Journal of the Mathematical Society of Japan Vol. 6, No. 1, March, 1954

On some matrix operators.

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(Received Nov. 16, 1953)

0. Introduction.

Let K be an arbitrary field of any characteristic $\chi(K)\,(=0$ or $p)$. We denote by $\mathfrak{gl}(K, n)$ the set of all matrices of degree n over K and by $GL(K, n)$ the set of all non-singular matrices in $\mathfrak{gl}(K, n)$. I_n and O_n mean the unit matrix and zero matrix of degree n respectively.

Besides ordinary operations on matrices, we consider the following three operators. For $A=(a_{ij})\in\mathfrak{gl}(K, n)$ and $B\in\mathfrak{gl}(K, m)$ we consider the direct sum:

$$A\dot{+}B=\begin{pmatrix} A & O \\ O & B \end{pmatrix}\in\mathfrak{gl}\,(K, n+m)\,,$$

the Kronecker product:

$$A\otimes B=\begin{pmatrix} a_{11}B, a_{12}B,\cdots, a_{1n}B \\ \cdot \\ a_{n1}B, a_{n2}B,\cdots, a_{nn}B \end{pmatrix}\in\mathfrak{gl}(K, nm)\,,$$

and the Kronecker sum: $A\oplus B=A\otimes I_m+I_n\otimes B\in\mathfrak{gl}(K, nm)$.
These operations $\dot{+}, \otimes, \oplus$ are non-commutative but associative.
Now we define two set-theoretical sums:

$$\mathfrak{R}=\mathfrak{R}(K)=\bigcup_{n=1}^{\infty}\mathfrak{gl}(K, n)\,, \qquad \mathfrak{S}=\mathfrak{S}(K)=\bigcup_{n=1}^{\infty}GL(K, n)\,.$$

For an element A in $\mathfrak{R}$, we denote by $d(A)$ its degree.

Now let L be a Lie algebra over K and $\mathfrak{R}_0\ni\rho_1, \rho_2,\cdots$ the set of representations of L. Between the elements of $\mathfrak{R}_0$, the operations such as $\rho_1\dot{+}\rho_2$, $\rho_1\oplus\rho_2$ are defined in the well-known way. We can also speak of the degree $d(\rho)$ of ρ, and of the transform $T\rho T^{-1}$ of ρ by an element T in $GL(K, d(\rho))$.

Harish-Chandra [1] has considered a mapping ζ of $\mathfrak{R}_0$ into $\mathfrak{R}$, satisfying the following conditions:

I′ $d(\zeta(\rho))=d(\rho)$ for every ρ in $\mathfrak{R}_0$,

II′ $\zeta(T\rho T^{-1})=T\zeta(\rho)T^{-1}$ for every ρ in $\mathfrak{R}_0$ and for every T in $GL(K, d(\rho))$,

III′ $\zeta(\rho_1 \dot{+} \rho_2)=\zeta(\rho_1)\dot{+}\zeta(\rho_2)$ for every ρ_1, ρ_2 in $\mathfrak{R}_0$,

IV′ $\zeta(\rho_1 \oplus \rho_2)=\zeta(\rho_1)\oplus\zeta(\rho_2)$ for every ρ_1, ρ_2 in $\mathfrak{R}_0$.

He called such a mapping ζ a representation of $\mathfrak{R}_0$, and denoted the set of all representations of $\mathfrak{R}_0$ by $\dot{L}$. Then $\dot{L}$ becomes a Lie algebra over K with respect to the following operations: if $\zeta_1, \zeta_2 \in \dot{L}$, $a_1, a_2 \in K$, then

$$(a_1\zeta_1+a_2\zeta_2)(\rho)=a_1\cdot\zeta_1(\rho)+a_2\cdot\zeta_2(\rho),$$

$$[\zeta_1, \zeta_2](\rho)=[\zeta_1(\rho), \zeta_2(\rho)]=\zeta_1(\rho)\zeta_2(\rho)-\zeta_2(\rho)\zeta_1(\rho).$$

Harish-Chandra has proved the following result analogous to Tannaka duality theorem: "If K is algebraically closed and $\chi(K)=0$, and if L is semi-simple, then L is isomorphic with $\dot{L}$ under the mapping $X\rightarrow\zeta_X (X\in L)$ defined as follows: $\zeta_X(\rho)=\rho(X)$ for every $\rho\in R_0$". However, if L is not semi-simple, the problem to determine the structure of $\dot{L}$ from that of L seems to be difficult. In this note we shall treat this problem in the simplest case, namely in case where L is a one-dimensional Lie algebra over K. We shall solve it completely, when K is algebraically closed (Theorem 1). It will turn out that $\dot{L}$ is an infinite dimensional abelian Lie algebra (Corollary to Theorem 1). Incidentally we shall obtain a characterization of the "replica" of matrices introduced by C. Chevalley [2] (Theorem 2). From now on, let L be a one-dimensional Lie algebra over K. Let X be a base of L over K. Then the set $\mathfrak{R}_0$ of all representations of L can be identified with $\mathfrak{R}$ by the one-to-one correspondence $\rho\leftrightarrows\rho(X)$ $(\rho\in\mathfrak{R}_0)$. Obviously, this correspondence preserves $d(\rho)$, $\dot{+}$, $\oplus$ and transforms. Thus, every element in $\dot{L}$ can be defined as a mapping (or an operator) of $\mathfrak{R}$ into $\mathfrak{R}$ satisfying the following conditions.

I. $d(\zeta(A))=d(A)$ for every A in $\mathfrak{R}$.

II. $\zeta(TAT^{-1})=T\zeta(A)T^{-1}$ for every A in $\mathfrak{R}$ and for every T in $GL(K, d(A))$.

III. $\zeta(A\dot{+}B)=\zeta(A)\dot{+}\zeta(B)$ for every A, B in $\mathfrak{R}$.

IV$_1$. $\zeta(A\oplus B)=\zeta(A)\oplus\zeta(B)$ for every A, B in $\mathfrak{R}$.

We call such an operator a *sum-sum* (abbr. s-s) *operator*. Replacing the last condition by one of the following ones, we define three other kinds of operators.

IV_2. $\zeta(A\oplus B)=\zeta(A)\otimes\zeta(B)$ for every A, B in $\mathfrak{R}$ (*sum-product* (s-p) *operator.*)

IV_3. $\zeta(A\otimes B)=\zeta(A)\oplus\zeta(B)$ for every A, B in $\mathfrak{R}$ (*product-sum* (p-s) *operator.*)

IV_4. $\zeta(A\otimes B)=\zeta(A)\otimes\zeta(B)$ for every A, B in $\mathfrak{R}$ (*product-product* (p-p) *operator.*)

The determination of p-p operators means to determine the dual of dual in the sense of Tannaka of the infinite cyclic group. We shall show that an analogous method to the one used in § 2 to determine s-s operators allows us also to determine s-p, p-s and p-p operator. (§ 3, Theorem 3–8)

The writer is grateful to Prof. S. Iyanaga for his suggestions and remarks during the preparation of this note.

1. Preliminaries.

In this section we shall prove some lemmas which we shall need later. In what follows, K is supposed as algebraically closed (except in Appendix)

LEMMA 1. *For every matrix A in $\mathfrak{gl}(K, n)$ there exist two matrices S, N in $\mathfrak{gl}(K, n)$ such that*

$$A=S+N, \qquad SN=NS,$$

S: a semi-simple matrix[1] *(or an s-matrix),*

N: a nilpotent matrix (or an n-matrix).

S and N are determined by A uniquely, and can be expressed as polynomials in A without constant terms.

PROOF. Though this is a well known fact, we shall give here a proof which is valid whenever A has only separable eigen-values over K.

Let $\mathfrak{A}$ be the associative subalgebra of $\mathfrak{gl}(K, n)$ generated by A. By Wedderburn's theorem $\mathfrak{A}$ can be decomposed into the direct sum

1) A matrix is called semi-simple if its minimum polynomial has only simple roots.

of the radical $\mathfrak{N}$ and a semi-simple subalgebra $\mathfrak{K}$: $\mathfrak{A}=\mathfrak{N}+\mathfrak{K}$. Therefore A can be written as

$$A=S+N, \qquad S\in\mathfrak{K}, \qquad N\in\mathfrak{N}.$$

It is easily verified that $SN=NS$ and also that S, N are respectively s-matrix and n-matrix. S and N, being in $\mathfrak{A}$, can be expressed as polynomials in A without constant term.

Now let S_1 and N_1 be respectively s-matrix and n-matrix such that

$$A=S_1+N_1, \qquad S_1N_1=N_1S_1.$$

Then, S_1, N_1 are commutative with A. So they commute with S, N. Therefore $S-S_1$ and $N-N_1$ are respectively s-matrix and n-matrix. On the other hand

$$S-S_1=N_1-N,$$

so that we have $S=S_1$ and $N=N_1$.
We shall write $S=A^{(s)}$, $N=A^{(n)}$, and call them the semi-simple and the nilpotent part of A respectively (or the s-part and n-part of A).

In parallelism to Lemma 1, we have the following

LEMMA 2. *For every matrix A in $GL(K, n)$ there exist two matrix S, U in $GL(K,n)$ such that*

$$A=SU=US,$$

S: an s-matrix, U: a matrix of which all eigen-values are equal to 1[2] *(or an u-matrix),*

PROOF. Take S as $A^{(s)}$ and U as $AA^{(s)-1}$. Then S and U satisfy above conditions. Uniqueness is shown similarly as in Lemma 1.
We shall write $U=A^{(u)}$ and call it the u-part of A.

Now let $\mathfrak{X}$ be an arbitrary non-empty set in $\mathfrak{gl}(K, n)$. We denote the commutator algebra of $\mathfrak{X}$ in $\mathfrak{gl}(K, n)$ by $Z(\mathfrak{X})$: $Z(\mathfrak{X})=\{A;\ A\in\mathfrak{gl}(K,n),\ AX=XA$ for every $X\in\mathfrak{X}\}$, then we have

LEMMA 3. *If we put $\mathfrak{X}_1=GL(K,n)\frown Z(\mathfrak{X})$, then*

$$Z(\mathfrak{X}_1)=Z(Z(\mathfrak{X})).$$

PROOF. Obviously we have $Z(\mathfrak{X}_1)\supset Z(Z(\mathfrak{X}))$. Now let C be a matrix belonging to $Z(\mathfrak{X}_1)$. Let $A_1,\cdots,A_r$ $(A_1=I_n)$ be a base of $Z(\mathfrak{X})$ over K,

2) A matrix U is a u-matrix if and only if $U-I$ is an n-matrix. $(n=d(U))$.

and let $\xi_1,\cdots,\xi_r$ be independent variables over K. We denote by $\psi_{ij}(\xi_1,\cdots,\xi_r)$ the (i,j) component of the matrix $C(\sum_{i=1}^{r}\xi_i A_i)-(\sum_{i=1}^{r}\xi_i A_i)C$, and by $\varphi(\xi_1,\cdots,\xi_r)$ the determinant of $\sum_{i=1}^{r}\xi_i A_i$. Suppose that $C\notin Z(Z(\mathfrak{T}))$. Then there exist $\lambda_1,\cdots,\lambda_r$ in K and indices i,j, such that $\psi_{ij}(\lambda_1,\cdots,\lambda_r)\neq 0$, so that we have $\psi_{ij}(\xi_1,\cdots,\xi_r)\neq 0$.

On the other hand, since $\varphi(1,0,\cdots,0)=\det. I_n\neq 0$, we have $\varphi(\xi_1,\cdots,\xi_r)\neq 0$. Now K being an infinite field, there exist $\mu_1,\cdots,\mu_r$ in K such that

$$\psi_{ij}(\mu_1,\cdots,\mu_r)\,\varphi(\mu_1,\cdots,\mu_r)\neq 0\,.$$

Then $B=\sum_{i=1}^{r}\mu_i A_i$ is in $\mathfrak{T}_1$ and $BC\neq CB$. This contradicts the fact $C\in Z(\mathfrak{T}_1)$.

REMARK. Lemmas 1–3 hold for any infinite perfect field K.

2. Determination of s-s operators.

Let ζ be an s-s operator from $\mathfrak{N}$ in $\mathfrak{N}$. From condition II, we see in particular that $TAT^{-1}=A$ implies $T\zeta(A)T^{-1}=\zeta(A)$.

In other words, $\zeta(A)\in Z(GL(K,n)\cap Z(A))$. Therefore by Lemma 3, we have $A\in Z(Z(A))$.

Now as is well-known,[3], $Z(Z(A))$ coincides with the set of all polynomials in A. So we have

$$\zeta(A)=\alpha_0 I_n+\alpha_1 A+\cdots+\alpha_n A^n \qquad (n=d(A))\,,$$

$$\alpha_i\in K \qquad (0\leqq i\leqq n)\,.$$

Here α_i may depend on A.

Now let N, M be two n-matrices of degree n, m respectively. Let r, s be their respective indices: $N^{r-1}\neq O_n$, $M^{s-1}\neq O_m$, $N^r=O_n$; $M^s=O_m$. Then we have

$$\zeta(N)=\sum_{i=0}^{r-1}\nu_i N^i\,, \qquad \zeta(M)=\sum_{i=0}^{s-1}\mu_i M^i$$

$$\zeta(N\dot{+}M)=\sum_{i=0}^{t-1}\lambda_i(N\dot{+}M)^i \qquad (t=\mathrm{Max}\,(r,s))\,,$$

3) See for example, Wedderburn, Lectures on matrices, p. 106. Theorem 2.

and from condition III follow identities

$$\sum_{i=0}^{r-1} \nu_i N^i = \sum_{i=0}^{t-1} \lambda_i N^i , \qquad \sum_{i=0}^{s-1} \mu_i M^i = \sum_{i=0}^{t-1} \lambda_i M^i .$$

From the linear independence of $I_n, N, \cdots, N^{r-1}$ and of $I_m, M, \cdots, M^{s-1}$, we have $\nu_i = \lambda_i = \mu_i \qquad (0 \leqq i \leqq \mathrm{Min.}(r, s)-1)$.

In other words, there exists a sequence $c_0, c_1, \cdots$, of elements in K, such that for any n-matrix N in $\Re$

$$\zeta(N) = \sum_{i=0}^{\infty} c_i N^i \qquad \text{(finite series!)}.$$

($c_0, c_1, \cdots$ depend only on the mapping ζ). As is easily seen, the choice of $c_0, c_1, \cdots$ is unique.

Now, putting

$$\zeta(\alpha I_1) = g(\alpha) I_1 ,$$

we define the mapping g from K to K. From III it follows that for a diagonal matrix $D = (\lambda_i \delta_{ij})$, we have

$$\zeta(D) = (g(\lambda_i) \delta_{ij}) .$$

We remark here that the determinations of $c_0, c_1, \cdots$ and of the mapping g are derived only from conditions I, II, III. The condition IV_1 will specify now the c_i's and g.

Now applying ζ on both sides of $\alpha I_1 \oplus \beta I_1 = (\alpha + \beta) I_1$, we obtain

$$g(\alpha + \beta) = g(\alpha) + g(\beta) .$$

So g is a homomorphism of the additive group of K into itself. Consequently if k_0 is the prime field of K, g is a k_0-linear mapping from K into itself.

Now, let us seek conditions which will characterize the sequence c_i. For the above n-matrices N, M, we have from IV_1, ($N \oplus M$ is also an n-matrix!)

$$\sum_{i=0}^{\infty} c_i (N \oplus M)^i = \Big(\sum_{i=0}^{\infty} c_i N^i \Big) \oplus \Big(\sum_{i=0}^{\infty} c_i M^i \Big) ,$$

that is

$$\sum_{i=0}^{\infty} c_i \sum_{k=0}^{i} \binom{i}{k} N^k \otimes M^{i-k} = \sum_{i=0}^{\infty} c_i \, (N^i \otimes I_m + I_n \otimes M^i) .$$

From the linear independence of $N^i \otimes M^j$ $(0 \leqq i \leqq r-1,\ 0 \leqq j \leqq s-1)$.

we have $c_0=0$, $c_i\binom{i}{k}=0$, $(2\leqq i\leqq \mathrm{Min.}(r, s)-1,\ 1\leqq k\leqq i-1)$.
If $\chi(K)=0$, then all c_i's except c_1, are zero, and we have

$$\zeta(N)=cN \qquad (c=c_1)\,.$$

To treat the case of $\chi(K)=p$, we prove the following

LEMMA 4. *Let p be a prime number and l a positive integer. Then*

i) *the greatest common divisor of* $\binom{l}{1}, \binom{l}{2},\cdots,\binom{l}{l-1}$ *is*

$$\begin{cases} 1, \text{ if } l \text{ is not a power of a prime number,} \\ p, \text{ if } l=p^e\,. \end{cases}$$

Let i, j, t be three integers such that $0\leqq i\leqq j$, $0\leqq t\leqq ip$. Then

ii) $$\frac{(jp+t)!}{t!(jp-ip+t)!\,(ip-t)!} \equiv \begin{cases} 0, \text{ mod. } p, \text{ if } t \text{ is not a multiple of } p, \\ \dfrac{(j+t')!}{t'!\,(j-i+t')!\,(i-t')!}\,, \text{ mod. } p, \text{ if } t=t'p\,. \end{cases}$$

PROOF. i) If l is not a power of a prime number q, then let β_f be the first non-vanishing coefficient in the q-adic expression of $l=\beta_0+\beta_1 q+\cdots+\beta_r q^r$ $(0\leqq \beta_i\leqq q-1,\ 0\leqq i\leqq r)$. Then we have $f<r$ or $f=r$, $\beta_f>1$. On the other hand, as is easily seen

$$\binom{qx}{qy}\equiv\binom{x}{y} \text{ mod. } q\,.$$

So we have $\binom{l}{q^f}\equiv\binom{\beta_f+\cdots+\beta_r q^{r-f}}{1}$, mod. q and $1\leqq q^f\leqq l-1$. Therefore we get the first part of i).

Next if $l=p^e(e\geqq 1)$, then let i be an arbitrary integer such that $1\geqq i\geqq p-1$. The p-exponent of $p^e!$ is then given by

$$\sum_{\nu=1}^{\infty}\left[\frac{p^e}{p^\nu}\right]=p^{e-1}+\cdots+1=\frac{p^e-1}{p-1}\,.$$

Similarly the p-exponents of $(ip^{e-1})!$ and of $\big((p-i)p^{e-1}\big)!$ are given by $i\,\dfrac{p^{e-1}-1}{p-1}$, $(p-i)\,\dfrac{p^{e-1}-1}{p-1}$ respectively. So the p-exponent of $\binom{p^e}{ip^{e-1}}$ is equal to 1, and the G.C.M. of $\binom{l}{1}, \binom{l}{2},\cdots,\binom{l}{l-1}$ has also the same

p-exponent 1. As was shown above, the G. C. M. of $\binom{l}{1},\cdots,\binom{l}{l-1}$ cannot be divided by any other prime number. Therefore we get the second part of i).

ii) Let $t=t'p+q$, $0<q<p$. Then

$$(jp+t)!=p^{j+t'}(j+t')!\,\alpha_1,\quad t!=p^{t'}t'!\,\alpha_2,$$

$$(jp-ip+t)!=p^{j-i+t'}(j-i+t')!\,\alpha_3,\quad (ip-t)!=p^{i-t'-1}(i-t'-1)!\,\alpha_4,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are integers such that $\alpha_i \not\equiv 0 \mod. p.$ $(1\leqq i\leqq 4)$. so we have

$$\frac{(jp+t)!}{t!\,(jp-ip+t)!(ip-t)!}=p\frac{(j+t')!}{t'!(j-i+t')!\,(i-t'-1)!}\,\frac{\alpha_1}{\alpha_2\alpha_3\alpha_4}\equiv 0 \pmod{p}.$$

Next let $t=t'p$, then by similar calculation as above,

$$\frac{(jp+t)!}{t!(jp-ip+t)!\,(ip-t)!}\equiv\frac{(j+t')!}{t'!\,(j-i+t')!\,(i-t')!} \pmod{p}.$$

Now let us return to the determination of s-s operator in case $\chi(K)=p$. By Lemma 4, (i) and $c_i\binom{i}{k}=0$ $(1\leqq k\leqq i-1)$, we have

$$c_i=0 \text{ (if } i \text{ is not a power of } p),$$

$$\zeta(N)=c_1N+c_pN^p+c_{p^2}N^{p^2}+\cdots \qquad (N:\ n\text{-matrix}).$$

Now let A be any matrix in $\mathfrak{gl}(K,n)$. Transform A by a suitable matrix T into Jordan's normal form:

$$TAT^{-1}=\sum_{i=1}^{\nu}\dot{+}(\alpha_i\,Id_i+N_i),\quad d_i=d(N_i),\quad N_i:\ n\text{-matrix}.$$

Since $\alpha I_d+N=\alpha I_1\oplus N$, we have

$$T\,\zeta(A)T^{-1}=\sum_{i=1}^{r}\dot{+}\big(g(\alpha_i)I_1\oplus\zeta(N_i)\big)$$

$$=\sum_{i=1}^{r}\dot{+}\big(g(\alpha_i)I_{d_i}+\zeta(N_i)\big).$$

On the other hand, we have $TA^{(s)}T^{-1}=\sum_{i=1}^{r}\dot{+}\ \alpha_i\,I_{d_i}$, $TA^{(n)}T^{-1}=\sum_{i=1}^{r}\dot{+}N_i$, as is easily seen. Therefore we have

$$T\zeta(A)T^{-1}=T\zeta(A^{(s)})T^{-1}+T\zeta(A^{(n)})T^{-1},$$

that is

$$\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(n)}).$$

We shall call (C) the condition

$$\text{(C)}\begin{cases} c_i=0 \quad \text{for all} \quad i\neq 1 \quad \text{in case} \quad \chi(K)=0, \\ c_i=0 \quad \text{for all} \quad i\neq p^{\nu} \quad \text{in case} \quad \chi(K)=p \end{cases}$$

for the sequence $c_0, c_1, \cdots$ of elements in K, and any such sequence satisfying this condition a C-sequence.

We have seen that for any s-s operator ζ, there correspond an endomorphism g of the additive group of K, and a C-sequence $c_0, c_1, \cdots$, which in turn determine ζ uniquely. We shall say that ζ has as its invariants g and the C-sequence $c_0, c_1, \cdots$.

Conversely, let g be any endomorphism of the additive group of K, and $c_0, c_1, \cdots$ be any C-sequence. Let us show that there exists an s-s operator ζ which has g and $c_0, c_1, \cdots$ as its invariants.

First let S be any s-matrix of degree n. We transform S into the diagonal form

$$TST^{-1}=(\alpha_i\,\delta_{ij}),$$

and then we define

$$\zeta(S)=T^{-1}\big(g(\alpha_i)\,\delta_{ij}\big)\,T.$$

Now we must show that $\zeta(S)$ is thus well defined. Let T_1 be a matrix such that

$$T_1ST_1^{-1}=(\alpha_{p_i}\,\delta_{ij}),$$

where $(p_1, \cdots, p_n)$ is a permutation of $(1, \cdots, n)$. Then we must show that

$$T^{-1}\big(g(\alpha_i)\delta_{ij}\big)T=T_1^{-1}\big(g(\alpha_{p_i})\delta_{ij}\big)T_1.$$

To show this, take a permutation matrix P such that

$$(\alpha_{p_i}\,\delta_{ij})=P(\alpha_i\,\delta_{ij})P^{-1}.$$

Then we have

$$S=T^{-1}(\alpha_i\,\delta_{ij})T=T_1^{-1}P(\alpha_i\,\delta_{ij})P^{-1}T_1,$$

that is, $TT_1^{-1}P$ commutes with $(\alpha_i \delta_{ij})$. On the other hand, as $\big(g(\alpha_i)\delta_{ij}\big)$ is a polynomial in $(\alpha_i \delta_{ij})$, $TT_1^{-1}P$ commutes with $\big(g(\alpha_i)\delta_{ij}\big)$:

$$T^{-1}\big(g(\alpha_i)\delta_{ij}\big)T = T_1^{-1}P\big(g(\alpha_i)\delta_{ij}\big)P^{-1}T_1 = T_1^{-1}\big(g(\alpha_{p_i})\delta_{ij}\big)T_1 .$$

This is what we had to show.

Next we define for any n-matrix N,

$$\zeta(N) = \sum_{i=0}^{\infty} c_i N^i ,$$

and for any matrix A, we define

$$\zeta(A) = \zeta(A^{(s)}) + \zeta(A^{(n)}) .$$

Now let us show that the mapping ζ defined above satisfies the conditions I–IV$_1$. I is obvious. II follows immediately for s-matrix and n-matrix from the definition. For general matrices it follows from the fact that $(TAT^{-1})^{(s)} = TA^{(s)}T^{-1}$, $(TAT^{-1})^{(n)} = TA^{(n)}T^{-1}$ and the definition of ζ. III follows immediately if A and B are both s-matrices or both n-matrices. For general case, we have

$$\zeta(A \dot{+} B) = \zeta(A^{(s)} \dot{+} B^{(s)}) + \zeta(A^{(n)} \dot{+} B^{(n)}) = \{\zeta(A^{(s)}) \dot{+} \zeta(B^{(s)})\}$$

$$+ \{\zeta(A^{(n)} \dot{+} \zeta(B^{(n)})\} = \zeta(A) \dot{+} \zeta(B) .$$

To show IV$_1$, remark that $(A \oplus B)^{(s)} = A^{(s)} \oplus B^{(s)}$, $(A \oplus B)^{(n)} = A^{(n)} \oplus B^{(n)}$. So we have only to show IV$_1$, under the assumption that A, B are both s-matrices or both n-matrices.

Let A, B be both s-matrices. Choose matrices T_1, T_2 so that $T_1AT_1^{-1} = (\alpha_i \delta_{ij})$, $T_2BT_2^{-1} = (\beta_i \delta_{ij})$, and put $T_3 = T_1 \otimes T_2$, then we have

$$T_3(A \oplus B)T_3^{-1} = \sum_{i,j} \dot{+} (\alpha_i I_1 \oplus \beta_j I_1) = \sum_{i,j} \dot{+} (\alpha_i + \beta_j) I_1 .$$

Now from the additiveness of g and from the definition of ζ, we have

$$T_3\zeta(A \oplus B)T_3^{-1} = \sum_{i,j} \dot{+} \big(g(\alpha_i) + g(\beta_j)\big)I_1 = \big(g(\alpha_i)\delta_{ij}\big) \oplus \big(g(\beta_i)\delta_{ij}\big)$$

$$= \big(T_1 \zeta(A) T_1^{-1}\big) \oplus \big(T_2 \zeta(B) T_2^{-1}\big) = T_3\big(\zeta(A) \oplus \zeta(B)\big)T_3^{-1} .$$

Next let A, B be both n-matrices. Then we have

$$\zeta(A \oplus B) = \sum_{i=0}^{\infty} c_i (A \otimes I_m + I_n \otimes B)^i = \zeta(A) \oplus \zeta(B) .$$

Thus we have proved the following

THEOREM 1. *Let K be any algebraically closed field. For any s-s operator ζ from $\mathfrak{R}(K)$ into $\mathfrak{R}(K)$, there correspond uniquely an endomorphism g of the additive group K and a* C*-sequence $c_0, c_1, \cdots$ which we have called the invariants of ζ.*
They are connected with ζ as follows:

$$\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(n)}),$$

where

$$\zeta(A^{(s)})=T^{-1}\big(g(\alpha_i)\delta_{ij}\big)T \qquad \textit{with} \quad (\alpha_i\,\delta_{ij})=TA^{(s)}T^{-1},$$

$$\zeta(A^{(n)})=\sum_{i=0}^{\infty} c_i\, A^{(n)i}.$$

Conversely, for any endomorphism g of the additive group K and for any C*-sequence $c_0, c_1, \cdots$ there is one and only one s-s operator having them as invariants.*

COROLLARY. *Let L be a 1-dimensional Lie algebra over an algebraically closed field K. Then $\dot{L}$ is an infinite dimensional abelian Lie algebra over K.*

PROOF. As was shown in the introduction, $\dot{L}$ is isomorphic to the Lie algebra consisting of s-s operators. If ζ_1, ζ_2 are any two s-s operators, we have $[\zeta_1(A), \zeta_2(A)]=0$ for every A in $\mathfrak{R}$ since $\zeta_i(A)$ is a polynomial in A, $i=1,2$. Thus, $\dot{L}$ is abelian. Now the set F of all endomorphism of the additive group K becomes a linear space over K in the natural way. As can be seen easily, $\dim F/K=\infty$. From this, we can conclude that $\dot{L}$ is infinite dimensional over K, q. e. d.

Now we give here some properties of s-s operators:

THEOREM 2. *Let ζ be any s-s operator from $\mathfrak{R}$ into $\mathfrak{R}$. Then:*

α) *If $AB=BA$, then $\zeta(A+B)=\zeta(A)+\zeta(B)$.*

β) *$\zeta(-{}^tA)=-{}^t\zeta(A)$, where ${}^tA$ denotes the transposed matrix of A.*

γ) *A matrix B is a replica*[4] *of a matrix A if and only if there exists an s-s operator ζ such that $\zeta(A)=B$.*

PROOF. α) From $AB=BA$ follows easily that $(A+B)^{(s)}=A^{(s)}+B^{(s)}$, $(A+B)^{(n)}=A^{(n)}+B^{(n)}$, and that the four matrices $A^{(s)}$, $A^{(n)}$, $B^{(s)}$, $B^{(n)}$ commute with each other. Consequently there is a matrix T such that $TA^{(s)}T^{-1}=(\alpha_i\,\delta_{ij})$, $TB^{(s)}T^{-1}=(\beta_i\,\delta_{ij})$, and we have

4) Cf. C. Chevalley [2].

$$\begin{aligned}\zeta(A+B)&=\zeta(A^{(s)}+B^{(s)})+\zeta(A^{(n)}+B^{(n)})\\&=T^{-1}\big(g(\alpha_i+\beta_i)\delta_{ij}\big)T+\zeta(A^{(n)}+B^{(n)})\\&=\zeta(A^{(s)})+\zeta(B^{(s)})+\zeta(A^{(n)}+B^{(n)})\,.\end{aligned}$$

On the other hand, we have by the property of the C-sequence $c_0, c_1, \cdots$,

$$\zeta(A^{(n)}+B^{(n)})=\sum_{i=0}^{\infty} c_i(A^{(n)}+B^{(n)})^i=\sum_{i=0}^{\infty} c_i\, A^{(n)i}+\sum_{i=0}^{\infty} c_i\, B^{(n)i}\,.$$

Thus, we have

$$\zeta(A+B)=\zeta(A)+\zeta(B)\,.$$

β) From α) and $\zeta(O_n)=O_n$, we have $\zeta(-A)=-\zeta(A)$. Now, as ${}^tA$ and A have the same elementary divisors, there is a matrix T such that $TAT^{-1}={}^tA$. On the other hand $\zeta(A)$ is a polynomial in A:

$$\zeta(A)=\sum_{i=0}^{n}\alpha_i\, A^i \qquad \big(n=d(A)\big)\,,$$

so we have

$$\zeta({}^tA)=\zeta(TAT^{-1})=T\zeta(A)T^{-1}=\sum_{i=0}^{n}\alpha_i(TAT^{-1})^i=\sum_{i=0}^{n}\alpha_i\, {}^tA^i={}^t\zeta(A)\,.$$

Thus we have

$$\zeta(-{}^tA)=-\zeta({}^tA)=-{}^t\zeta(A)\,.$$

γ) Let $B=\zeta(A)$. Take a matrix T such that $TA^{(s)}T^{-1}=(\alpha_i\,\delta_{ij})$. Let g and $c_0, c_1, \cdots$ be the invariants of ζ. Then we have by Theorem 1,

$$TB^{(s)}T^{-1}=\big(g(\alpha_i)\,\delta_{ij}\big)\,,$$

$$B^{(n)}=\sum_{i=0}^{\infty} c_i\, A^{(n)i}\,.$$

Now, as g is an endomorphism of the additive group K, it follows that for any integers $m_1, \cdots, m_n$ $\big(n=d(A)\big)$ such that $\sum_{i=1}^{n}\alpha_i\, m_i=0$, we have $\sum_{i=1}^{n} m_i\, g(\alpha_i)=0$. From this we can conclude easily that $B^{(s)}$ is a replica of $A^{(s)}$ [5]. By the above formula for $B^{(n)}$, and the property of $c_0, c_1 \cdots$, $B^{(n)}$ is a replica of $A^{(n)}$. So it follows that[5] B is a replica of A.

Conversely, let B be a replica of A. Take a matrix T such that $TA^{(s)}T^{-1}=(\alpha_i\,\delta_{ij})$. As B is a polynomial in A[5], we have then $TB^{(s)}T^{-1}=(\beta_i\,\delta_{ij})$. As is known,[5] any linear relation between the α_i's with

5) See § 4.

integral coeffients, $\sum_{i=0}^{n} m_i \alpha_i = 0$, holds also for the β_i: $\sum_{i=1}^{n} m_i \beta_i = 0$, so there is a k_0-linear mapping g' from the k_0-module generated by $\alpha_1, \cdots, \alpha_n$ into the k_0-module generated by $\beta_1, \cdots, \beta_n$ such that $g(\alpha_i) = \beta_i$ $(1 \leqslant i \leqslant n)$. Then we can extend g' to a k_0-linear mapping g from the k_0-module K into itself.

Next, as is known,[5] there exsists a C-sequence $c_0, c_1, \cdots$ in K such that

$$B^{(n)} = \sum_{i=0}^{\infty} c_i A^{(n)i},$$

and that only a finite number of the c_i's are non-vanishing. We now construct an s-s operator ζ having g and $c_0, c_1 \cdots$ as invariants. Then as can be seen easily, we have $\zeta(A) = B$.

REMARK. If B is a replica of A, there are infinitely many s. s. operators ζ such that $\zeta(A) = B$.

3. Determination of s-p, p-s and p-p operators.

For s-p, p-s and p-p operators almost the same discussion as in §2 applies. First, for given ζ satisfying also I, II, III, we define elements c_i $(0 \leqslant i < \infty)$ in K and a mapping from K into K by the formulas:

$$\zeta(N) = \sum_{i=0}^{\infty} c_i N^i \qquad \text{(for any } n\text{-matrix } N \text{ in } R),$$

$$\zeta(\alpha I_1) = g(\alpha) I_1 \qquad \text{(for any element } \alpha \text{ in } K).$$

Now, let ζ be an s. p. operator, then condition IV_2 implies as in §2 that

$$c_i c_j = \binom{i+j}{i} c_{i+j} \qquad (0 \leqslant i, j < \infty),$$

$$g(\alpha + \beta) = g(\alpha) g(\beta) \qquad \text{(for every } \alpha, \beta \text{ in } K).$$

Then a simple calculation shows that

$$\begin{array}{llll} \chi(K)=0: & c_i = 0 \ (0 \leqslant i < \infty) & \text{that is,} & \zeta(N) = O_n \quad \big(n = d(N)\big), \\ & \text{or } c_0 = 1,\ c_i = c_1^i / i! & \text{that is,} & \zeta(N) = \exp c_1 N, \\ \chi(K)=p: & c_i = 0 \ (0 \leqslant i < \infty) & \text{that is,} & \zeta(N) = O_n \quad \big(n = d(N)\big), \\ & \text{or } c_0 = 1,\ c_i = 0 \ (i \geqslant 1) & \text{that is,} & \zeta(N) = I_n \quad \big(n = d(N)\big). \end{array}$$

Next, consider the mapping g. From the above formula we have $g(\alpha)=0$ (for every α in K) or $g(\alpha)\neq 0$ (for every α in K). In the latter case, g is a homomorphism of the additive group K into the multiplicative group K^* of K. However, if $\chi(K)=p$, we have for every α in K,

$$g(\alpha)^p=g(p\alpha)=1\,, \qquad \text{so that} \qquad g(\alpha)=1\,.$$

Thus we have the following theorem by a similar discussion as in Theorem 1.

THEOREM 3. *Let ζ be an s-p operator from $\mathfrak{R}$ into $\mathfrak{R}$. Then we have*

i) $\chi(K)=p$: $\zeta(A)=O_n$ *for every matrix A in $\mathfrak{R}$, $n=d(A)$,*
or $\zeta(A)=I_n$ *for every matrix A in $\mathfrak{R}$, $n=d(A)$,*
$\chi(K)=0$: $\zeta(A)=O_n$ *for every matrix A in $\mathfrak{R}$, $n=d(A)$,*

or ii) *ζ has as invariants a homomorphism g from K into K^* and an element c in K. They are connected with ζ as follows:*

$$\zeta(A)=\zeta(A^{(s)})\,\zeta(A^{(n)}) \qquad \textit{for every } A \textit{ in } R,$$

where

$$\zeta(A^{(s)})=T\big(g(\alpha_i)\,\delta_{ij}\big)T^{-1} \qquad \textit{with } (\alpha_i\,\delta_{ij})=TA^{(s)}T^{-1}$$

$$\zeta(A^{(n)})=\exp cA^{(n)}\,.$$

Conversely, for every homomorphism g from K into K^* and for every element c in K, there is one and only one s-p operator from $\mathfrak{R}$ into $\mathfrak{R}$ having them as invariants.

The s-p operator ζ, $\zeta(A)=O_n$ (for every A in $\mathfrak{R}$, $n=d(A)$) is called *singular.* Other s-p operators will be called non-singular, i.e. those which map $\mathfrak{R}$ into $\mathfrak{S}$.

THEOREM 4. *An s-p operator ζ has the following properties:*

α) *If $AB=BA$, then $\zeta(A+B)=\zeta(A)\zeta(B)$.*

β) *If ζ is non-singular, then $\zeta(-{}^tA)={}^t\zeta(A)^{-1}$.*

These are proved as in the proof of Theorem 2. (We shall discuss on an analogy of γ) in the next section.)

Now, let ζ be a p-s operator, from $\mathfrak{R}$ into $\mathfrak{R}$. Then condition IV_3 gives as in § 2 that

$c_i=0$ $(0\leqslant i<\infty)$, that is, $\zeta(N)=O_n$ for every n-matrix N in $\mathfrak{R}$, $n=d(N)$.

$$g(\alpha\beta)=g(\alpha)+g(\beta) \qquad (\text{for every } \alpha,\ \beta \text{ in } K)\,.$$

In particular, we have $g(0)=g(1)=0$. Furthermore, if $\chi(K)=p$, then every element α in algebraically closed field K can be written as $\alpha=\gamma^p$, so we have

$$g(\alpha)=pg(\gamma)=0\,.$$

Now let A be any matrix of degree n in $\mathfrak{R}$ and N be any n-matrix of degree m in $\mathfrak{R}$. Then, $N\otimes A$ being an n-matrix, we have

$$O_{mn}=\zeta(N\otimes A)=\zeta(N)\oplus\zeta(A)=O_{mn}+I_m\otimes\zeta(A)\,.$$

Hence we have

$$\zeta(A)=O_n\,.$$

This shows that every p-s operator ζ from $\mathfrak{R}$ into $\mathfrak{R}$ is a trivial one: $\zeta(A)=O_n$ (for every A in $\mathfrak{R}$). So we shall consider p-s operators from $\mathfrak{S}=\bigcup_{n=1}^{\infty} GL(K,n)$ into $\mathfrak{R}$. Let ζ be such an operator. For every n-matrix N of degree n, we define $\bar{\zeta}$ as

$$\bar{\zeta}(N)=\zeta(I_n+N)\,.$$

Then $\bar{\zeta}$ is a mapping defined on the set of all n-matrices in $\mathfrak{R}$ with values in $\mathfrak{R}$, and as is seen easily, $\bar{\zeta}$ satisfies the conditions I, II, III in §1. Then $\bar{\zeta}$ determines the elements $d_i(0\leqslant i<\infty)$ in K such that

$$\bar{\zeta}(N)=\sum_{i=0}^{\infty} d_i N^i \qquad \text{(for every n-matrix N in $\mathfrak{R}$)}\,.$$

Now, as ζ satisfies IV_3, we have for any n-matrix N and M,

$$\bar{\zeta}(N\otimes I_m+I_n\otimes M+N\otimes M)=\bar{\zeta}(N)\oplus\bar{\zeta}(M) \qquad \big(n=d(N),\ m=d(M)\big)\,,$$

from which we have

$$\sum_{0\leq i<j<\infty}\left\{\sum_{t=0}^{i}\varDelta_{ijt}\,d_{j+t}\right\}(N^i\otimes M^j+N^j\otimes M^i)+\sum_{i=0}^{\infty}\left\{\sum_{t=0}^{i}\varDelta_{iit}\,d_{i+t}\right\}(N^i\otimes M^i)$$
$$=\sum_{i=0}^{\infty} d_i\,(N^i\otimes I_m+I_n\otimes M^i)\,,$$

where

$$\varDelta_{ijt}=(j+t)!/t!\,(j-i+t)!\,(i-t)!\,.$$

Comparing the coefficients of $N^i\otimes M^i$ in both sides of the equality, we have (since the indices of N and M can be preassigned to be any positive integer)

$$d_0=0\,,$$

and also that

$$\sum_{t=0}^{i} \Delta_{ijt}\, d_{j+t}=0 \qquad (1\leqslant i\leqslant j<\infty)\,.$$

In particular, putting $i=1$, we obtain

$$jd_j+(j+1)\,d_{j+1}=0 \qquad (1\leqslant j<\infty)\,.$$

In case $\chi(K)=0$, we have

$$d_j=(-1)^{j+1}\, d_1/j \qquad (1\leqslant j<\infty)\,,$$

and

$$\zeta(I_n+N)=d_1 \log(I_n+N)\,.$$[6]

In case $\chi(K)=p$, we have

$$d_i=0\,, \quad \text{if} \quad j\not\equiv 0 \bmod. p\,.$$

Hence we have

$$\sum_{i=1}^{\infty} d_{ip}(N\otimes I_m+I_n\otimes M+N\otimes M)^{ip}=\sum_{i=1}^{\infty} d_{ip}(N^{ip}\otimes I_m+I_n\otimes M^{ip})\,.$$

Therefore, putting $d_{ip}=e_i$ $(i=1,2,\cdots)$,

$$\sum_{i=1}^{\infty} e_i\,(N^p\otimes I_m+I_n\otimes M^p+N^p\otimes M^p)^i=\sum_{i=1}^{\infty} e_i\,(N^{pi}\otimes I_m+I_n\otimes M^{pi})\,.$$

Thus we have as above

$$\sum_{t=0}^{i} \Delta_{ijt}\, e_{j+t}=0 \qquad (1\leqslant i\leqslant j<\infty)\,.$$

Then, as above, we obtain

$$e_j=0\,, \quad \text{if} \quad j\not\equiv 0 \bmod. p\,.$$

Proceeding similarly, we have

$$d_i=0 \qquad (i=0,1,2,\cdots)\,.$$

Now, for any matrix A in $\mathfrak{S}$, we have $\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(u)})$ as in § 2. Thus, we have the following

THEOREM 5. i) *Let ζ be a p-s operator from $\mathfrak{R}$ into $\mathfrak{R}$. Then*

$$\zeta(A)=O_n \text{ for every } A \text{ in } \mathfrak{R},\ n=d(A)\,.$$

6) If N is an n-matrix of degree n, then $\log(I_n+N)$ is defined as

$$\log(I_n+N)=\sum_{i=1}^{\infty}(-1)^i\,\frac{N^i}{i} \qquad \text{(finite series)}.$$

ii) *Let ζ be a p-s operator from $\mathfrak{S}$ into $\mathfrak{N}$. Then,*
In case $\chi(K)=p$: $\zeta(A)=O_n$ for every A in $\mathfrak{N}$, $n=d(A)$.
In case $\chi(K)=0$: ζ has as invariants a homomorphism g from the multiplicative group K^* into the additive group K and an element d in K. They are connected with ζ as follows:

$$\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(u)}) \text{ for every } A \text{ in } \mathfrak{S},$$

where $$\zeta(A^{(s)})=T\big(g(\alpha_i)\delta_{ij}\big)T^{-1} \quad \text{with } (\alpha_i\,\delta_{ij})=TA^{(s)}T^{-1},$$

$$\zeta(A^{(u)})=d\log A^{(u)}.$$

Conversely, for any homomorphism from K^* into K and an element d in K, there is one and only one p-s operator from $\mathfrak{S}$ into $\mathfrak{N}$ having them as invariants.

THEOREM 6. *Let ζ be a p-s operator from $\mathfrak{S}$ into $\mathfrak{N}$. Then,*
α) *if $AB=BA$ then $\zeta(AB)=\zeta(A)+\zeta(B)$.*
β) *$\zeta({}^tA^{-1})=-{}^t\zeta(A)$ for every matrix A in $\mathfrak{S}$.*

Proof is almost the same as that of Theorem 2.

Next, let us consider p-p operators from $\mathfrak{N}$ into $\mathfrak{N}$. The condition IV_4 implies as in § 2 that

$$g(\alpha\beta)=g(\alpha)g(\beta) \quad \text{for every} \quad \alpha,\beta \text{ in } K,$$

$$c_i^2=c_i\,, \qquad c_ic_j=0 \qquad (i\neq j) \qquad (0<i,\, j<\infty)\,.$$

Thus we have $c_i=0$ $(0\leqslant i<\infty)$ or $c_i=1$ for some i and all other c_j's are zero. We are thus in one of the following two cases:
Case A) $\zeta(N)=O_n$ for every n-matrix N in $\mathfrak{N}$, $n=d(N)$.
Case B) $\zeta(N)=N^i$ for every n-matrix N in $\mathfrak{N}$.
Ad case A). For every matrix A in $\mathfrak{N}$ there are matrices T, N, A_0 such that

$$A=T(N+A_0)T^{-1},$$

N: an n-matrix, A_0: a non-singular matrix.
(Consider for example Jordan's normal form of A.) N and A_0 are uniquely determined by A upto similar matrices. Then we have

$$\zeta(A)=T\big(\zeta(N)\dot{+}\zeta(A_0)\big)T^{-1} \qquad \big(m=d(N)\big)\,.$$

Accordingly ζ is determined completely by its contraction on $\mathfrak{S}$.

Conversely, let ζ' be any p-p operator from $\mathfrak{S}$ into $\mathfrak{N}$. Define ζ

for A in $\mathfrak{N}$ by $\zeta(A)=T\big(O_m\dot{+}\zeta(A_0)\big)T^{-1}$, where A is decomposed as above: $A=T(N\dot{+}A_0)T^{-1}$. Then we may verify as in the proof of Theorem 1 that ζ is uniquely defined and satisfies the conditions I–III and IV_4. Thus for case A) our problem is reduced to determine p-p operators from $\mathfrak{S}$ into $\mathfrak{N}$.

Ad case B). Take an n-matrix N such that $N^i \neq O_m$ $\big(m=d(N)\big)$. Then for any matrix A in $\mathfrak{N}$, we have

$$\zeta(N\otimes A)=(N\otimes A)^i=\zeta(N)\otimes\zeta(A)$$

or

$$N^i\otimes A^i=N^i\otimes\zeta(A).$$

So we have

$$\zeta(A)=A^i, \qquad g(\alpha)=\alpha^i \quad (\text{for every } \alpha \text{ in } K).$$

Now, returning to case A), let us consider a p-p operator ζ from $\mathfrak{S}$ into $\mathfrak{N}$. Define $\bar{\zeta}$ as

$$\bar{\zeta}(N)=\zeta(I_n+N) \qquad (N\text{: any } n\text{-matrix of degree } n).$$

Then as in the case of p-s operators, $\bar{\zeta}$ determines the elements d_i $(0\leqslant i<\infty)$ such that

$$\bar{\zeta}(N)=\sum_{i=0}^{\infty} d_iN^i \qquad (\text{for every } n\text{-matrix } N).$$

As ζ satisfies IV_4 we have for any n-matrices N and M,

$$\bar{\zeta}(N\otimes I_m+I_n\otimes M+N\otimes M)=\bar{\zeta}(N)\otimes\bar{\zeta}(M) \qquad \big(n=d(N),\ m=d(M)\big).$$

From this follows, as in the case of p-s operators,

$$(1) \qquad d_id_j=\sum_{t=0}^{i} \varDelta_{ijt}\, d_{j+t} \qquad (0\leqslant i\leqslant j<\infty).$$

Putting $i=0$, we have

$$d_0d_j=d_j \qquad (0\leqslant j<\infty).$$

Hence we have $d_0=1$ or $d_i=0$ $(0\leqslant i<\infty)$. In the latter case we have

$$\zeta(I_n+N)=O_n.$$

Accordingly,

$$g(1)=0,$$

and hence $g(\alpha)=0$ (for all α in K).

Then, by the formula $\zeta(A)=\zeta(A^{(s)})\zeta(A^{(u)})$, we have

$$\zeta(A)=O_n \quad \text{for all } A \text{ in } \mathfrak{S} \quad \big(n=d(A)\big).$$

Now let us suppose that $d_0=1$. Putting $i=1$ in (1) we have

$$(2) \qquad d_1 d_j = j\cdot d_j+(j+1)d_{j+1} \qquad (1\leqslant j<\infty).$$

Hence, if $\chi(K)=0$,

$$d_j=d_1(d_1-1)\cdots(d_1-j+1)/j!=\binom{d_1}{j},$$

$$\zeta(I_n+N)=I_n+\binom{d_1}{1}N+\binom{d_1}{2}N^2+\cdots.$$

Now put $\log(I_n+N)=M$. Then a simple calculation shows

$$\zeta(I_n+N)=\exp d_1 M.$$

Next let $\chi(K)=p$. From the above relations (2) we have

$$d_1\begin{pmatrix} d_{ip+1}\\ d_{ip+2}\\ \vdots\\ \vdots\\ d_{ip+(p-1)}\end{pmatrix}=\begin{pmatrix}1&2&0\cdots0\\0&2&3\cdots0\\0&0&3\cdots0\\ \cdots&\cdots&\cdots\\0&0&0\cdots p-1\\0&0&0\cdots p-1\end{pmatrix}\begin{pmatrix} d_{ip+1}\\ d_{ip+2}\\ \vdots\\ \vdots\\ d_{ip+(p-1)}\end{pmatrix} \qquad (0\leqslant i<\infty).$$

Hence, if $d_1\notin(1,2,\cdots,p-1)$,[7] then

$$d_k=0 \text{ for all } k,\quad k\not\equiv 0 \pmod{p},$$

and we have

$$\zeta(I_n+N)=\sum_{i=0}^{\infty} d_{ip}N^{ip}.$$

If $d_1\in(1,2,\cdots,p-1)$ we have (regarding d_1 as a positive integar)

$$d_{ip+1}=d_{ip}\binom{d_1}{1},\qquad d_{ip+2}=d_{ip}\binom{d_1}{2},\cdots,\qquad d_{ip+(p-1)}=d_{ip}\binom{d_1}{p-1}\ (0\leqslant i<\infty).$$

Hence it follows that

$$\zeta(I_n+N)=\left\{I_n+\binom{d_1}{1}N+\binom{d_1}{2}N^2+\cdots+\binom{d_1}{p-1}N^{p-1}\right\}\sum_{i=0}^{\infty} d_{ip}N^{ip}$$

$$=(I_n+N)^{d_1}\sum_{i=0}^{\infty} d_{ip}N^{ip}.$$

7) $(1,2,\cdots,p-1)$ means the set of non zero elements of the prime field of K.

Now let us define for any element x in K $(\chi(K)=p)$ and for any n-matrix U of degree n

$$U^x=\begin{cases} I_n & \text{if} \quad x \notin (1,2,\cdots,p-1) \\ \text{the power of } U \text{ where the exponent } x \text{ is regarded as a positive} \\ \text{integer if } x\in(1,2,\cdots,p-1). \end{cases}$$

Then the above result can be written in the form:

$$\zeta(I_n+N)=(I_n+N)^{d_1}\sum_{i=0}^{\infty} d_{ip}N^{ip}.$$

Now by Lemma 4, ii), $e_i=d_{ip}$ $(0\leqslant i<\infty)$ satisfy the relations (1), hence we have similarly as above

$$\sum_{i=0}^{\infty} d_{ip}N^i=(I_n+N^p)^{d_p}\sum_{i=0}^{\infty} d_{ip^2}N^{ip^2}.$$

Take an integer f such that p^f becomes larger than the index of N, then we have

$$\zeta(I_n+N)=(I_n+N)^{d_1}(I_n+N^p)^{d_p}\cdots(I_n+N^{p^f})^{d_{p^f}},$$

which can be written as

$$\zeta(I_n+N)=\prod_{i=0}^{\infty}(I_n+N^{p^i})^{d_{p^i}} \qquad \text{(finite product !)}.$$

Thus we have the following

THEOREM 7. *Any p-p operator ζ from $\mathfrak{R}$ into $\mathfrak{R}$ is either one of the following types:*

i) *$\zeta(A)=O_n$ for every matrix A in $\mathfrak{R}$, $n=d(A)$.*

ii) *$\zeta(A)=A^i$ for every matrix A in $\mathfrak{R}$, where i is a non-negative integer independent of A.*

iii) *ζ has as invariants a mapping g from K into K such that*

(3) $\qquad g(0)=0, \quad g(\alpha\beta)=g(\alpha)g(\beta), \quad g(\alpha)\neq 0 \quad (\text{for } \alpha\neq 0).$

and an element d in K, $(\chi(K)=0)$, or a sequence of elements d_i in K $(0\leqslant i<\infty)$ $(\chi(K)=p)$ respectively.

They are connected with ζ as follows:

$$\zeta(A)=\zeta(A^{(s)})\,\zeta(A^{(u)}) \quad \textit{if } A \textit{ is non-singular,}$$

where

$$\zeta(A^{(s)})=T^{-1}\big(g(\alpha_i)\,\delta_{ij}\big)T \qquad \textit{with } (\alpha_i\,\delta_{ij})=TA^{(s)}T^{-1},$$

$$\zeta(A^{(u)})=\begin{cases}\exp\,(d\log A^{(u)}) & (\chi(K)=0)\\ \prod\limits_{i=0}^{\infty}(I_n+N^{p^i})^{d_i} & (\chi(K)=p),\ \textit{where}\ A^{(u)}=I_n+N,\ n=d(A).\end{cases}$$

And for the general matrix $A=T(N\dot{+}A_0)T^{-1}$ (N*: n-matrix,* A_0*: non-singular*)

$$\zeta(A)=T\big(O_m\dot{+}\zeta(A_0)\big)T^{-1}\qquad \big(m=d(N)\big).$$

Conversely, for any given invariants consisting of g and d $\big(\chi(K)=0\big)$ *or* d_i $\big(\chi(K)=p\big)$, *there is one and only one p-p operator* ζ *having them as invariants.*

We shall call the p-p operators belonging to ii) or iii) in Theorem 7, i. e. those which have the property

$$\zeta(\mathfrak{S})\subset\mathfrak{S}$$

non-singular.

THEOREM 8. *Let* ζ *be a p-p operator from* $\mathfrak{N}$ *into* $\mathfrak{N}$.

α) *If* $AB=BA$, *then* $\zeta(AB)=\zeta(A)\zeta(B)$.

β) *If* ζ *is non-singular and* A *is in* $\mathfrak{S}$, *then* $\zeta({}^tA^{-1})={}^t\zeta(A)^{-1}$.

4. On the concept of replica.

As was stated in Theorem 2 γ), the concept of replica introduced by C. Chevalley [2] is in a close relation with s-s operators, so that it may be called s-s-replica. We shall now define other kinds of replicas, which we shall call s-p-, p-s- and p-p-replicas, and which are in the same relation to the corresponding operators as s-s-replicas to s-s-operators.

In the following, K need not be algebraically closed.

Let M be an n-dimensional vector space over K. We denote by $\mathfrak{gl}(M)$ the set of all linear endomorphisms of M over K, and by $GL(M)$ the set of non-singular ones in $\mathfrak{gl}(M)$. Let M^* be the dual space of M. We write (x,ξ) for the inner product of vectors $x\in M$ and $\xi\in M^*$. We shall denote by $M_{r,s}$ the set of (r,s)-tensors, i. e. the tensor product

$$\underbrace{M\otimes\cdots\otimes M}_{r}\otimes\underbrace{M^*\otimes\cdots\otimes M^*}_{s}.$$

For every $A\in\mathfrak{gl}(M)$, the transposed of A is denoted by ${}^tA\big(\in\mathfrak{gl}(M^*)\big)$, and $A_{r,s}\in\mathfrak{gl}(M_{r,s})$ is defined by

$$A_{r,s}=\underbrace{A\oplus\cdots\oplus A}_{r}\oplus\underbrace{(-{}^{t}A)\oplus(-{}^{t}A)\oplus\cdots\oplus(-{}^{t}A)}_{s}.$$

For every $A\in GL(M)$ we define $A_{(r,s)}$ by

$$A_{(r,s)}=\underbrace{A\oplus\cdots\oplus A}_{r}\oplus\underbrace{({}^{t}A^{-1})\oplus\cdots\oplus({}^{t}A^{-1})}_{s}.$$

Let $x\otimes\xi$ be an element in $M\otimes M^{*}$ ($x\in M$, $\xi\in M^{*}$). Then define an element A in $\mathfrak{gl}(M)$ by $Ay=(y,\xi)x$ for every y in M. It is easy to see that this mapping $x\otimes\xi\to A$ is a linear isomorphism from $M\otimes M^{*}$ onto $\mathfrak{gl}(M)$. We identify them under this isomorphism, then we have easily

$$A_{1,1}(X)=AX-XA=[A,X] \qquad \bigl(A\in\mathfrak{gl}(M)\quad X\in\mathfrak{gl}(M)\bigr)$$

$$A_{(1,1)}(X)=AXA^{-1} \qquad \bigl(A\in GL(M),\quad X\in\mathfrak{gl}(M)\bigr).$$

DEFINITON. Let A, B be in $\mathfrak{gl}(M)$ or in $GL(M)$.[8] We shall say that

B is an *s-s-replica* of A (in symbol: $A\underset{\text{s-p}}{\to}B$) if $\mathfrak{X}\in M_{r,s}$, $A_{r,s}\mathfrak{X}=0$ implies $B_{r,s}\mathfrak{X}=0$,

B is an *s-p-replica* of A ($A\underset{\text{s-p}}{\to}B$) if $\mathfrak{X}\in M_{r,s}$, $A_{r,s}\mathfrak{X}=0$ implies $B_{(r,s)}\mathfrak{X}=\mathfrak{X}$,

B is a *p-s-replica* of A ($A\underset{\text{p-s}}{\to}B$) if $\mathfrak{X}\in M_{r,s}$, $A_{(r,s)}\mathfrak{X}=\mathfrak{X}$ implies $B_{r,s}\mathfrak{X}=0$, and

B is a *p-p-replica* of A ($A\underset{\text{p-p}}{\to}B$) if $\mathfrak{X}\in M_{r,s}$, $A_{(r,s)}\mathfrak{X}=\mathfrak{X}$ implies $B_{(r,s)}\mathfrak{X}=\mathfrak{X}$.

where the implication must hold for all integers $r,s\geqslant 0$, $r+s>0$.

In the following we discuss in detail only on the p-p-replica. For simplicity, we write $\to$ for $\underset{\text{p-p}}{\to}$. Now we have

PROPOSITION. 1°) $(A_{(r,s)})_{(u,v)}=A_{(ru+sv,\,rv+su)}$.

2°) *$\to$ is a reflexive and transitive relation.*

3°) *If $A\to B$, then $A_{(r,s)}\to B_{(r,s)}$ for every r,s ($\geqslant 0$, $r+s>0$).*

4°) *The set of all p-p-replicas of A: $\{A\}_{\text{p-p}}=\{B; A\to B\}$ is a subgroup of $GL(M)$.*

5°) $(A^{(s)})_{(p,q)}=(A_{(p,q)})^{(s)}$, $(A^{(u)})_{(p,q)}=(A_{(p,q)})^{(u)}$.

6°) *Let N be a subspace of M such that $AN\subset N$. We denote by*

8) We do not define $A_{(r,s)}$ for a singular matrix A.

A_N, $A_{M/N}$ the linear endomorphisms induced by A on N and M/N respectively. Then

$$(A_N)^{(s)}=(A^{(s)})_N, \qquad (A_{M/N})^{(s)}=(A^{(s)})_{M/N},$$

$$(A_N)^{(u)}=(A^{(u)})_N, \qquad (A_{M/N})^{(u)}=(A^{(u)})_{M/N}.$$

All this is easy to prove.

PROPOSITION 2. 1°) *$A\to A^{(s)}$, $A\to A^{(u)}$ for every A in $GL(M)$.*
2°) *If $AB=BA$, then $(AB)^{(s)}=A^{(s)}B^{(s)}$, $(AB)^{(u)}=A^{(u)}B^{(u)}$.*
3°) *If $A\to B$, then B is a polynomial in A without constant term.*

PROOF. 1°) Let $x\in M$, $Ax=x$. Then $(A-I_n)x=0$. As $(A-I_n)^{(s)}$ is a polynomial in $A-I_n$ without constant term, we have $(A-I_n)^{(s)}x=0$, hence, $A^{(s)}x=x$. Therefore from Prop. 1, 5°) we have $A\to A^{(s)}$. Then $A^{(u)}=AA^{(s)-1}$ is in $\{A\}_{\text{p-p}}$ by Prop. 1, 4°). 2°) is obvious. 3°) $AXA^{-1}=X$ implies $BXB^{-1}=X$, hence $B-Z\{Z(A)\}$. Therefore B is a polynomial in A. As A is non-singular, I_n is a linear combination of A, $A^2,\cdots, A^n$. $\big(n=d(A)\big)$.

PROPOSITION 3. *If A is an s-matrix (u-matrix) and $A\to B$, then B is also an s-matrix (u-matrix).*

PROOF. If A is an s-matrix, then from Prop. 2, 3°) B is also an s-matrix. If A is a u-matrix, put $A=I_n+N$, (N: n-matrix), then from Prop. 2, 3°) there are $f+1$ elements $\alpha_0, \alpha_1, \cdots, \alpha_f$ in K such that

$$B=\alpha_0 I_n+\alpha_1 N+\cdots+\alpha_f N^f.$$

Take a vector $x\in M$ such that $x\neq 0$, $Nx=0$. Then $Ax=x$ implies that $Bx=x$. Hence $\alpha_0=1$ and B is a u-matrix.

PROPOSITION 4. *$A\to B$ holds if and only if both $A^{(s)}\to B^{(s)}$ and $A^{(u)}\to B^{(u)}$ hold.*

PROOF. Suppose $A^{(s)}\to B^{(s)}$ and $A^{(u)}\to B^{(u)}$. Then from Prop. 1, 4°), Prop. 2, 1°) we have $A\to B$. Conversely, let $A\to B$. We shall show first that $A^{(s)}x=x$ implies $B^{(s)}x=x$. Let N be the subspace of M defined by $N=\{x';\ x'\in M,\ A^{(s)}x'=x'\}$. Then we have $AN\subset N$, hence $A_N=A_N^{(u)}\to B_N$. Therefore B_N is a u-matrix by Prop. 3. Hence we have $B_N^{(s)}=I_N$, that is, $B^{(s)}x=x$. From this and Prop. 1, 3°), 5°), it follows that $A^{(s)}_{(p,q)}\mathfrak{X}=\mathfrak{X}$ implies $B^{(s)}_{(p,q)}\mathfrak{X}=\mathfrak{X}$, that is $A^{(s)}\to B^{(s)}$. Similarly we have $A^{(u)}\to B^{(u)}$.

PROPOSITION 5. *Taking a base in M, let $A=(\alpha_i\,\delta_{ij})$, $B=(\beta_i\,\delta_{ij})$. Then for $A\to B$, it is necessary and sufficient that for every set of* *integers* $m_1, \cdots, m_n$ *such that* $\prod_{i=1}^{\infty} \alpha_i^{m_i}=1$, *we have* $\prod_{i=1}^{n} \beta_i^{m_i}=1$.

PROOF. As is seen easily, we have, for $A=\alpha_1 I_1^{(1)} \dot{+} \alpha_2 I_1^{(2)} \dot{+} \cdots \dot{+} \alpha_n I_1^{(n)}$,

$$A_{(r,s)}=\sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_s=1}^{n} (\alpha_{i_1} \cdots \alpha_{i_r} \alpha_{j_1}^{-1} \cdots \alpha_{j_s}^{-1}) I_1^{(i_1)} \oplus \cdots \oplus I_1^{(i_r)} \oplus {}^t I_1^{(j_1)} \oplus \cdots \oplus {}^t I_1^{(j_s)}.$$

Then the very definition of $A \rightarrow B$ gives us the result.

PROPOSITION 6. *Let N be an n-matrix. Then for $A=I_n+N \rightarrow B$ it is necessary and sufficient that*

i) *if $\chi(K)=0$, there exists an element c in K such that $B=\exp\left(c \cdot \log(I_n+N)\right)$*

ii) *if $\chi(K)=p$, there exist element $f+1$ $c_0, c_1, \cdots, c_f$ in K such that*

$$B=\prod_{i=1}^{f} (I_n+N^{p^i})^{c_i}.$$

PROOF. *Sufficiency.* i) $\chi(K)=0$. Put $\log(I_n+N)=M$. Then we have $(I_n+N)_{(r,s)}=\exp M_{r,s}$ and $A_{(r,s)}\mathfrak{X}=\mathfrak{X}$ holds if and only if $M_{r,s}\mathfrak{X}=0$. Thus we have $I_n+N \rightarrow B$.

ii) $\chi(K)=p$. Sufficiency is obvious from $I_n+N \rightarrow I_n+N^{p^i}$.

Necessity. Let $I_n+N \rightarrow B$. Then B is a polynomial in N: $B=\sum_{i=0}^{\infty} c_i N^i$. Similarly, $(I_n+N)_{(2,0)} \rightarrow B_{(2,0)}$ implies that $B \otimes B$ is a polynomial in $(I_n+N) \otimes (I_n+N)-I_{n^2}$: $B \otimes B=\sum_{i=1}^{\infty} d_i (N \otimes I_n+I_n \otimes N+N \otimes N)^i$.

As B is a u-matrix we have $c_0=d_0=1$. Let the index of N be r. Then the same calculation as in § 3 shows that

$$c_i c_j=\sum_{t=0}^{i} \Delta_{ijt}\, d_{j+t} \qquad (0 \leqslant i \leqslant j \leqslant r-1)$$

Putting $i=0$, we have $c_0 c_j=d_j$, hence $c_j=d_j$. Therefore the above equations become of the same type as (1), hence our conclusion follows.

From the above propositions and analogous propositions on s-p- and p-s-replicas, which are proved similary, follows the

THEOREM 9. i) *$A \underset{\text{p-p}}{\rightarrow} B$ holds if and only if there exists a non-singular p-p operator ζ such that $\zeta(A)=B$.*

ii) *$A \underset{\text{s-p}}{\rightarrow} B$ holds if and only if there exists a non-singular s-p operator ζ such that $\zeta(A)=B$.*

iii) *If* $\chi(K)=0$, *then* $A \underset{\text{p-s}}{\rightarrow} B$ *holds if and only if there exists a p-s operator* ζ *from* $\mathfrak{S}$ *into* $\mathfrak{N}$ *such that* $\zeta(A)=B$.

REMARK. If $\chi(K)=p$, then $\zeta(A)=B$ implies $A \underset{\text{p-s}}{\rightarrow} B$. But the converse is not true. In fact, take an element α in K which is not a root of unity and an element $\beta\neq 0$ in K. Then we have $\alpha I_1 \underset{\text{p-s}}{\rightarrow} \beta I_1$, but there exists no p-s operator ζ such that $\zeta(\alpha I_1)=\beta I_1$.

Appendix.

In this appendix we shall examine the case in which K is not algebraically closed.

When K is not algebraically closed, s-matrices are not necessarily transformed into the diagonal form, and above discussions in § 2, 3 do not apply. We did not succeed in complete determination of s-s, s-p, p-s and p-p operators in this case, but some remarks about this case will be given below.

Let K be any infinite perfect field and K be its algebraic closure. We shall discuss only s-s operators because other operators can be treated almost similarly, Let ζ be an s-s operator from $\mathfrak{N}(k)$ into $\mathfrak{N}(k)$. k being perfect, $A^{(s)}$ and $A^{(n)}$ belong to $\mathfrak{N}(k)$ with A. As was remarked in § 1, 2 we have the following

PROPOSITION 7. i) *If* $A\in\mathfrak{N}(k)$, *then* $\zeta(A)$ *is a polynomial in* A *with coefficients in* k.

ii) ζ *determines an endomorphism* g *of the additive group* K *and a* C-*sequence* $c_0, c_1,\cdots$ *in* k. *They are connected with* ζ *as follows:*

$$\zeta(\alpha I_1)=g(\alpha)I_1 \qquad \textit{for every } \alpha \textit{ in } k,$$

$$\zeta(N)=\sum_{i=1}^{\infty} c_i N^i \qquad \textit{for every } n\textit{-matrix } N \textit{ in } \mathfrak{N}(k).$$

(*We shall call* g *and* $c_0, c_1,\cdots$ *the invariants of* ζ).

Furthermore we have

iii) $\zeta(A)^{(s)}=\zeta(A^{(s)})$ *for every matrix* A *in* $\mathfrak{N}(k)$.

PROOF OF iii). We shall denote elements in $\mathfrak{N}(K)$ by $\widetilde{A}, \widetilde{B},\cdots$. Take a matrix $\widetilde{P}$ such that

$$\widetilde{P}A\widetilde{P}^{-1}=(\alpha_1 I_{d_1}+\widetilde{N}_1)\dot{+}\cdots\dot{+}(\alpha_r I_{d_r}+\widetilde{N}_r),$$

$\tilde{N}_i$: n-matrix of degree d_i.

Now there are polynomials f, h such that $\zeta(A)=f(A)$, $\zeta(A^{(s)})=h(A^{(s)})$. Then we have

$$\tilde{P}\,\zeta(A)^{(s)}\tilde{P}^{-1}=f(\alpha_1)\,I_{d_1}\dot{+}\cdots\dot{+}f(\alpha_r)\,I_{d_r},$$

$$\tilde{P}\,\zeta(A^{(s)})\tilde{P}^{-1}=h(\alpha_1)\,I_{d_1}\dot{+}\cdots\dot{+}h(\alpha_r)\,I_{d_r}.$$

On the other hand, there is a polynomial φ such that $\zeta(A\dot{+}B^{(s)})=\varphi(A\dot{+}A^{(s)})$, hence $(\tilde{P}\dot{+}\tilde{P})\,\big(\zeta(A\dot{+}A^{(s)})\big)^{(s)}(\tilde{P}\dot{+}\tilde{P})^{-1}=\varphi(\alpha_1)\,I_{d_1}\dot{+}\cdots\dot{+}\varphi(\alpha_r)I_{d_r}\dot{+}\varphi(\alpha_1)\,I_{d_1}\dot{+}\cdots\dot{+}\varphi(\alpha_r)\,I_{d_r}$. Since $\zeta(A\dot{+}A^{(s)})^{(s)}=\zeta(A)^{(s)}\dot{+}\zeta(A^{(s)})$ we have

$$f(\alpha_i)=\varphi(\alpha_i)=h(\alpha_i)\qquad(1\leqslant i\leqslant r).$$

Hence we have $\zeta(A^{(s)}=\zeta(A)^{(s)}$.

Now we shall need the following

LEMMA 5. *For every s-s operator ζ from $\Re(k)$ into $\Re(k)$ there is one and only one s-s operator $\bar{\zeta}$ from $\Re(K)$ into $\Re(K)$ such that*

$$\zeta(A)=\bar{\zeta}(A)\quad\textit{for every s-matrix } A \textit{ or n-matrix } A \textit{ in } \Re(K).$$

PROOF. Let ζ have the invariants g and $c_0, c_1,\cdots$. Let us define the invariants $\bar{g}$ and $\bar{c}_0, \bar{c}_1,\cdots$ of $\bar{\zeta}$. Put $\bar{c}_i=c_i$ $(i=0, 1,\cdots)$.

Next let us define $\bar{g}$. First, for α in k we put $\bar{g}(\alpha)=g(\alpha)$. If ω is in K but not in k, denote the set of distinct k-conjugates of ω by

$$\omega_1,\cdots,\omega_n\qquad(\omega_1=\omega),$$

and define a matrix $T(\omega)=T(\omega_1,\cdots,\omega_n)=(\xi_{ij})$ in $\Re(K)$ of degree n as follows:

$$\begin{cases}\xi_{1j}=\omega_1\omega_2\cdots\hat{\omega}_j\cdots\omega_n, & (\wedge \text{ means that } \omega_j \text{ should be omitted.})\\ \xi_{2j}=\sum\limits_{\nu\neq j}^{n}\omega_1\cdots\hat{\omega}_\nu\cdots\hat{\omega}_j\cdots\omega_n, & \\ \cdots\cdots\cdots\cdots\cdots, & \\ \xi_{n-1,j}=\omega_1+\cdots+\hat{\omega}_j+\cdots+\omega_n, & \\ \xi_{n,j}=1. & \end{cases}$$

Now let us denote the minimum equation over k for ω by $x^n+a_1x^{n-1}+\cdots+a_n=0$. Then a simple calculation shows that

$$T(\omega)\begin{pmatrix}\omega_1 & & \\ & \ddots & \\ & & \omega_n\end{pmatrix}T(\omega)^{-1}=\begin{pmatrix}0 & & & 0 & (-1)^n a_n\\ -1 & 0 & & & \vdots \\ & -1 & \ddots & & \vdots \\ & & \ddots & 0 & a_2 \\ 0 & & & -1 & -a_1\end{pmatrix},$$

where $\det T(\omega)=\pm\prod_{i<j}(\omega_i-\omega_j)\neq 0$.

Thus we have established that for given $\omega_1,\cdots,\omega_n$ there are matrices $\tilde{P}$ in $\mathfrak{M}(K)$ and Ω in $\mathfrak{M}(K)$ such that $(\omega_j\,\delta_{ij})=\tilde{P}\Omega\tilde{P}^{-1}$.

Now, $\zeta(\Omega)$ being a polynomial in Ω, $\tilde{P}\zeta(\Omega)\tilde{P}^{-1}$ is also a diagonal matrix: $\tilde{P}\zeta(\Omega)\tilde{P}=(\eta_i\,\delta_{ij})$.

We define $\bar{g}$ by $\bar{g}(\omega_i)=\eta_i \quad (1\leqslant i\leqslant n)$.

In general, if $\omega_1,\cdots,\omega_n$ are in K and if there are matrices $\tilde{P}\in\mathfrak{M}(K)$, $\Omega\in\mathfrak{M}(k)$ such that $(\omega_i\delta_{ij})=\tilde{P}\Omega\tilde{P}^{-1}$ holds, we define $\bar{g}$ as above $\big(\bar{g}(\omega_i)\delta_{ij}\big)=\tilde{P}\zeta(\Omega)\tilde{P}^{-1}$. We must now show that the definition of $\bar{g}(\omega)$ is independent on $\omega_2,\cdots,\omega_n$, $\tilde{P}$ and Ω.

First we show that it does not depend on $\tilde{P}$ and Ω. If

$$(\omega_i\,\zeta_{ij})=\tilde{P}\Omega\tilde{P}^{-1}=\tilde{Q}W\tilde{Q}^{-1} \qquad \big(\Omega,\,W\in\mathfrak{M}(k)\big),$$

then Ω and W are similar in K, hence similar in k. Thus there is a matrix T in $\mathfrak{M}(k)$ such that $W=T\Omega T^{-1}$. Then, $\tilde{P}^{-1}\tilde{Q}T$ being commutative with Ω it is also commutative with $\zeta(\Omega)$ (by prop. 7, i)) $\tilde{P}\zeta(\Omega)\tilde{P}^{-1}=\tilde{Q}T\zeta(\Omega)T^{-1}\tilde{Q}^{-1}$. On the other hand we have $\zeta(W)=T\zeta(\Omega)T^{-1}$, so that we have $\tilde{P}\zeta(\Omega)\tilde{P}^{-1}=\tilde{Q}\zeta(W)\tilde{Q}^{-1}$ which was to show.

Next let us show that $\bar{g}(\omega)$ does not depend on $\omega_2,\cdots,\omega_n$. Let $\omega_1=\theta_1$ and

(*) $\quad (\omega_i\,\delta_{ij})=\tilde{P}\Omega\tilde{P}^{-1},\quad \Omega\in\mathfrak{M}(k),\quad \big(\bar{g}_1(\omega_i)\delta_{ij}\big)=\tilde{P}\zeta(\Omega)\tilde{P}^{-1},\quad d(\Omega)=n,$

(**) $\quad (\theta_i\,\delta_{ij})=\tilde{Q}\Theta\tilde{Q}^{-1},\quad \Theta\in\mathfrak{M}(k),\quad \big(\bar{g}_2(\theta_i)\delta_{ij}\big)=\tilde{Q}\zeta(\Theta)\tilde{Q}^{-1},\quad d(\Theta)=m.$

Then we have

$$(\omega_i\,\delta_{ij})\dot{+}(\theta_i\,\delta_{ij})=(\tilde{P}\dot{+}\tilde{Q})\,(\Omega\dot{+}\Theta)\,(\tilde{P}\dot{+}\tilde{Q})^{-1},$$

$$\left(\bar{g}_3(\omega_i)\partial_{ij}\right)\dot{+}\left(\bar{g}_3(\theta_i)\delta_{ij}=(\tilde{P}\dot{+}\tilde{Q})\,\zeta(\Omega\dot{+}\Theta)\,(\tilde{P}\dot{+}\tilde{Q})^{-1}\right..$$

Now, $\zeta(\Omega\dot{+}\Theta)=\zeta(\Omega)\dot{+}\zeta(\Theta)$ implies that

$$\bar{g}_1(\omega_1)=\bar{g}_3(\omega_1)=\bar{g}_3(\theta_1)=\bar{g}_2(\theta_1)\,,$$

which was to show.

Next let us show that $\bar{g}$ is additive. From (*), (**) we have

$$(\omega_i\delta_{ij})\oplus(\theta_i\delta_{ij})=\dot{\sum_{i,j}}(\omega_i+\theta_j)I_1=(\tilde{P}\oplus\tilde{Q})\,(\Omega\oplus\Theta)\,(\tilde{P}\oplus\tilde{Q})^{-1}\,.$$

Now $\zeta(\Omega+\Theta)=\zeta(\Omega)\oplus\zeta(\Theta)$ implies that

$$\bar{g}(\omega_1+\theta_1)=\bar{g}(\omega_1)+\bar{g}(\theta_1)\,.$$

Thus the invariants $\bar{g}$ and $\bar{c}_0,\bar{c}_1,\cdots$ are defined. Let $\bar{\zeta}$ be an s-s operator having them as invariants. Then it is easy to verify that $\bar{\zeta}$ is a desired s. s. operators. The construction of $\bar{g}$ shows also that $\bar{\zeta}$ is unique. (Remark that $\bar{\zeta}$ is not necessarily an extension of ζ).

Now using this lemma we shall prove the following

THEOREM 10. *Let ζ be an s-s operator from $\mathfrak{N}(k)$ into $\mathfrak{N}(k)$. Then the following conditions are equivalent to each other.*

1) *ζ can be extended to an s-s operator $\bar{\zeta}$ from $\mathfrak{N}(K)$ into $\mathfrak{N}(K)$.*
2) *For any A, B in $\mathfrak{N}(k)$ such that $AB=BA$, we have $\zeta(A)+\zeta(B)=\zeta(A+B)$.*
3) *$\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(n)})$ for every A in $\mathfrak{N}(k)$.*
4) *$\zeta(A)^{(n)}=\zeta(A^{(n)})$ for every A in $\mathfrak{N}(k)$.*
5) *$A\underset{\text{s-s}}{\rightarrow}\zeta(A)$ for every A in $\mathfrak{N}(k)$.*
6) *For every A in $\mathfrak{N}(k)$ there are elements $\alpha_0,\alpha_1,\cdots,\alpha_r$ in k such that*

$$\zeta(A)^{(n)}=\sum_{i=0}^{r}\alpha_i\,A^{(n)i}\,.$$

PROOF. By Lemma 5 and Proposition 7, iii), implications

$$1)\to2)\to3)\to4)\to5)\to6)\,.$$

are obvious. By proposition 7, iii) we have moreover 4)$\to$3), and by Lemma 5, 3)$\to$1). So it is sufficient to show 6)$\to$4). This is shown as follows. Remark that the mapping $A\to\zeta(A)^{(n)}$ is also an s-s operator from $\mathfrak{N}(k)$ into $\mathfrak{N}(k)$. Then the same discussion as in the proof of Theorem 1 shows that there are elements $\gamma_0,\gamma_1,\cdots$ in k which are independent on A, satisfying

$$\zeta(A)^{(n)}=\sum_{i=0}^{\infty}\gamma_i A^{(n)i} \qquad \text{for every matrix } A \text{ in } R(k).$$

Hence we have $\gamma_0=0$ and

$$\zeta(A)^{(n)}=\zeta(A^{(n)})^{(n)}=\zeta(A^{(n)}).$$

REMARK. Perhaps the conditions in Theorem 10 are satisfied by every s-s operator from $\mathfrak{R}(k)$ in $\mathfrak{R}(k)$, but we can neither prove nor disprove it.

Finally, ζ being an s-s operator from $\mathfrak{R}(K)$ into $\mathfrak{R}(K)$, we give a condition that ζ maps $\mathfrak{R}(k)$ into $\mathfrak{R}(k)$. Let the invariants of ζ be g and $c_0, c_1\cdots$ and G be the Galois group of K/k. Then we have

THEOREM 11. *It is necessary and sufficient for* $\zeta(\mathfrak{R}(k))\subset\mathfrak{R}(k)$ *that* $c_0, c_1,\cdots$ *belong to* k *and* $\sigma(g(\omega))=g(\sigma(\omega))$ *for every* ω *in* K *and for every* σ *in* G.

PROOF. *Necessity.* Obviously $c_0, c_1,\cdots$ must belong to k. Next let $\omega\in K$. Denote the all distinct k-conjugates of ω by $\omega_1,\cdots,\omega_n(\omega_1=\omega)$, and by $x^n+a_1x^{n-1}+\cdots+a_n=0$ the minimum equation for ω over k. Then

$$T(\omega_1,\cdots,\omega_n)\,(\omega_i\delta_{ij})\,T(\omega_1,\cdots,\omega_n)^{-1}=\begin{pmatrix} 0 & & 0 & (-1)^n a_n \\ -1 & 0 & & \vdots \\ & -1 & \ddots & \vdots \\ & & \ddots\,0 & a_2 \\ 0 & & -1 & -a_1 \end{pmatrix}=A\in\mathfrak{R}(k).$$

Let $\sigma\in G$ and $\sigma(\omega_1,\cdots,\omega_n)=(\omega_{p_1},\cdots,\omega_{p_n})$. We denote by (σ) the permutation $\begin{pmatrix}1 & 2 & \cdots & n\\ p_1 & p_2 & \cdots & p_n\end{pmatrix}$ and define a matrix $P_{(\sigma)}$ by

$$P_{(\sigma)}=(e_{p_1},\cdots,e_{p_n}), \qquad e_i=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}(i \qquad (i\text{-th unit vector}).$$

Then we have

$$\sigma T(\omega)=T(\omega_{p_1},\cdots,\omega_{p_n})\,P_{(\sigma)}, \qquad P_{(\sigma)}(\omega_{p_i}\delta_{ij})\,P_{(\sigma)}^{-1}=(\omega_i\delta_{ij}).$$

Now, $\zeta(A)=T(\omega)\big(g(\omega_i)\delta_{ij}\big)T(\omega)^{-1}\in\mathfrak{R}(k)$ implies that $\sigma\zeta(A)=\zeta(A)$, hence we have

$$P_{(\sigma)}\big(\sigma(g(\omega_i)\delta_{ij}\big)\,P_{(\sigma)}^{-1}=\big(g(\omega_i)\delta_{ij}\big),$$

that is $\sigma\big(g(\omega_1)\big)=g(\omega_{p_1})=g\big(\sigma(\omega_1)\big)$.

Sufficiency. If we follow the above discussion in the converse direction, we see that for every s-matrix A in $\mathfrak{R}(k)$ having irreducible minimum equation over k, we have $\zeta(A)\in\mathfrak{R}(k)$. However, every s-matrix B in $\mathfrak{R}(k)$ can be expressed as a direct sum of such matrices A, hence $\zeta(B)\in\mathfrak{R}(k)$. Now, if N is any n-matrix in $\mathfrak{R}(k)$, we have

$$\zeta(N)=\sum_{i=1}^{\infty} c_i N^i \in \mathfrak{R}(k)\,.$$

Thus for every matrix A in $\mathfrak{R}(k)$ we have

$$\zeta(A)=\zeta(A^{(s)})+\zeta(A^{(n)})\in\mathfrak{R}(k)\,.$$

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