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A remark on Hilbert's Nullstellensatz.

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In [3] Zariski gave a proof of the Nullstellensatz based on the following lemma.

Let k and K be fields such that $K=k[x_1,\cdots,x_n]$. Then K is a finite extension of k.

Besides the proof of this lemma that Zariski gave, there is another proof in Artin and Tate [1] and a further proof is indicated in the exercises in Bourbaki [2, p. 87, exercise 4 and p. 106, exercise 12]. It is our purpose to give still another proof of this lemma.

Our proof is based on the following well-known results: (1) If o is an integral domain with quotient field k, if $[K:k]=n$, and if O is the set of all elements of K which are integral over o, then O is an integral domain and each element of K can be written in the form A/a with A in O and a in o. (2) The field norm $N_{K/k}$ is multiplicative. (3) If A is in O and o is integrally closed, then $N_{K/k}A$ is in o.

Here is the proof. If each x_j is algebraic over k, we are finished. The case in which $x_1,\cdots,x_n$ are independent transcendentals is clearly impossible since the polynomial ring $k[x_1,\cdots,x_n]$ is not a field. If neither of these cases prevails, then we may assume that $x_1,\cdots,x_r$ form a transcendence basis of K over k for some r with $1\leqq r<n$, set $F=k(x_1,\cdots,x_r)$, and have K a finite extension of F, with say $[K:F]=m$. Let $o=k[x_1,\cdots,x_r]$ which is isomorphic to the polynomial ring $k[X_1,\cdots,X_r]$ over k. By (1) above, there is an element $f=f(x_1,\cdots,x_r)$ in o such that for each j, $j=r+1,\cdots,n$, $z_j=fx_j$ is integral over o. We select a non-constant polynomial $g(X_1,\cdots,X_r)$ which is relatively prime to $f(X_1,\cdots,X_r)$ (for example $g(X)=X_1f(X)+1$) and consider $w=1/g(x_1,\cdots,x_r)$. Since w is in $K=k[x_1,\cdots,x_n]$, there is a polynomial $H(X)$ in $k[X_1,\cdots,X_n]$ such that $w=H(x_1,\cdots,x_n)$. Multiplying this last relation with a sufficiently high power of f yields a relation of the form $f^sw=H_1(x_1,\cdots,x_r,z_{r+1},\cdots,z_n)$, where $H_1(X_1,\cdots,X_r,Z_{r+1},\cdots,Z_n)$ is in

the polynomial ring $k[X_1,\cdots,Z_n]$. By (1) again, the set of integral elements over o is closed under addition and multiplication, hence $H_1(x_1,\cdots,z_n)$ is integral over o; we write $f^s w=y$, with y integral over o, or equivalently, $f^s=yg$. We now apply the norm $N_{K/F}$ to obtain $f^{ms}=hg^m$, where by (3), $h=N_{K/F}(y)$ is in $o=k[x_1,\cdots,x_r]$. We thus have a polynomial equation

$$[f(X_1,\cdots,X_r)]^{ms}=h(X_1,\cdots,X_r)[g(X_1,\cdots,X_r)]^m,$$

which is impossible since $f(X)$ and $g(X)$ are relatively prime.

The crux of the above proof is the elimination of the algebraic quantities $x_{r+1},\cdots,x_n$ by the expedient formation of the norm. We may remark that the use of a transcendence basis can easily be replaced by induction on n.

References.

[1] E. Artin and J.T. Tate, A note on finite ring extensions, Jour. Math. Soc. Japan 3, (1951), 74-77.

[2] N. Bourbaki, Algèbre, Chapt. 5: Corps commutatifs, Paris, 1950.

[3] O. Zariski, A new proof of Hilbert's Nullstellensatz, Bull. A. M. S, 53 (1947), 362-368. Especially Theorem H_3^n, p. 363.

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