Journal of the Mathematical Society of Japan Vol. 6, No. 2, September, 1954

On the fundamental theorem of algebra.

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(Received Jan. 14, 1954)

In this note we give an elementary proof for the fundamental theorem of algebra that the complex number field C is algebraically closed, using a normed-ring-theoretic method.

For this purpose let $C[x]$ be the polynomial ring over C. We define the absolute value of $f\in C[x]$ by $|f|=|a_{0}|+\cdots+|a_{m}|$ where $f=a_{0}+\cdots+a_{m}x^{m}$, so that $|f|\geqq 0$ always, and $|f|=0$ if and only if $f=0$. (This symbol is clearly compatible with the usual absolute value when $f\in C$.) It follows easily that for $f, g\in C[x]$ and $z\in C$

$$|f+g|\leqq|f|+|g|, \qquad |fg|\leqq|f|\cdot|g|, \qquad |zf|=|z|\cdot|f|.$$

Suppose that $\phi\in C[x]$ is a fixed monic polynomial of degree $n\geqq 1$. We define an operator Φ for $f\in C[x]$ as follows. If $f=0$ or $\deg f<n$, we put $\Phi f=f$; if $m=\deg f\geqq n$ and $f=a_{0}+\cdots+a_{m}x^{m}$, then we put $\Phi f=f-a_{m}x^{m-n}\phi$, so that $\deg\Phi f<\deg f$ for $\Phi f\neq 0$ in this latter case. Then we clearly have always

$$|\Phi f|\leqq|f|+|f|\cdot|\phi|=M|f| \qquad (M=|\phi|+1),$$

and so $|\Phi^{n}f|\leqq M^{n}|f|$.

Now let ϕ considered above be irreducible over C. To prove the theorem, it then suffices to show that $n=1$. The residue-classes of $C[x]$ modulo ϕ form a field E, which contains C as a subfield if we identify each $z\in C$ with the residue-class containing z. Let $\theta\in E$ be the residue-class represented by x, then $\phi(\theta)=0$ and for each $\alpha\in E$ there is a uniquely determined polynomial $f_{\alpha}\in C[x]$ such that $\alpha=f_{\alpha}(\theta)$ and that the degree of f_{α} is $<n$ when $\alpha\neq 0$.

This being so, we define $|\alpha|=|f_{\alpha}|$ for $\alpha\in E$, so that $|\alpha|\geqq 0$ always, $|\alpha|=0$ if and only if $\alpha=0$, and $|z\alpha|=|z|\cdot|\alpha|$ for $z\in C$. (This symbol coincides with the usual absolute value when $\alpha\in C$.) If $\alpha, \beta\in E$, it is easily seen that $f_{\alpha+\beta}=f_{\alpha}+f_{\beta}$, whence $|\alpha+\beta|\leqq|\alpha|+|\beta|$. Further,

$f_{\alpha\beta} \equiv f_\alpha f_\beta \pmod{\phi}$ and so $f_{\alpha\beta} = \Phi^n(f_\alpha f_\beta)$, since $f_\alpha f_\beta$ is not of degree higher than $2n-2$. Therefore

$$|\alpha\beta| = |\Phi^n(f_\alpha f_\beta)| \leqq M^n |f_\alpha f_\beta| \leqq M^n |f_\alpha| \cdot |f_\beta| = M^n |\alpha| \cdot |\beta| .$$

The field E now becomes a normed field over C if we define the norm of $\alpha \in E$ by $\|\alpha\| = M^n |\alpha|$. For it readily follows from what has been said above that

1. $\|\alpha\| \geqq 0, \qquad \|\alpha\| = 0 \leftrightarrows \alpha = 0,$
2. $\|\alpha + \beta\| \leqq \|\alpha\| + \|\beta\|,$
3. $\|z\alpha\| = |z| \cdot \|\alpha\| \qquad (z \in C),$
4. $\|\alpha\beta\| \leqq \|\alpha\| \cdot \|\beta\| .$

But the fundamental theorem of normed rings (Mazur-Gelfand) asserts that every normed field K over C coincides with Ce, where e is the identity element of K. Hence $E = C$, q. e. d.

REMARK. Kametani, Journ. Math. Soc. Jap. 4 (1952), pp. 96–99, gave a neat proof for the Mazur-Gelfand theorem, which neither assumes the completeness of K nor uses any function-theoretic properties of C but the notion of continuity.

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