f(a, c), a)$ and consequently, after easy calculation $\forall w(\mathfrak{A}_2(w, a) \vdash\!\dashv w = f(a, c))$. Therefore making use of $\mathfrak{F}(f(a, c), \{x, y\}$ $(\mathfrak{A}_2(x, y) \wedge y < a))$ again, the following formulas can be proved

$\forall z(z<a\vdash g(f(z,c),c)\leqq f(a,c))\wedge\mathfrak{B}(f(a,c),c)$,

$\forall z(z<a\vdash g(f(z,c)\,c,)\leqq b)\wedge\mathfrak{B}(b,c)\vdash f(a,c)\leqq b$.

Hence 7.1 and 7.2 are provable.

7.3 and the second part of the lemma are easy consequences of 7.1 and 7.2, q. e. d..

In particular, we can introduce a new function $*_1+*_2$ satisfying 8.1–8.4

8.1 $\forall x\forall y(x\leqq x+y)$

8.2 $\forall x\forall y\forall z(z<y\vdash(x+z)'\leqq x+y)$

8.3 $\forall x\forall y\forall u\{\forall z(z<y\vdash(x+z)'\leqq u)\wedge x\leqq u\vdash x+y\leqq u\}$

8.4 $\forall x\forall y(x+y'=(x+y)')$

Whence follows easily (making use of the mathematical induction and the transfinite induction)

8.5 $\forall x(x+0=x)$

8.6 $\forall x(0+x=x)$

8.7 $\forall x\forall y(x<\omega\wedge y<\omega\vdash x+y=y+x)$

8.8 $\forall x\forall y(x<\omega\wedge y<\omega\vdash x+y<\omega)$

8.9 $\forall x\forall y\forall z(y<z\vdash\dashv x+y<x+z)$

8.10 $\forall x\forall y\forall z\{(x+y)+z=x+(y+z)\}$

We introduce a new function $*_1\times *_2$ satisfying

9.1 $\forall x\forall y\forall z\{z<y\vdash(x\times z)+x\leqq x\times y\}$

9.2 $\forall x\forall y\forall u[\forall z\{z<y\vdash(x\times z)+x\leqq u\}\vdash x\times y\leqq u]$

9.3 $\forall x\forall y\{(x\times y)+x=x\times y'\}$

Whence we have

9.4 $\forall x(x\times 0=0)$

9.5 $\forall x(0\times x=0)$

9.6 $\forall x\forall y\forall z\{x\times(y+z)=(x\times y)+(x\times z)\}$

9.7 $\forall x\forall y(x<\omega\wedge y<\omega\vdash x\times y<\omega)$

9.8 $\forall x\forall y(x<\omega\wedge y<\omega\vdash x\times y=y\times x)$

We make use of the usual convention concerning the adhering

force of $\times$ and $+$; for instance, $a\times b+c$ means $(a\times b)+c$. $a\times a$ will sometimes be denoted by a^2.
Uniq (α_2) denotes a formula $\forall x\forall y\forall z(\alpha_2[x, z]\wedge\alpha_2[y, z]\vdash x=y)$.
We take a new function $N(*)$. $N(a)$ will mean
$\mathrm{Min}\,(v)\forall\varphi_2\{\mathrm{Uniq}\,(\varphi_2)\vdash\exists x\{x<v\wedge\forall y\neg(\varphi_2[x,y]\wedge y<a)\}\}$.
From the axiom of minimum and the axiom of cardinal number we have

10.1 $\mathrm{Uniq}\,(\{x,y\}\mathfrak{A}(x,y))\rightarrow\exists x\{x<N(a)\wedge\forall y\neg(\mathfrak{A}(x,y)\wedge y<a)\}$.

10.2 $\forall\varphi_2\{\mathrm{Uniq}\,(\varphi_2)\vdash\exists x\{x<b\wedge\forall y\neg(\mathfrak{A}(x,y)\wedge y<a)\}\}\rightarrow N(a)\leqq b$.

10.3 $b<N(a)\rightarrow\exists\varphi_2\{\mathrm{Uniq}\,(\varphi_2)\wedge\forall x\{x<b\vdash\exists y(\varphi_2[x,y]\wedge y<a)\}\}$

Whence we have

10.4.1 $\forall x(x<N(x))$

10.4.2 $\forall x\forall y(x<N(y)\vdash N(x)\leqq N(y))$.

In the same way as in Lemma on the recursive function, we can introduce a new function $\aleph(a)$ satisfying

$\forall x[\forall y\{y<x\vdash N(\aleph(y))\leqq\aleph(x)\}\wedge\omega\leqq\aleph(x)]$,
$\forall x\forall z[y\{y<x\vdash N(\aleph(y))\leqq z\}\wedge\omega\leqq z\vdash\aleph(x)\leqq z]$,
$\forall x\forall y(x<y\vdash\aleph(x)\leqq\aleph(y))$.

Whence follow

11.1 $\aleph(0)=\omega$

11.2 $\aleph(a')=N(\aleph(a))$

11.3 $b<a\rightarrow\aleph(b)<\aleph(a)$

11.4 $\forall x(x<a\vdash\aleph(x)<b)\rightarrow\aleph(a)\leqq b$

We take now two new predicates $Le(*, *, *, *)$ and $R(*, *, *, *)$, $Le(a, b, c, d)$ will mean $b<d\vee(b=d\wedge a<c)$ and $R(a, b, c, d)$ will mean $\max(a, b)<\max(c, d)\vee\{\max(a, b)=\max(c, d)\wedge Le(a, b, c, d)\}$.
Then we have

12.1 $\rightarrow R(a, b, c, d),\ R(c, d, a, b),\ a=c\wedge b=d$

12.2 $R(a, b, c, d),\ R(c, d, e, f)\rightarrow R(a, b, e, f)$

12.3 $R(a, b, c, d),\ R(c, d, a, b)\rightarrow$

12.4 $R(a, b, c, d), a=c\wedge b=d\rightarrow$

Using repeatedly the axiom of the transfinite induction we have

12.5 $\forall\varphi_2[\mathsf{E}u\mathsf{E}v\varphi_2[u,v]\vdash\mathsf{E}x\mathsf{E}y\{\varphi_2[x,y]\wedge\forall u\forall v(\varphi_2[u,v]$
$\vdash R(x,y,u,v)\vee(x=u\wedge y=v))\}]$

This will be called' the axiom of the transfinite induction with respect to $R(*,*,*,*)$.

We take further a new function $j(*,*)$. $j(a,b)$ will be an abbreviation of

$$\begin{aligned}&\mathrm{Min}(z)\mathsf{E}\varphi_3[\forall u\forall v\{R(u,v,a,b)\vee(u=a\wedge v=b)\vdash\mathsf{E}s\varphi_3[s,u,v]\}\\&\wedge\forall t\forall u\forall v\forall s\forall x\forall y\{\varphi_3[t,u,v]\wedge\varphi_3[s,x,y]\vdash(R(u,v,x,y)\mapsto t<s)\}\\&\wedge\varphi_3[0,0,0]\wedge\varphi_3[z,a,b]].\end{aligned}$$

Clearly we have $j(0,0)=0$.

We shall study the properties of the function $j(*,*)$. To prepare for it, we define another new function $j'(*,*)$, so that

13.1 $a<\omega,\ b<\omega,\ a>b\rightarrow j'(a,b)=a^2+b$

13.2 $a<\omega,\ b<\omega,\ a\leqq b\rightarrow j'(a,b)=b^2+b+a$

Then we have evidently

13.3 $a<\omega,\ b<\omega,\ c<\omega,\ d<\omega\rightarrow R(a,b,c,d)\mapsto j'(a,b)<j'(c,d)$

13.4 $a<\omega,\ b<\omega,\ c<\omega,\ d<\omega\rightarrow j'(a,b)=j'(c,d)\mapsto a=c\wedge b=d$

13.5 $a<\omega\rightarrow\mathsf{E}x\mathsf{E}y\{x<\omega\wedge y<\omega\wedge a=j'(x,y)\}$

13.6 $j'(0,0)=0$

By the definition of $j(a,b)$, it is easily proved that

14.1 $a<\omega,\ b<\omega\rightarrow j(a,b)=j'(a,b)$

14.2 $j(\omega,0)=\omega$

Hence we have by the mathematical induction

14.3 $a<\omega\rightarrow j(\omega,a)=\omega+a$

14.4 $j(0,\omega)=2\omega$

Together with the mathematical induction this implies

14.5 $a<\omega\rightarrow j(a,\omega)=2\omega+a$

14.6 $j(\omega,\omega)=3\omega$

Therefore we can see easily

14.7 $a\leqq\omega,\ b\leqq\omega,\ c\leqq\omega,\ d\leqq\omega\rightarrow R(a,b,c,d)\mapsto j(a,b)<j(c,d)$

14.8 $a \leqq \omega,\ b \leqq \omega,\ c \leqq \omega,\ d \leqq \omega \rightarrow j(a, b) = j(c, d) \leftrightarrow a = c \wedge b = d$

14.9 $a \leqq 3\omega \rightarrow \mathrm{E}x\mathrm{E}y\{x \leqq \omega \wedge y \leqq \omega \wedge a = j(x, y)\}$

where $2a$, $3a$ are the abbreviations of $a+a$, $a+a+a$.

Now let $\mathfrak{A}(a)$ be

$$\forall u \forall v \forall x \forall y\{u \leqq \aleph(a) \wedge v \leqq \aleph(a) \wedge x \leqq \aleph(a) \wedge y \leqq \aleph(a) \vdash (R(u, v, x, y) \leftrightarrow j(u, v) < j(x, y))\}$$
$$\wedge \forall x\{x \leqq 3\aleph(a) \vdash \mathrm{E}y\mathrm{E}z(y \leqq \aleph(a) \wedge z \leqq \aleph(a) \wedge x = j(y, z))\}$$
$$\wedge j(\aleph(a), 0) = \aleph(a) \wedge j(\aleph(a), \aleph(a)) = 3\aleph(a)$$

and $\mathfrak{B}(a)$ be

$$\forall u \forall v \forall x \forall y\{u < \aleph(a) \wedge v < \aleph(a) \wedge x < \aleph(a) \wedge y < \aleph(a) \vdash (R(u, v, x, y) \leftrightarrow j(u, v) < j(x, y))\}$$

Moreover let $\mathfrak{C}(a)$ be

$$\forall x\{x < \aleph(a) \vdash \mathrm{E}y\mathrm{E}z(y < \aleph(a) \wedge z < \aleph(a) \wedge x = j(y, z))\} \wedge \mathfrak{B}(a)$$

and $\mathfrak{D}(a)$ be

$$\forall x \forall y\{x < \aleph(a) \wedge y < \aleph(a) \vdash j(x, y) < \aleph(a)\} \wedge \mathfrak{C}(a).$$

By the transfinite induction we have following sequences in succession

15.1 $\mathfrak{B}(a),\ b < \aleph(a),\ c < \aleph(a) \rightarrow j(b, c) \geqq \max(b, c)$

15.2 $\mathfrak{D}(a) \rightarrow j(\aleph(a), 0) = \aleph(a)$

15.3 $\mathfrak{D}(a), b < \aleph(a) \rightarrow j(\aleph(a), b) = \aleph(a) + b$

15.4 $\mathfrak{D}(a) \rightarrow j(0, \aleph(a)) = 2\aleph(a)$

15.5 $\mathfrak{D}(a), b < \aleph(a) \rightarrow j(b, \aleph(a)) = 2\aleph(a) + b$

15.6 $\mathfrak{D}(a) \rightarrow j(\aleph(a), \aleph(a)) = 3\aleph(a)$

15.7 $\mathfrak{D}(a) \rightarrow \mathfrak{A}(a)$

Since $\forall x(x \geqq \omega \vdash x - 1 = x)$, we have

16.1 $\mathrm{E}\varphi_2\{\mathrm{Uniq}(\varphi_2) \wedge \forall x(x \leqq \aleph(a) \vdash \mathrm{E}y(y < \aleph(a) \wedge \varphi_2[x, y]))\}$.

This implies, as $j(\aleph(a), 0) = \aleph(a)$ follows $\mathfrak{A}(a)$,

16.2 $\mathfrak{A}(a) \rightarrow \mathrm{E}\varphi_2\{\mathrm{Uniq}(\varphi_2) \wedge \forall x \forall y(x \leqq \aleph(a) \wedge y \leqq \aleph(a) \vdash \mathrm{E}z(z < \aleph(a) \wedge \varphi_2[j(x, y), z]))\}$

As $b < \aleph(a')$ and $c < \aleph(a')$ mean

$\mathrm{E}\varphi_2\{\mathrm{Uniq}(\varphi_2) \wedge \forall x(x \leqq b \vdash \mathrm{E}y(y \leqq \aleph(a) \wedge \varphi_2[x, y]))\}$ and

$\mathrm{E}\varphi_2\{\mathrm{Uniq}(\varphi_2)\wedge\forall x(x\leqq c\vdash\mathrm{E}y(y\leqq\aleph(a)\wedge\varphi_2[x,y]))\}$, we have from 16.2

16.3 $b<\aleph(a'),\ c<\aleph(a'),$

$$\mathfrak{A}(a)\to\mathrm{E}\varphi_2\{\mathrm{Uniq}(\varphi_2)\wedge\forall x\forall y(R(x,y,b,c)\vee(x=b\wedge y=c)$$
$$\vdash\mathrm{E}z(z<\aleph(a)\wedge\varphi_2[j(x,y),z]))\}$$

and so $b<\aleph(a'),\ c<\aleph(a'),\ \mathfrak{A}(a),\ \mathfrak{C}(a')\to j(b,c)<\aleph(a')$.

Hence we have

17. $\mathfrak{A}(a),\ \mathfrak{C}(a')\to\mathfrak{D}(a')$.

Now we shall prove

18.0 $\mathfrak{A}(a),\ \mathfrak{B}(a')\to\mathfrak{C}(a')$

If $\mathfrak{B}(a')$ and $\neg\mathfrak{C}(a')$, then it is proved by the transfinite induction that there exists an ordinal number b satisfying

18.1 $b<\aleph(a')$

18.2 $\forall x\{x<b\vdash\mathrm{E}y\mathrm{E}z(y<\aleph(a')\wedge z<\aleph(a')\wedge x=j(y,z))\}$

18.3 $\forall x\forall y\neg(b=j(x,y)\wedge x<\aleph(a')\wedge y<\aleph(a'))$

From 15.1 we have

$\mathfrak{B}(a'),\ b<\aleph(a')\to\mathrm{E}y\mathrm{E}z(y<\aleph(a')\wedge z<\aleph(a')\wedge b<j(y,z))$.

So we have by the transfinite induction with respect to $R(*,*,*,*)$ two ordinal numbers c,d satisfying

18.4 $c<\aleph(a')\wedge d<\aleph(a')$

18.5 $\forall x\forall y(R(x,y,c,d)\vdash j(x,y)<b)$

18.6 $j(c,d)>b$

But if we take $\alpha_3[*,*,*]$ satisfying

$$\forall x\forall y\forall z\{\alpha_3[x,y,z]\vdash\dashv(R(y,z,c,d)\vdash x=j(y,z))$$
$$\wedge(R(y,z,c,d)\vee(y=c\wedge z=d))\wedge(y=c\wedge z=d\vdash x=b),$$

then we see easily

$$\forall u\forall v\{R(u,v,c,d)\vee(u=c\wedge v=d)\vdash\mathrm{E}s\alpha_3[s,u,v]\}$$
$$\wedge\forall t\forall u\forall v\forall s\forall x\forall y\{\alpha_3[t,u,v]\wedge\alpha_3[s,x,y]\vdash(R(u,v,x,y)\vdash\dashv t<s)\}$$
$$\wedge\alpha_3[0,0,0]\wedge\alpha_3[b,c.d].$$

So by the definition of $j(*,*)$ we have $j(c,d)\leqq b$.

This is a contradiction. q.e.d.

Now we shall prove by the transfinite induction

19. $\forall x \mathfrak{A}(x)$

From 14 we have clearly $\mathfrak{A}(0)$.

Next we shall prove $\mathfrak{A}(a) \rightarrow \mathfrak{A}(a')$

From 15.7, 17 and 18 we have only to prove $\mathfrak{A}(a) \rightarrow \mathfrak{B}(a')$ Clearly we have

$$\neg \mathfrak{B}(a') \rightarrow \mathrm{E}u\mathrm{E}v\mathrm{E}x\mathrm{E}y\{u < \aleph(a') \wedge v < \aleph(a') \wedge x < \aleph(a') \wedge y < \aleph(a) \\ \wedge \neg (R(u, v, x, y) \vdash\!\dashv j(u, v) < j(x, y))\}$$

and so

$$\neg \mathfrak{B}(a') \rightarrow \mathrm{E}u\mathrm{E}v\mathrm{E}x\mathrm{E}y\{x < \aleph(a') \wedge y < \aleph(a') \wedge R(u, v, x, y) \\ \wedge j(u, v) \geqq j(x, y)\}\,.$$

So we can see by the transfinite induction with respect to $R(*, *, *, *)$ that there exist two ordinal numbers b, c satisfying

19.1 $b < \aleph(a') \wedge c < \aleph(a')$

19.2 $\mathrm{E}u\mathrm{E}v(R(u, v. b, c) \wedge j(u, v) \geqq j(b, c))$

19.3 $\forall u \forall v \forall x \forall y\{R(u, v, b, c) \wedge R(x, y, b. c) \vdash (R(u, v, x, y) \vdash\!\dashv j(u, v) < j(x, y))\}$

But if $\mathfrak{A}(a)$, there exists in virtue of 16.3 and the axiom of substitution an ordinal number d satisfying

$\forall u \forall v(R(u, v, b, c) \vdash j(u, v) < d)$.

So if we take $\alpha_3[*, *, *]$ satisfying

$$\forall x \forall y \forall z\{\alpha_3[x, y, z] \vdash\!\dashv ((b=y \wedge c=z) \vee R(y, z, b, c)) \\ \wedge (R(y, z, b, c) \vdash x=j(y, z)) \wedge (y=b \wedge z=c \vdash x=d)\}\,,$$

then we can see easily

$$\forall u \forall v\{R(u, v, b, c) \vee (u=b \wedge v=c) \vdash \mathrm{E}s\alpha_3[s, u, v]\} \\ \wedge \forall t \forall u \forall v \forall s \forall x \forall y\{\alpha_3[t, u, v] \wedge \alpha_3[s, x, y] \vdash (R(u, v, x, y) \vdash\!\dashv t < s)\} \\ \wedge \alpha_3[0, 0, 0] \wedge \alpha_3[d, b, c]\,.$$

So by the definition of $j(*, *)$ we have

19.4 $$\mathrm{E}\varphi_3(\forall u \forall v\{R(u, v, b, c) \vee (u=b \wedge v=c) \vdash \mathrm{E}s\varphi_3[s, u, v]\} \\ \wedge \forall t \forall u \forall v \forall s \forall x \forall y\{\varphi_3[t, u, v] \wedge \varphi_3[s, x, y] \vdash (R(u, v, x, y) \vdash\!\dashv t < s)\} \\ \wedge \varphi_3[0, 0, 0] \wedge \varphi_3[j(b, c), b, c])$$

and so $\forall u \forall v(R(u, v, b, c) \vdash j(u, v) < j(b, c))$.

This is a contradiction.

Therefore we have only to prove

$\lim(a),\ \forall x(x<a\vdash\mathfrak{A}(x))\to\mathfrak{A}(a)$, where $\lim(a)$ is an abbreviated notation for $\forall x\neg(x'=a)$.

But we have easily $\lim(a),\ \forall x(x<a\vdash\mathfrak{A}(x))\to\mathfrak{D}(a)$.

So from 15.7 holds $\lim(a),\ \forall x(x<a\vdash\mathfrak{A}(x))\to\mathfrak{D}(a)$. q. e. d.

From 19 we have directly the following properties

20.1 $R(a,b,c,d)\vdash\dashv j(a,b)<j(c,d)$

20.2 $a=c\wedge b=d\vdash\dashv j(a,b)=j(c,d)$

20.3 $\forall x \mathrm{E} y \mathrm{E} z(x=j(y,z))$

20.4 $\forall x\forall y(j(x,y)\geqq\max(x,y))$

20.5 $j(\aleph(a),0)=\aleph(a)$

20.6 $j(\aleph(a),\aleph(a))=3\aleph(a)$

20.7 $b<\aleph(a),\ c<\aleph(a)\to b+c<\aleph(a)$

From 20.3 we can define two new functions $g^1(*)$ and $g^2(*)$ satisfying

20.8 $\forall x(x=j(g^1(x),g^2(x)))$

20.9 $\forall x(x\geqq g^1(x))$

20.10 $\forall x(x\geqq g^2(x))\wedge\forall x(x>0\vdash x>g^2(x))$

Now we define successively $0'=1,\ 1'=2,\ 2'=3,\ 3'=4,\ 4'=5,\ 5'=6,\ 6'=7,\ 7'=8,\ 8'=9,\ 9'=10,\ 10'=11,\ 11'=12$ and take a new predicate $S(*,*,*,*,*,*)$ with the following sense.

$S(i,a,b,k,c,d)$ will mean $R(a,b,c,d)\vee\{j(a,b)=j(c,d)\wedge i<k\}$

In the same way as before we can define four new functions $j(*,*,*)$, $g_0(*)$, $g_1(*)$ and $g_2(*)$ satisfying

21.1 $\forall i\forall k\forall u\forall v\forall x\forall y\{i<9\wedge k<9\vdash(S(i,u,v,k,x,y)\vdash\dashv j(i,u,v)$
$<j(k,x,y))\}$

21.2 $\forall x(g_0(x)<9)$

21.3 $\forall x(j(g_0(x),g_1(x),g_2(x))=x)$

21.4 $\forall x(x\geqq g_1(x))$

21.5 $\forall x(x\geqq g_2(x))\wedge\forall x(x>0\vdash g_2(x))$

After these preparations we shall begin to construct Gödel's model Δ.

First, we define formulas of the form $\mathfrak{F}(\{x\}(\alpha[x]\wedge x<a))$

Let $\overset{\alpha,a}{\leq}(b,c)$ be $b>c\wedge\alpha[j(b,c)]\wedge j(b,c)<a$

$\overset{\alpha,a}{\equiv}(b,c)$ be $(b\leqq c\wedge\alpha[j(b,c)]\wedge j(b,c)<a)$

$\qquad\vee(b\geqq c\wedge\alpha[j(c,b)]\wedge j(c,b)<a)$

$\overset{\alpha,a}{\leqq}(b,c)$ be $\mathsf{E}x\{\overset{\alpha,a}{\equiv}(x,c)\wedge\overset{\alpha,a}{\leq}(b,x)\}$

$\overset{\alpha,a}{\equiv}(b;\{c;d\})$ be $\forall x[x<b\vdash\{\overset{\alpha,a}{\leqq}(b,x)\vdash\dashv\overset{\alpha,a}{\equiv}(x,c)\vee\overset{\alpha,a}{\equiv}(x,d)\}]$

$\qquad\wedge\mathsf{E}x(x<b\wedge\overset{\alpha,a}{\equiv}(x,c))\wedge\mathsf{E}x(x<b\wedge\overset{\alpha,a}{\equiv}(x,d))$

$\overset{\alpha,a}{\leqq}(b;\{c;d\})$ be $\mathsf{E}x\{x<b\wedge\overset{\alpha,a}{\leqq}(b,x)\wedge\overset{\alpha,a}{\equiv}(x;\{c;d\})\}$

$\overset{\alpha,a}{\equiv}(b;<c;d>)$ be $\mathsf{E}x\mathsf{E}y\{x<b\wedge y<b\wedge\overset{\alpha,a}{\equiv}(b;\{x;y\})$

$\qquad\wedge\overset{\alpha,a}{\equiv}(x;\{c;c\})\wedge\overset{\alpha,a}{\equiv}(y;\{c;d\})\}$

$\overset{\alpha,a}{\leqq}(b;<c;d>)$ be $\mathsf{E}x\{x<b\wedge\overset{\alpha,a}{\leqq}(b,x)\wedge\overset{\alpha,a}{\equiv}(x;<c;d>)\}$

$\overset{\alpha,a}{\equiv}(b;<c;d;e>)$ be $\mathsf{E}x\{x<b\wedge\overset{\alpha,a}{\equiv}(b;<c;x>)\wedge\overset{\alpha,a}{\equiv}(x;<d;e>)\}$

$\overset{\alpha,a}{\leqq}(b;<c;d;e>)$ be $\mathsf{E}x\{x<b\wedge\overset{\alpha,a}{\leqq}(b,x)\wedge\overset{\alpha,a}{\equiv}(x;<c;d;e>)\}$

Moreover let

$H_1(\alpha,a)$ be $\overset{\alpha,a}{\equiv}(g^2(a),g_1(g^1(a)))\vee\overset{\alpha,a}{\equiv}(g^2(a),g_2(g^1(a)))$

$H_2(\alpha,a)$ be $\overset{\alpha,a}{\leqq}(g_1(g^1(a)),g^2(a))$

$\qquad\wedge\mathsf{E}x\mathsf{E}y[x<g^2(a)\wedge y<g^2(a)\wedge\overset{\alpha,a}{\leqq}(y,x)\wedge\overset{\alpha,a}{\equiv}(g^2(a);<x;y>)]$

$H_3(\alpha,a)$ be $\overset{\alpha,a}{\leqq}(g_1(g^1(a)),g^2(a))\wedge\neg\overset{\alpha,a}{\leqq}(g_2(g^1(a)),g^2(a))$

$H_4(\alpha,a)$ be $\overset{\alpha,a}{\leqq}(g_1(g^1(a)),g^2(a))\wedge\mathsf{E}x\mathsf{E}y[x<g^2(a)\wedge y<g^2(a)$

$\qquad\wedge\overset{\alpha,a}{\equiv}(g^2(a);<x;y>)\wedge\overset{\alpha,a}{\leqq}(g_2(g^1(a)),y)]$

$H_5(\alpha,a)$ be $\mathsf{E}x[x<g_1(g^1(a))\wedge\overset{\alpha,a}{\leqq}(g_1(g^1(a));<x;g^2(a)>)]$

$H_6(\alpha,a)$ be $\mathsf{E}x\mathsf{E}y[x<g_1(g^1(a))\wedge y<g_1(g^1(a))$

$\qquad\wedge\overset{\alpha,a}{\leqq}(g_1(g^1(a));<x;y>)\wedge\overset{\alpha,a}{\equiv}(g^2(a);<y;x>)]$

$H_7(\alpha,a)$ be $\mathsf{E}x\mathsf{E}y\mathsf{E}z[x<g_1(g^1(a))\wedge y<g_1(g^1(a))\wedge z<g_1(g^1(a))$

$\qquad\wedge\overset{\alpha,a}{\leqq}(g_1(g^1(a));<x;y;z>)\wedge\overset{\alpha,a}{\equiv}(g^2(a);<y;z;x>)$

$H_8(\alpha,a)$ be $\mathsf{E}x\mathsf{E}y\mathsf{E}z[x<g_1(g^1(a))\wedge y<g_1(g^1(a))\wedge z<g_1(g^1(a))$

$\qquad\wedge\overset{\alpha,a}{\leqq}(g_1(g^1(a));<x;y;z>)\wedge\overset{\alpha,a}{\equiv}(g^2(a);<x;z;y>)]$

$H_9(\alpha,a)$ be $\forall x[x<g^2(a)\vdash\{\overset{\alpha,a}{\leqq}(g^1(a),x)\vdash\dashv\overset{\alpha,a}{\leqq}(g^2(a),x)\}]$

Finally let $\mathfrak{F}(\{x\}(\alpha[x]\wedge x<a))$ be

$[g^1(a)>g^2(a)\vdash\{(g_0(g^1(a))=1\vdash H_1(\alpha,a))$

$\qquad\wedge(g_0(g^1(a))=2\vdash H_2(\alpha,a))$

$$\wedge(g_0(g^1(a))=3 \vdash H_3(\alpha. a))$$
$$\wedge(g_0(g^1(a))=4 \vdash H_4(\alpha, a))$$
$$\wedge(g_0(g^1(a))=5 \vdash H_5(\alpha, a))$$
$$\wedge(g_0(g^1(a))=6 \vdash H_6(\alpha, a))$$
$$\wedge(g_0(g^1(a))=7 \vdash H_7(\alpha, a))$$
$$\wedge(g_0(g^1(a))=8 \vdash H_8(\alpha, a))\}]$$

$$\wedge(g^1(a) \leqq g^2(a) \vdash H_9(\alpha, a))$$

And according to Lemma on the recursive predicate, let $f\dot{n}(*)$ be a new predicate satisfying

22. $\forall x\{fn(x) \vdash\!\dashv \mathfrak{F}(\{y\}(\alpha[y] \wedge y < x))\}$

Now, let $\overset{a}{\leq}(b, c)$, $\overset{a}{=}(b, c)$, $\overset{a}{\leqq}(b, c)$, $\overset{a}{=}(b; \{c; d\})$, $\overset{a}{\leqq}(b; \{c; d\})$, $\overset{a}{=}(b; <c; d>)$, $\overset{a}{\leqq}(b; <c; d>)$, $\overset{a}{=}(b, <c; d; e>)$, $\overset{a}{\leqq}(b; <c; d; e>)$ mean a formula which we obtain by substituting $\{x\}fn(x)$ for α in $\overset{\alpha, a}{\leq}(b, c)$, $\overset{\alpha, a}{=}(b, c)$, $\overset{\alpha, a}{\leqq}(b, c)$, $\overset{\alpha. a}{=}(b; \{c; d\})$, $\overset{\alpha, a}{\leqq}(b; \{c; d\})$, $\overset{\alpha, a}{=}(b; <c; d>)$, $\overset{\alpha, a}{\leqq}(b; <c; d>)$, $\overset{\alpha. \alpha}{=}(b; <c; d; e>)$, $\overset{\alpha, a}{\leqq}(b; <c; d; e>)$ respectively.

We write $H_i(a)$ for $H_i(\{x\}fn(x), a)$ $(1 \leqq i \leqq 9)$, where $H_i(a)$ is obtained by substituting $\overset{a}{\leq}(b, c)$, $\overset{a}{=}(b, c), \cdots$ for all $\overset{\alpha, a}{\leq}(b, c)$, $\overset{\alpha, a}{=}(b, c), \cdots$ in $H_i(\alpha, a)$. Then 22 means that following ten sequences are provable.

22.1 $g^1(a) > g^2(a),\ g_0(g^1(a))=0 \rightarrow fn(a)$

22.2 $g^1(a) > g^2(a),\ g_0(g^1(a))=1 \rightarrow fn(a) \vdash\!\dashv H_1(a)$

22.3 $g^1(a) > g^2(a),\ g_0(g^1(a))=2 \rightarrow fn(a) \vdash\!\dashv H_2(a)$

22.4 $g^1(a) > g^2(a),\ g_0(g^1(a))=3 \rightarrow fn(a) \vdash\!\dashv H_3(a)$

22.5 $g^1(a) > g^2(a),\ g_0(g^1(a))=4 \rightarrow fn(a) \vdash\!\dashv H_4(a)$

22.6 $g^1(a) > g^2(a),\ g_0(g^1(a))=5 \rightarrow fn(a) \vdash\!\dashv H_5(a)$

22.7 $g^1(a) > g^2(a),\ g_0(g^1(a))=6 \rightarrow fn(a) \vdash\!\dashv H_6(a)$

22.8 $g^1(a) > g^2(a),\ g_0(g^1(a))=7 \rightarrow fn(a) \vdash\!\dashv H_7(a)$

22.9 $g^1(a) > g^2(a),\ g_0(g^1(a))=8 \rightarrow fn(a) \vdash\!\dashv H_8(a)$

22.10 $g^1(a) \leqq g^2(a) \rightarrow fn(a) \vdash\!\dashv H_9(a)$

Now let $<(b, c)$ be $b > c \wedge fn(j(b, c))$

$=(b, c)$ be $(b \leqq c \wedge fn(j(b, c))) \vee (b \geqq c \wedge fn(j(c, b)))$

$\leqq(b, c)$ be $\mathrm{E}x(=(x, c) \wedge <(b, x))$

$$=(b;\{c;d\}) \text{ be } \forall x[x<b\vdash\{\leqslant(b,x)\vdash\dashv=(x,c)\vee=(x,d)\}]$$
$$\wedge \mathrm{E}x(x<b\wedge=(x,c))\wedge \mathrm{E}x(x<b\wedge=(x,d))$$
$$\leqslant(b;\{c;d\}) \text{ be } \mathrm{E}x\{x<b\wedge\leqslant(b,x)\wedge=(x;\{c;d\})$$
$$=(b;<c;d>) \text{ be } \mathrm{E}x\mathrm{E}y\{x<b\wedge y<b\wedge=(b;\{x;y\})$$
$$\wedge=(x;\{c;c\})\wedge=(y;\{c;d\})\}$$
$$\leqslant(b;<c;d>) \text{ be } \mathrm{E}x\{x<b\wedge\leqslant(b,x)\wedge=(x;<c;d>)\}$$
$$=(b;<c;d;e>) \text{ be } \mathrm{E}x\{x<b\wedge=(b;<c;x>)\wedge=(x;<d;e>)\}$$
$$\leqslant(b;<c;d;e>) \text{ be } \mathrm{E}x\{x<b\wedge\leqslant(b,x)\wedge=(x;<c;d;e>)\}$$

Clearly we have $j(b,c)<a\rightarrow<(b,c)\vdash\dashv\overset{a}{\leqslant}(b,c)$, and
$j(b,c)<a,\ j(c,b)<a\rightarrow=(b,c)\vdash\dashv\overset{a}{=}(b,c)$
As $\mathrm{E}x[\{(x\leqq c\wedge fn(j(x,c))\wedge j(x,c)<a)\vee(x\geqq c\wedge fn(j(c,x))\wedge j(c,x)<a)\}$
$\wedge\{b>x\wedge fn[j(b,x)]\wedge j(b,x)<a\}]$ is $\overset{a}{\leqslant}(b,c)$, we have easily
$j(b,c)\leqq a,\ j(b,b)\leqq a\rightarrow\leqslant(b,c)\vdash\dashv\overset{a}{\leqslant}(b,c)$.

Hence we have the following sequences in succession,

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,$
$\quad j(c,b)\leqq a,\ j(d,b)\leqq a,\rightarrow=(b;\{c;d\})\vdash\dashv\overset{a}{=}(b;\{c;d\})$

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,$
$\quad j(c,b)\leqq a,\ j(d,b)\leqq a\rightarrow\leqslant(b;\{c;d\})\vdash\dashv\overset{a}{\leqslant}(b;\{c;d\})$

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,$
$\quad j(c,b)\leqq a,\ j(d,b)\leqq a\rightarrow=(b;<c;d>)\vdash\dashv\overset{a}{=}(b;<c;d>)$

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,$
$\quad j(c,b)\leqq a,\ j(d,b)\leqq a\rightarrow\leqslant(b;<c;d>)\vdash\dashv\overset{a}{\leqslant}(b;<c;d>)$

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,\ j(b,e)\leqq a,$
$j(c,b)\leqq a,\ j(d,b)\leqq a,\ j(e,b)\leqq a\rightarrow=(b;<c;d;e>)\vdash\dashv\overset{a}{=}(b;<c;d;e>)$

$j(b,b)\leqq a,\ j(b,c)\leqq a,\ j(b,d)\leqq a,\ j(b,e)\leqq a,$
$j(c,b)\leqq a,\ j(d,b)\leqq a,\ j(e,b)\leqq a\rightarrow\leqslant(b;<c;d;e>)\vdash\dashv\overset{a}{\leqslant}(b;<c;d;e>)$

Hence we have directly from 22

23.1 $g^1(a)>g^2(a),\ g_0(g^1(a))=0\rightarrow fn(a)$

23.2 $g^1(a)>g^2(a),\ g_0(g^1(a))=1\rightarrow fn(a)\vdash\dashv=(g^2(a),g_1(g^1(a)))$
$$\vee=(g^2(a),g_2(g^1(a)))$$

23.3 $g^1(a)>g^2(a),\ g_0(g^1(a))=2\rightarrow fn(a)\vdash\dashv\leqslant(g_1(g^1(a)),g^2(a))$
$\wedge\mathrm{E}x\mathrm{E}y[x<g^2(a)\wedge y<g^2(a)\wedge\leqslant(y,x)$
$\wedge=(g^2(a);\langle x;y\rangle)]$

23.4 $g^1(a)>g^2(a),\ g_0(g^1(a))=3\rightarrow fn(a)\vdash\dashv\leqslant(g_1(g^1(a)),g^2(a))$
$\wedge\neg\leqslant(g_2(g^1(a)),g^2(a))$

23.5 $g^1(a)>g^2(a),\ g_0(g^1(a))=4\rightarrow fn(a)\vdash\dashv\leqslant(g_1(g^1(a)),g^2(a))$
$\wedge\mathrm{E}x\mathrm{E}y\{x<g^2(a)\wedge y<g^2(a)$
$\wedge=(g^2(a);\langle x;y\rangle)\wedge\leqslant(g_2(g^1(a)),y)\}$

23.6 $g^1(a)>g^2(a),\ g_0(g^1(a))=5\rightarrow fn(a)\vdash\dashv\mathrm{E}x\{x<g_1(g^1(a))$
$\wedge\leqslant(g_1(g^1(a));\langle x;g^2(a)\rangle)\}$

23.7 $g^1(a)>g^2(a),\ g_0(g^1(a))=6\rightarrow fn(a)\vdash\dashv\mathrm{E}x\mathrm{E}y\{x<g_1(g^1(a))\wedge y<g_1(g^1(a))$
$\wedge\leqslant(g_1(g^1(a));\langle x;y\rangle)$
$\wedge=(g^2(a);\langle x;y\rangle)$

23.8 $g^1(a)>g^2(a),\ g_0(g^1(a))=7\rightarrow fn(a)\vdash\dashv\mathrm{E}x\mathrm{E}y\mathrm{E}z\{x<g_1(g^1(a))$
$\wedge y<g_1(g^1(a))\wedge z<g_1(g^1(a))$
$\wedge\leqslant(g_1(g^1(a));\langle x;y;z\rangle)$
$\wedge=(g^2(a);\langle y;z;x\rangle)\}$

23.9 $g^1(a)>g^2(a),\ g_0(g^1(a))=8\rightarrow fn(a)\vdash\dashv\mathrm{E}x\mathrm{E}y\mathrm{E}z\{x<g_1(g^1(a))$
$\wedge y<g_1(g^1(a))\wedge z<g_1(g^1(a))$
$\wedge\leqslant(g_1(g^1(a));\langle x;y;z;\rangle)$
$\wedge=(g^2(a);\langle x;z;y\rangle)\}$

23.10 $g^1(a)\leqq g^2(a)\rightarrow fn(a)\vdash\dashv\forall x[x<g^2(a)\vdash\{\leqslant(g^1(a),x)\vdash\dashv\overset{a}{\leqslant}(g^2(a),x)\}]$

Now we shall simplify 23.10

Since $\overset{a}{\leqslant}(g^2(a),x)$ is

$\mathrm{E}y[\{(y\leqq x\wedge fn(j(y,x))\wedge j(y,x)<a)\vee(y\geqq x\wedge fn(j(x,y))\wedge j(x,y)<a)\}$
$\wedge\{g^2(a)>y\wedge fn(j(g^2(a),y))\wedge j(g^2(a),y)<a\}]$,

and $g^1(a)\leqq g^2(a),\ y\leqq x,\ x<g^2(a)\rightarrow j(y,x)<a,$
$g^1(a)\leqq g^2(a),\ g^2(a)>y,\ y\geqq x\rightarrow j(x,y)<a,$

and $g^1(a)\leqq g^2(a),\ g^2(a)>y\rightarrow j(g^2(a),y)<j(0,g^2(a))\leqq a$;

we have clearly 23.10.

23.11 $g^1(a)\leqq g^2(a)\rightarrow fn(a)\vdash\dashv\forall x[x<g^2(a)\vdash\{\leqslant(g^1(a),x)\vdash\dashv\leqslant(g^2(a),x)\}]$

In particular we have $fn(j(a, a))$. Hence

23.12 $\rightarrow =(a, a)$

Hence we have

24. $=(b, c) \mapsto \forall x[x < \max(b, c) \vdash \{\leqq(b, x) \mapsto \leqq(c, x)\}]$.

Moreover we shall prove

25. $=(b, c) \mapsto \forall x\{\leqq(b, x) \mapsto \leqq(c, x)\}$

To prove this proposition we first prove some simple propositions.

25.1 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$,
$\leqq(b, c)$, $=(c, d)$, $b < a$, $c < a$, $d < a \rightarrow \leqq(b, d)$

PROOF. This is clear by the fact $\leqq(b, c) \mapsto \mathrm{E}x(<(b, x) \wedge =(x, c))$.

25.2 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$,
$\leqq(b, c)$, $=(b, d)$, $b < a$, $c < a$, $d < a \rightarrow \leqq(d, c)$

PROOF. $\leqq(b, c) \mapsto \mathrm{E}x(<(b, x) \wedge =(x, c)) \mapsto \mathrm{E}x(<(b, x) \wedge =(x, c) \wedge x < b)$ and so $\leqq(b, c)$, $=(b, d) \rightarrow \mathrm{E}x(\leqq(d, x) \wedge =(x, c) \wedge x < b)$.
Hence from 25.1 the proposition is clear.

25.3 $b < a$, $c < a$, $=(b, a)$, $=(a, c) \rightarrow =(b, c)$

PROOF. This is clear from 24.

25.4 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$,
$=(a, b)$, $=(b, c)$, $\leqq(a, d)$, $b < a$, $c < a$, $d < a \rightarrow \leqq(c, d)$

PROOF. From 24 we have $d < a$, $=(a, b)$, $\leqq(a, d) \rightarrow \leqq(b, d)$, and the proposition is clear from 25.2.

25.5 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$,
$=(a, b)$, $=(b, c)$, $\leqq(c, d)$, $b < a$, $c < a$, $d < a \rightarrow \leqq(a, d)$.

PROOF. This is clear from 25.2 and 24.

25.6 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$,
$=(a, b)$, $=(b, c)$, $b < a$, $c < a \rightarrow =(a, c)$.

PROOF. The proposition means 25.4 and 25.5 from 24.

25.7 $\forall x \forall y \forall z\{x < a \wedge y < a \wedge z < a \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$
$\rightarrow \forall x \forall y \forall z\{x < a' \wedge y < a' \wedge z < a' \wedge =(x, y) \wedge =(y, z) \vdash =(x, z)\}$

PROOF. This is clear from 25.3 and 25.6.

26. $\forall x \forall y \forall z\{=(x,y)\wedge=(y,z)\vdash=(x,z)\}$

PROOF. In virtue of 25.7, this is easily proved by the transfinite induction on a in the formula

$$\forall x \forall y \forall z\{x<a\wedge y<a\wedge z<a\wedge=(x,y)\wedge=(y,z)\vdash=(x,z)\}.$$

Together with 25.1 and 25.2 this implies

27. $\leqslant(b,c),\ =(c,d)\rightarrow\leqslant(b,d)$

28. $\leqslant(b,c),\ =(b,d)\rightarrow\leqslant(d,c)$

From 28 we have easily $=(b,d)\rightarrow\forall x(\leqslant(b,x)\mapsto\leqslant(d,x))$

Hence 25 is clear. q. e. d.

As usual we write $c\in b$ or $b\ni c$ for $\leqslant(b,c)$ and $b\equiv c$ for $=(b,c)$ and simplify 23 using 25, 26, 27 and 28.

First 23.2 means $b>c,\ g_0(b)=1\rightarrow fn(j(b,c))\mapsto c\equiv g_1(b)\vee c\equiv g_2(b)$ and $b>c,\ g_0(b)=1\rightarrow<(b,c)\mapsto c\equiv g_1(b)\vee c\equiv g_2(b)$

As $b\ni c$ is $\mathrm{E}x(c\equiv x\wedge<(b,x))$, we have

29. $g_0(b)=1\rightarrow b\ni c\mapsto c\equiv g_1(b)\vee c\equiv g_2(b)$

We write $\{b,c\}$ for $j(1,b,c)$

Then we have clearly

30.1 $\{b,c\}\ni d\mapsto b\equiv d\vee c\equiv d$

30.2 $a\equiv b,\ c\equiv d\rightarrow\{a,c\}\equiv\{b,d\}$

By similar calculations as above we obtain

30.3 $=(b;\{c;d\})\mapsto b\equiv\{c,d\}$

because $\forall x[x<b\vdash\{b\ni x\mapsto x\equiv c\vee x\equiv d\}]\wedge\mathrm{E}x(x<b\wedge x\equiv c)$
$\wedge\mathrm{E}x(x<b\wedge x\equiv d)\mapsto\forall x\{b\ni x\mapsto x\equiv c\vee x\equiv d\}$

30.4 $\leqslant(b;\{c;d\})\mapsto b\ni\{c,d\}$

because $\mathrm{E}x\{x<b\wedge b\ni x\wedge x\equiv\{c,d\}\}\mapsto\mathrm{E}x(b\ni x\wedge x\equiv\{c,d\})$.

Moreover we writer $<c,d>$ for $\{\{c,c\},\{c,d\}\}$ and $<c,d,e>$ for $<c,<d,e>>$.

Then we have easily

30.5 $a\equiv c\wedge b\equiv d\mapsto<a,b>\equiv<c,d>$

30.6 $a\equiv b\wedge c\equiv d\wedge e\equiv f\mapsto<a,c,e>\equiv<b,d,f>$

30.7 $=(b;<c;d>)\mapsto b\equiv<c,d>$

because $\mathsf{E}x\mathsf{E}y(x<b \wedge y<b \wedge b\equiv\{x,y\} \wedge x\equiv\{c,c\} \wedge y\equiv\{c,d\})$
$\mapsto \mathsf{E}x\mathsf{E}y(b\equiv\{x,y\} \wedge x\equiv\{c,c\} \wedge y\equiv\{c,d\})$

30.8 $\leqslant(b;<c;d>) \mapsto b\ni<c,d>$
30.9 $=(b;<c;d;e>) \mapsto b\equiv<c,d,e>$
30.10 $\leqslant(b;<c;d;e>) \mapsto b\ni<c,d,e>$

Now we shall simplify 23.

31.1 $g_0(b)=0\rightarrow a\in b \mapsto \mathsf{E}x(x\equiv a \wedge x<b)$
31.2 $g_0(b)=2\rightarrow b\ni c \mapsto c\in g_1(b) \wedge \mathsf{E}x\mathsf{E}y\{y\ni x \wedge c\equiv<x,y>\}$

PROOF. From 23.3 we have

$b>c,\ g_0(b)=2\rightarrow fn(j(b,c)) \mapsto g_1(b)\ni c$
$\wedge \mathsf{E}x\mathsf{E}y(x<c \wedge y<c \wedge y\ni x \wedge c\equiv<x,y>)$

$b>c,\ g_0(b)=2\rightarrow <(b,c) \mapsto g_1(b)\ni c$
$\wedge \mathsf{E}x\mathsf{E}y(x<c \wedge y<c \wedge y\ni x \wedge c\equiv<x,y>)$

As $\mathsf{E}x\mathsf{E}y(x<c \wedge y<c \wedge y\ni x \wedge c\equiv<x,y>) \mapsto \mathsf{E}x\mathsf{E}y(y\ni x \wedge c\equiv<x,y>)$, we have

$b>c,\ g_0(b)=2\rightarrow <(b,c) \mapsto \mathsf{E}x\mathsf{E}y(y\ni x \wedge c\equiv<x,y>) \wedge g_1(b)\ni c$

Hence $g_0(b)=2\rightarrow b\ni c \mapsto \mathsf{E}x\mathsf{E}y(y\ni x \wedge c\equiv<x,y>) \wedge g_1(b)\ni c.$ q. e. d.

We have in the same way

31.3 $g_0(b)=3\rightarrow b\ni c \mapsto g_1(b)\ni c \wedge \neg g_2(b)\ni c$
31.4 $g_0(b)=4\rightarrow b\ni c \mapsto g_1(b)\ni c \wedge \mathsf{E}x\mathsf{E}y(c\equiv<x,y> \wedge g_2(b)\ni y)$
31.5 $g_0(b)=5\rightarrow b\ni c \mapsto \mathsf{E}x(g_1(b)\ni<x,c>)$
31.6 $g_0(b)=6\rightarrow b\ni c \mapsto \mathsf{E}x\mathsf{E}y(g_1(b)\ni<x,y> \wedge c\equiv<y,x>)$
31.7 $g_0(b)=7\rightarrow b\ni c \mapsto \mathsf{E}x\mathsf{E}y\mathsf{E}z(g_1(b)\ni<x,y,z> \wedge c\equiv<y,z,x>)$
31.8 $g_0(b)=8\rightarrow b\ni c \mapsto \mathsf{E}x\mathsf{E}y\mathsf{E}z(g_1(b)\ni<x,y,z> \wedge c\equiv<x,z,y>)$

Now we can continue our proof in the same way as Gödel (1).

First we define new functions $Od(a)$, $C(a)$, $b-c$ and $b\cdot c$ as abbreviations of $\mathrm{Min}(z)(z\equiv a)$, $\mathrm{Min}(z)(z\in a)$, $j(3,b,c)$ and $b-(b-c)$ respectively. Then we have the following sequences.

32.1 $a\equiv Od(a)$
32.2 $a\equiv b\rightarrow b\geqq Od(a)$
32.3 $a\in b\rightarrow Od(a)<Od(b)$

because $a \in Od(b)$ means $\mathrm{E}x\{<(Od(b), x) \wedge a \equiv x\}$

32.4 $\mathrm{E}x(x \in a) \rightarrow C(a) \in a \wedge \forall x(x \in a \vdash x \geqq C(a))$

32.5 $\neg(a \in a)$

because $Od(a) \in Od(a)$ means $Od(a) < Od(a)$.

32.6 $a \in (b-c) \vdash\dashv a \in b \wedge \neg(a \in c)$

32.7 $a \in (b \cdot c) \vdash\dashv a \in b \wedge a \in c$

Moreover we define $a \subseteq b$ as $\forall x(x \in a \vdash x \in b)$ and $a \subset b$ as $a \subseteq b \wedge \neg (a \equiv b)$. Then we have easily

33.1 $0 \in \omega \wedge \forall x\{x \in \omega \vdash \mathrm{E}y(y \in \omega \wedge x \subset y)\}$

PROOF. This is clear from 31.1 and 32.3.

33.2 $\forall x \neg(x \in 0)$

33.3 $\forall y \forall z(z \in y \wedge y \in a \vdash z \in j(0, a, a))$

because $c \in b \wedge b \in a \rightarrow c \equiv Od(c) < Od(b) < Od(a) \leqq a \leqq j(0, a, a)$.

Now we denote with $cl(\alpha)$ the formula $\forall x \mathrm{E}y \forall z(\alpha[z] \wedge z \in x \vdash\dashv z \in y)$ for a variable α of the type 1.
It follows directly
$\forall x(\mathfrak{A}(x) \vdash\dashv \mathfrak{B}(x)),\ cl(\{x\}\mathfrak{A}(x)) \rightarrow cl(\{x\}\mathfrak{B}(x))$.
And we have easily

34.1 $\forall x \neg \alpha[x] \rightarrow cl(\alpha)$

34.2 $\forall x \alpha[x] \rightarrow cl(\alpha)$

34.3 $cl(\alpha),\ \alpha[a],\ a \equiv b \rightarrow \alpha[b]$

because $cl(\alpha) \rightarrow \mathrm{E}y \forall z(\alpha[z] \wedge z \in \{a, b\} \vdash\dashv z \in y)$
Further by the same calculations as in the chapter V p. 40 in Gödel (1) we have the following sequences

34.4 $\rightarrow cl(\{x\}(x \in a))$

34.5 $cl(\alpha),\ cl(\beta) \rightarrow cl(\{x\}(a[x] \wedge \neg \beta[x]))$

34.6 $cl(\alpha),\ cl(\beta) \rightarrow cl(\{x\}(\alpha[x] \wedge \beta[x]))$

34.7 $cl(\alpha),\ cl(\beta) \rightarrow cl(\{x\}(\alpha[x] \vee \beta[x]))$

34.8 $cl(\alpha) \rightarrow cl(\{x\}(\mathrm{E}y \mathrm{E}z(x \equiv <y, z> \wedge \alpha[x])))$

34.9 $cl(\alpha) \rightarrow cl(\{x\}(\mathrm{E}y \alpha[<y, x>]))$

34.10 $cl(\alpha) \rightarrow cl(\{u\}(\mathrm{E}x \mathrm{E}y(\alpha[<x, y>] \wedge u \equiv <y, x>)))$

34.11 $cl(\alpha)\rightarrow cl(\{u\}(\mathrm{E}x\mathrm{E}y\mathrm{E}z(\alpha[<x,y,z>]\wedge u\equiv<y,z,x>)))$

34.12 $cl(\alpha)\rightarrow cl(\{u\}(\mathrm{E}x\mathrm{E}y\mathrm{E}z(\alpha[<x,y,z>]\wedge u\equiv<x,z,y>)))$

34.13 $cl(\alpha)\rightarrow cl(\{u\}(\mathrm{E}x\mathrm{E}y(u\equiv<y,x>\wedge\alpha[y])))$

34.14 $cl(\alpha),\ cl(\beta)\rightarrow cl(\{x\}(\mathrm{E}y\mathrm{E}z(x\equiv<y,z>\wedge\alpha[y]\wedge\beta[z])))$

34.15 $cl(\alpha)\rightarrow cl(\{x\}(\mathrm{E}y\alpha[<x,y>]))$

34.16 $cl(\alpha),\ cl(\beta)\rightarrow cl(\{x\}(\mathrm{E}y(\alpha[<x,y>]\wedge\beta[y])))$

From 34.4 and 34.16 we have

34.17 $cl(\alpha)\rightarrow cl(\{x\}(\mathrm{E}y(\alpha[<x,y>]\wedge y\in a)))$

and so $cl(\alpha)\rightarrow cl(\{x\}(\mathrm{E}y(\alpha[<x,y>]\wedge y\in\{a\})))$

This implies together with 34.3

34.18 $cl(\alpha)\rightarrow cl(\{x\}(\alpha[<x,a>]))$

As in chapter VI p. 46 in Gödel (1) we have the following sequence

35. $cl(\alpha),\ \forall x\forall y\forall z(\alpha[<x,z>]\wedge\alpha[<y,z>]\vdash x\equiv y)$

$$\rightarrow\mathrm{E}x\forall y\{y\in x\dashv\vdash\mathrm{E}z(z\in a\wedge\alpha[<y,z>]\}$$

PROOF. From 34.17 we have $cl(\alpha)\rightarrow cl(\{x\}(\mathrm{E}y(\alpha[<x,y>]\wedge y\in a)))$
On the other hand, if we define $\{x,y\}\beta_2[x,y]$ as the abbreviation of $\{x,y\}(\alpha[<y,x>]\wedge\forall u(\alpha[<u,x>]\vdash u\geqq y)$, then we have clearly

$cl(\alpha),\ \forall x\forall y\forall z(\alpha[<x,z>]\wedge\alpha[<y,z>]\vdash x\equiv y)$

$$\rightarrow\forall x\forall y\forall z(\beta_2[x,z]\wedge\beta_2[y,z]\vdash x=y)$$

Hence we see by the axiom of substitution that there exists an ordinal number b satisfying $g_0(b)=0$ and

$\forall x\forall y(\beta_2[x,y]\wedge y<a\vdash x<b)$.

From $cl(\{x\}(\mathrm{E}y(\alpha[<x,y>]\wedge y\in a)))$ it follows that there exists an ordinal number c satisfying $\forall x(\mathrm{E}y(\alpha[<x,y>]\wedge y\in a)\wedge x\in b\dashv\vdash x\in c)$
Then we can see easily $\forall x\{x\in c\dashv\vdash\mathrm{E}y(y\in a\wedge\alpha[<x,y>])\}$. q. e. d.

Now we shall begin to prove

36. $\forall x\mathrm{E}y\forall z(z\leqq x\vdash z\in y)$

To prove the proposition we have only to prove

37. $\forall x\forall y(y\leqq\aleph(x)\vdash y\in\aleph(x'))$

In the same way as in the proof of 12.3 in the Chapter VIII in Gödel (1) (p. p. 54–61) we have first

38. $\forall x(\alpha[x] \vdash \alpha[C(x)] \wedge \alpha[g_1(x)] \wedge \alpha[g_2(x)])$,
$\forall i \forall x \forall y(i < 9 \wedge \alpha[x] \wedge \alpha[y] \vdash \alpha[j(i, x, y)])$,
$\forall u \forall v \forall x \forall y\{\alpha_2[x, y] \wedge \alpha_2[u, v] \vdash (x < u \vdash\!\dashv y < v)\}$,
$\forall x \mathsf{E} y(\alpha[x] \vdash y < a \wedge \alpha_2[y, x])$,
$\forall x \mathsf{E} y(x < a \vdash \alpha[y] \wedge \alpha_2[x, y])$,
$\alpha[b], \alpha[c], \alpha_2[d, b], \alpha_2[e, c] \rightarrow b \ni c \vdash\!\dashv d \ni e$.

39. $\mathsf{E}\varphi_2[\forall u \forall v \forall x \forall y\{\varphi_2[v, u] \wedge \varphi_2[x, y] \vdash (u < x \vdash\!\dashv v < y)\}$
$\wedge \forall x\{\alpha[x] \vdash\!\dashv \mathsf{E} y \varphi_2[y, x]\}$
$\wedge \forall x \forall y \forall v\{\varphi_2[y, x] \wedge v < y \vdash \mathsf{E} u \varphi_2[u, v]\}]$

PROOF. Let $\mathfrak{F}_1(\alpha_2)$ be $\forall x \forall y[\alpha_2[y, x] \vdash\!\dashv \alpha[x] \wedge \forall u \forall v(\alpha_2[v, u] \wedge v < y \vdash u < x) \wedge \forall z(\alpha[z] \wedge \forall u \forall v(\alpha_2[v, u] \wedge v < y \vdash u < x) \vdash x \leqq z\}]$
Then by Lemma on the recursive predicate we have $\mathsf{E}\varphi_2 \mathfrak{F}_1(\varphi_2)$
We have directly

39.1 $\mathfrak{F}_1(\alpha_2),\ \alpha_2[b, a] \rightarrow \alpha[a]$

39.2 $\mathfrak{F}_1(\alpha_2),\ \alpha_2[b, a],\ \alpha_2[d, c],\ c < a \rightarrow d < b$

39.3 $\mathfrak{F}_1(\alpha_2),\ \alpha_2[b, a],\ \alpha_2[e, a] \rightarrow b = e$

and so (from 39.2 and 39.3)

39.4 $\mathfrak{F}_1(\alpha_2),\ \alpha_2[b, a],\ \alpha_2[d, c],\ d < a \rightarrow c < a$.

If remains, therefore, only to prove

39.5 $\mathfrak{F}_1(\alpha_2),\ \alpha[a] \rightarrow \mathsf{E} y \alpha_2[y, a]$

39.6 $\mathfrak{F}_1(\alpha_2),\ \alpha_2[b, a],\ d < b \rightarrow \mathsf{E} u \alpha_2[d, u]$.

Let $\{z\}\mathfrak{A}(z)$ be $\{z\}(\forall x \forall y(\alpha_2[y, x] \wedge x < a \vdash y < z)$. Then from 39.3 and the axiom of substitution,
$\forall u \forall v(\alpha_2[v, u] \wedge u < a \vdash v < \mathrm{Min}(w)\mathfrak{A}(w))$ and consequently
$\mathfrak{F}_1(\alpha_2),\ \alpha[a] \rightarrow \alpha_2[\mathrm{Min}(w)\mathfrak{A}(w), a]$
which proves 39.5.

Let $\{x, y\}\mathfrak{B}(x, y)$ be $\{x, y\}(\alpha[x] \wedge \forall u \forall v(\alpha_2[v, u] \wedge v < y \vdash u < x))$.
Then $\mathfrak{F}_1(\alpha_2), \alpha_2[b, a], d < b \rightarrow \mathfrak{B}(a, d)$ and so
$\mathfrak{F}_1(\alpha_2), \alpha[b, a],\ d < b \rightarrow \mathsf{E} w\{\mathfrak{B}(w, d) \wedge \forall z(\mathfrak{B}(z, d) \vdash w \leqq z)\}$
which proves 39.6. q. e. d.

Now we consider

40. $\mathrm{E}\varphi_2\mathrm{E}\varphi[\forall x(x<\aleph(a)\vdash\varphi[x])\wedge\varphi[b]$
$\wedge\forall i\forall x\forall y(i<9\wedge\varphi[x]\wedge\varphi[y]$
$\vdash\varphi[C(x)]\wedge\varphi[g_1(x)]\wedge\varphi[g_2(x)]\wedge\varphi[j(i,x,y)])$
$\wedge\forall x\{\varphi[x]\vdash\mathrm{E}y(\varphi_2[x,y]\wedge y<\aleph(a))\}$
$\wedge\forall x\forall y\forall z(\varphi_2[x,z]\wedge\varphi_2[y,z]\vdash x=y)]$

We shall prove that from 40 we have 37.
For this purpose we have only to prove

41. $\forall x\{x\in b\vdash\mathrm{E}y(y<\aleph(a)\wedge x\equiv y)\}\to\mathrm{E}x(b\equiv x\wedge x<\aleph(a'))$

under the assumption 40.

From the assumption 40 we may assume that there exist variables $\alpha[*_i]$ and $\alpha_2[*_1, *_2]$ satisfying

41.1 $\forall x(x<\aleph(a)\vdash\alpha[x])\wedge\alpha[b]$

41.2 $\forall i\forall x\forall y(i<9\wedge\alpha[x]\wedge\alpha[y]$
$\vdash\alpha[C(x)]\wedge\alpha[g_1(x)]\wedge\alpha[g_2(x)]\wedge\alpha[j(i,x,y)])$

41.3 $\forall x\{\alpha[x]\vdash\mathrm{E}y(\alpha_2[x,y]\wedge y<\aleph(a))\}$

41.4 $\forall x\forall y\forall z(\alpha_2[x,z]\wedge\alpha_2[y,z]\vdash x=y)$.

Then by the axiom of substitution we have an ordinal number c satisfying

41.5 $\forall x(\alpha[x]\vdash x<c)$

Therefore we can see from 37 that there exists a variable $\beta_2[*_1, *_2]$ satisfying $\forall u\forall v\forall y\forall z\{\beta_2[u,v]\wedge\beta_2[y,z]\vdash(u<y\dashv\vdash v<z)\}$
$\wedge\forall u\{\alpha[u]\wedge u<c\vdash\mathrm{E}v\beta_2[v,u]\}$
$\wedge\forall u\forall v\forall y\{\alpha[u]\wedge u<c\wedge\beta_2[v,u]$
$\wedge y<v\vdash\mathrm{E}z(\beta_2[y,z]\wedge\alpha[z])\}$

and so from 41.5

41.6 $\forall u\forall v\forall y\forall z\{\beta_2[u,v]\wedge\beta_2[y,z]\vdash(u<y\dashv\vdash v<z)\}$
$\wedge\forall u\{\alpha[u]\vdash\mathrm{E}v\beta_2[v,u]\}$
$\wedge\forall u\forall v\forall y\{\alpha[u]\wedge\beta_2[v,u]\wedge y<v\vdash\mathrm{E}z(\beta_2[y,z]\wedge\alpha[z])\}$.

We shall define $\{x\}\beta[x]$ as $\{x\}(\mathrm{E}y(\alpha[y]\wedge\beta_2[x,y]))$ and $\{x,y\}\gamma_2[x,y]$ as $\{x,y\}(\mathrm{E}z(\beta_2[x,z]\wedge\alpha_2[z,y]))$.
As we see easily

$\forall x\forall y\forall z(\gamma_2[x,z]\wedge\gamma_2[y,z]\vdash x=y)$, it is easily proved by the axiom of substitution that there exists an ordinal number d satisfying

41.7 $\forall x\{\beta[x] \vdash\!\dashv x < d\}$

and so

41.8 $d < \aleph(a')$

We can see clearly $\mathrm{E}x(\beta_2[x, b] \wedge x < d)$.

Let $\hat{b}$ be an ordinal number satisfying $\beta_2[\hat{b}, b]$, then from 38 we have $\forall x \forall y\{\beta_2[x, y] \wedge \alpha[y] \vdash (b \ni y \vdash\!\dashv \hat{b} \ni x)\}$.

Since $\forall x(x < \aleph(x) \vdash \beta_2[x, x])$, we see easiy $\forall x\{x < \aleph(a) \vdash (\hat{b} \ni x \vdash\!\dashv b \ni x)\}$, and so $b \equiv \hat{b} \cdot \aleph(a)$. As $\hat{b} < \aleph(a')$, we see easily $\hat{b} \cdot \aleph(a) < \aleph(a')$. Hence 41 is proved.

Therefore to prove 36 we have only to prove 40.

For convenience' sake we write $h_0(*)$ for $g^1(g^2(*))$, $h_1(*)$ for $g^1(g^2(g^2(*)))$ and $h_2(*)$ for $g^2(g^2(g^2(*)))$.

Let $\mathfrak{A}^1(*_1, *_2, *_3, *_4)$ be $(*_1 = j(*_2, *_3, *_4) \wedge *_2 < 9)$
$\vee(*_1 = \mathrm{C}(*_3) \wedge *_2 = 9 \wedge *_3 = *_4) \vee (*_1 = g^1(*_3) \wedge *_2 = 10 \wedge *_3 = *_4)$
$\vee(*_1 = g^2(*_3) \wedge *_2 = 11 \wedge *_3 = *_4)$ and let $\mathfrak{A}^2(*_1, *_2, *_3, *_4)$ be
$(*_1 = b \wedge *_2 = 12 \wedge *_3 = 0 \wedge *_4 = 0) \vee (\neg *_1 = b \wedge *_1 < \aleph(a) \wedge *_2 = 12$
$\wedge *_3 = *_4 \wedge *_4 = 1 + *_1)$.

Then the sequence $\mathfrak{A}^2(c_1, c_2, c_3, c_4) \to c_3 < \aleph(a)$ can be proved without difficulty.

Let $\mathfrak{F}(c, \alpha_2)$ be
$\mathrm{E}u\mathrm{E}y_1\mathrm{E}y_2\{\mathfrak{A}^1(g^1(c), h_0(c), y_1, y_2) \wedge \alpha_2[j(y_1, h_1(c)), u] \wedge \alpha_2[j(y_2, h_2(c)), u]\}$
$\vee \mathfrak{A}^2(g^1(c), h_0(c), h_1(c), h_2(c))$, and let $\mathfrak{B}^1(c, d)$ be a formula satisfying $\forall z \forall v\{\mathfrak{B}^1(z, v) \vdash\!\dashv \mathfrak{F}(z, \mathfrak{B}^1(c, d) \wedge d < v)\}$ and let $\mathfrak{B}^2(d_1, d_2, d_3)$ be $\mathfrak{B}^1(j(d_1, d_2), d_3)$.

Then, of course, $\mathfrak{B}^2(d_1, d_2, d_3) \vdash\!\dashv$
$\mathrm{E}u\mathrm{E}y_1\mathrm{E}y_2\{\mathfrak{A}^1(d_1, g^1(d_2), y_1, y_2) \wedge \mathfrak{B}^2(y_1, g^1(g^2(d_2)), u) \wedge \mathfrak{B}^2(y_2, g^2(g^2(d_2)), u)$
$\wedge u < d_3\} \vee \mathfrak{A}^2(d_1, g^1(d_2), g^1(g^2(d_2)), g^2(g^2(d_2))$, and it follows
$\mathfrak{B}^2(d_1, d_2, d_3) \wedge d_3 < d \to \mathfrak{B}^2(d_1, d_2, d)$.

By the transfinite induction on d_3 we have
$\forall x \forall y \forall z\{\mathfrak{B}^2(y, x, d_3) \wedge \mathfrak{B}^2(z, x, d_3) \vdash y = z\}$.

As for the case $d_3 = 0$ we have obviously

$\mathfrak{B}^3(d_1, d_2, 0) \vdash\dashv \mathfrak{A}^2(d_1, g^1(d_2), g^1(g^2(d_2), g^2(g^2(d_2)))$.
Hence $\mathfrak{B}^2(d_1, d_2, 0) \vdash d_2 < \aleph(a)$ and, by the transfinite induction on d_3,
$\forall x \forall y \{\mathfrak{B}^2(y, x. d_3) \vdash x < \aleph(a)\}$
In particular, we have

42. $\mathfrak{B}^2(d_1, d_2, \omega) \vdash d_2 < \aleph(a)$

$\mathfrak{B}^2(d_1, d_2, \omega) \wedge \mathfrak{B}^2(d, d_2, \omega) \vdash d_1 = d$,

and it is easy to show

43. $\mathrm{E} x \mathfrak{B}^2(b, x, \omega)$,

$d < \aleph(a) \vdash \mathrm{E} x \mathfrak{B}^2(d, x, \omega)$;

furthermore, we have

44. $i < 9 \wedge \mathrm{E} x \mathfrak{B}^2(c, x, \omega) \wedge \mathrm{E} x \mathfrak{B}^2(d, x, \omega) \vdash$

$\mathrm{E} x \mathfrak{B}^2(j(i, c, d), x, \omega) \wedge \mathrm{E} x \mathfrak{B}^2(C(c), x, \omega) \wedge \mathrm{E} x \mathfrak{B}^2(g^1(c), x, \omega)$

$\wedge \mathrm{E} x \mathfrak{B}^2(g(c), x, \omega)$

since $\mathfrak{B}^2(d_1, d_2, \omega) \vdash\dashv \mathrm{E} w(w < \omega \wedge \mathfrak{B}^2(d_1, d_2, w))$.
And 40 follows from 42, 43 and 44. Thus 40 is proved. And so 36 is proved.

In the preceding we have shown that the following sequences are provable.

$\forall x \forall y \{x \equiv y \vdash\dashv \forall z(z \in x \vdash\dashv z \in y)\}$
$\forall x \forall y \forall v \{v \in \{x, y\} \vdash\dashv v \equiv x \wedge v \equiv y\}$
$\forall x \forall y (< x, y > \equiv \{\{x\} \{x, y\}\})$
$\forall x \forall y \forall z (< x, y, z > \equiv < x, < y, x >>)$
$\forall x \forall y \{x \subseteq y \vdash\dashv \forall z(z \in x \vdash z \in y)\}$
$\forall x \forall y \{x \subset y \vdash\dashv x \subseteq y \wedge \neg x \equiv y\}$
$0 \in \omega \wedge \forall x \{x \in \omega \vdash \mathrm{E} y(y \in \omega \wedge x \subset y)\}$
$\forall u \mathrm{E} x \forall y \forall z \{y \in z \wedge z \in u \vdash y \in x\}$
$\forall x \mathrm{E} y \forall z (z \subseteq x \vdash z \in y)$
$\forall \varphi \{cl(\varphi) \vdash\dashv \forall x \mathrm{E} y \forall z (\varphi[z] \wedge z \in x \vdash\dashv z \in y)\}$
$\forall x \mathrm{E} \varphi \{cl(\varphi) \wedge \forall y (\varphi[y] \vdash\dashv y \in x)\}$
$\forall x \forall y \forall \varphi \{x \equiv y \wedge cl(\varphi) \vdash \varphi[x] \vdash\dashv \varphi[y])\}$
$\forall \varphi \forall u [cl(\varphi) \wedge \forall x \forall y \forall z (\varphi[< x, z >] \wedge \varphi[< y, z >] \vdash x \equiv y)$

$$\vdash \mathrm{E}x\forall y\{y\in x \vdash\dashv \mathrm{E}z(z\in u \wedge \varphi[\langle y, z\rangle])\}]$$

$$\forall \varphi[\mathrm{E}x\varphi[x] \vdash \mathrm{E}u\{\varphi[u] \wedge \forall x \neg (x\in u \wedge \varphi[u])\}]$$

$$\forall x \neg (x\in 0)$$

$$\forall x[\neg x \equiv 0 \vdash C(x)\in x]$$

$$\mathrm{E}\varphi\{cl(\varphi) \wedge \forall x\forall y(\varphi[\langle x, y\rangle] \vdash\dashv x\in y)\}$$

$$\forall \varphi\forall \psi[cl(\varphi)\wedge cl(\psi) \vdash \mathrm{E}\xi\{cl(\xi)\wedge \forall u(\xi[u] \vdash\dashv \varphi[u]\wedge \psi[u])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall u(\psi[u] \vdash\dashv \neg\, \varphi[u])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall x(\psi[x] \vdash\dashv \mathrm{E}y\, \varphi[\langle y, x\rangle])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall x\forall y(\psi[\langle x, y\rangle] \vdash\dashv \varphi[x])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall x\forall y(\psi[\langle x. y\rangle] \vdash\dashv \varphi[\langle y, x\rangle])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall x\forall y\forall z(\psi[\langle x, y, z\rangle] \vdash\dashv \varphi[\langle y, z, x\rangle])\}]$$

$$\forall \varphi[cl(\varphi) \vdash \mathrm{E}\psi\{cl(\psi)\wedge \forall x\forall y\forall z(\psi[\langle x, y, z\rangle] \vdash\dashv \varphi[\langle x, z, y\rangle])\}]$$

Hence the consistency of the Fraenkel-v. Neumann's set theory is proved assuming the consistency of the ordinal number theory.

References

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